In these notes, I will provide some useful relations involving the generators of $\mathfrak{su}(N)$ Lie algebra. We employ the physicist's convention, where the N^2-1 generators in the defining representation of $\mathfrak{su}(N)$, denoted by T^a, serve as a basis for the set of traceless hermitian $N\times N$ matrices. The generators satisfy the commutation relations,

$$[T^a\,,\,T^b]=if_{abc}T^c\,,\qquad\qquad \text{where } a,b,c=1,2,\ldots,N^2-1. \tag{1}$$

In particular

$$\mathrm{Tr}\,T^a=0\,. \tag{2}$$

We employ the following normalization convention for the generators in the defining representation of $\mathfrak{su}(N)$,

$$\mathrm{Tr}(T^aT^b)=\tfrac{1}{2}\delta_{ab}\,. \tag{3}$$

In this convention, the f^{abc} are totally antisymmetric with respect to the interchange of any pair of indices.

Consider a d-dimensional irreducible representation, R^a of the generators of $\mathfrak{su}(N)$. The quadratic Casimir operator, $C_2\equiv R^aR^a$, commutes with all the $\mathfrak{su}(N)$ generators.[1] Hence in light of Schur's lemma, C_2 is proportional to the $d\times d$ identity matrix. In particular, the quadratic Casimir operator in the defining representation of $\mathfrak{su}(N)$ is given by

$$T^aT^a=C_F\mathbb{1}\,, \tag{4}$$

where $\mathbb{1}$ is the $N\times N$ identity matrix. To evaluate C_F, we take the trace of eq. (4) and make use of $\mathrm{Tr}\,\mathbb{1}=N$. Summing over a, we note that $\delta_{aa}=N^2-1$. Using the normalization of the generators specified in eq. (3), it follows that $\frac{1}{2}(N^2-1)=NC_F$. Hence,[2]

$$C_F=\frac{N^2-1}{2N}\,. \tag{5}$$

Next we quote an important identity involving the $\mathfrak{su}(N)$ generators in the defining representation,

$$T^a_{ij}T^a_{k\ell}=\frac{1}{2}\left(\delta_{i\ell}\delta_{jk}-\frac{1}{N}\delta_{ij}\delta_{k\ell}\right)\,, \tag{6}$$

[1]It is straightforward to show that C_2 commutes with all the generators of $\mathfrak{su}(N)$. In particular, using the commutation relations, $[R^a\,,\,R^b]=if_{abc}R^c$,

$$\big[R^aR^a\,,\,R^b\big]=R^a\big[R^a\,,\,R^b\big]+\big[R^a\,,\,R^b\big]R^a=if^{abc}(R^aR^c+R^cR^a)=0\,,$$

due to the antisymmetry of f^{abc} under the interchange of any pair of indices.

[2]In the older literature, the defining representation is (inaccurately) called the fundamental representation. It is for this reason that the Casimir operator in the defining representation is often denoted by C_F.

where the indices i, j, k and ℓ take on values from $1, 2, \ldots, N$. To derive eq. (6), we first note that any $N \times N$ complex matrix M can be written as a complex linear combination of the $N \times N$ identity matrix and the T^a,

$$M = M_0 \mathbb{1} + M_a T^a \,. \tag{7}$$

This can be regarded as a completeness relation on the vector space of complex $N \times N$ matrices. One can project out the coefficient M_0 by taking the trace of eq. (7). Likewise, one can project out the coefficients M_a by multiplying eq. (7) by T^b and then taking the trace of the resulting equation. Using eqs. (2) and (3), it follows that

$$M_0 = \frac{1}{N} \operatorname{Tr} M \,, \qquad\qquad M_a = 2 \operatorname{Tr}(M T^a) \,. \tag{8}$$

Inserting these results back into eq. (7) yields

$$M = \frac{1}{N} (\operatorname{Tr} M) \mathbb{1} + 2 \operatorname{Tr}(M T^a) T^a \,. \tag{9}$$

The matrix elements of eq. (9) are therefore

$$M_{ij} = \frac{1}{N} M_{kk} \delta_{ij} + 2 M_{\ell k} T^a_{k\ell} T^a_{ij} \,, \tag{10}$$

where the sum over repeated indices is implicit. We can rewrite eq. (10) in a more useful form,

$$\delta_{i\ell} \delta_{jk} M_{\ell k} = \left(\frac{1}{N} \delta_{ij} \delta_{k\ell} + 2 T^a_{ij} T^a_{k\ell} \right) M_{\ell k} \,. \tag{11}$$

It follows that

$$\left[T^a_{ij} T^a_{k\ell} - \frac{1}{2} \left(\delta_{i\ell} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{k\ell} \right) \right] M_{\ell k} = 0 \,. \tag{12}$$

This equation must be true for any arbitrary $N \times N$ complex matrix M. It follows that the coefficient of $M_{\ell k}$ in eq. (12) must vanish. This yields the identity states in eq. (6). The proof is complete.

Many important identities can be obtained from eq. (6). For example, multiplying eq. (6) by T^b_{jk} and summing over j and k yields

$$T^a T^b T^a = -\frac{1}{2N} T^a \,, \tag{13}$$

after employing eq. (2). If we now multiply eq. (13) by T^c and take the trace of both sides of the resulting equation, then the end result is

$$\operatorname{Tr}(T^a T^b T^a T^c) = -\frac{1}{4N} \delta_{bc} \,. \tag{14}$$

after using eq. (3). A more general expression for the trace of four generators (of which eq. (14) is a special case) is given in Appendix B.

In $\mathfrak{su}(N)$, one can also define a totally symmetric third rank tensor called d^{abc} via the relation,

$$T^a T^b = \frac{1}{2}\left[\frac{1}{N}\delta_{ab}\mathbb{1} + (d_{abc} + if_{abc})T^c\right], \tag{15}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix. Combining eqs. (1) and (15) yields the following anticommutation relation,

$$\{T^a\,,\,T^b\} \equiv T^a T^b + T^b T^a = \frac{1}{N}\delta_{ab}\mathbb{1} + d_{abc}T^c\,, \tag{16}$$

Using eqs. (3) and (16), one obtains an explicit expression,

$$d_{abc} = 2\,\mathrm{Tr}\big[\{T^a\,,\,T^b\}T^c\big]\,, \tag{17}$$

which can be taken as the definition of the d_{abc}. It then follows that $d_{aac} = 0$ (where a sum over the repeated index a is implicit). Indeed, since d_{abc} is a totally symmetric tensor, it follows that $d_{aca} = d_{caa} = 0$.

The case of $\mathfrak{su}(2)$ provides the simplest example. In this case, we identify $T^a = \frac{1}{2}\sigma^a$, where the σ^a (for $a = 1, 2, 3$) are the well-known Pauli matrices, and $f_{abc} = \epsilon_{abc}$ are the components of the Levi-Civita tensor. It is a simple matter to check that in the case of $\mathfrak{su}(2)$, we have $d_{abc} = 0$. In contrast, the d_{abc} are generally non-zero for $N \geq 3$.

Consider the trace identity obtained by multiplying eq. (15) by T^c and taking the trace. In light of eqs. (2) and (3),

$$\mathrm{Tr}(T^a T^b T^c) = \tfrac{1}{4}\left(d_{abc} + if_{abc}\right)\,. \tag{18}$$

It then follows that

$$f_{abd}\,\mathrm{Tr}(T^a T^b T^c) = \tfrac{1}{4} i f_{abc} f_{abd}\,, \tag{19}$$

$$d_{abd}\,\mathrm{Tr}(T^a T^b T^c) = \tfrac{1}{4} d_{abc} d_{abd}\,. \tag{20}$$

In obtaining eqs. (19) and (20), we used the fact that d_{abc} is symmetric and f_{abc} is antisymmetric under the interchange of any pair of indices, which implies that

$$f_{abc} d_{abd} = 0\,. \tag{21}$$

To evaluate the products $f_{abc}f_{abd}$ and $d_{abc}d_{abd}$, we proceed as follows. Using eqs. (1) and (16),

$$f_{abd}\,\mathrm{Tr}(T^a T^b T^c) = -i\,\mathrm{Tr}\big([T^b\,,\,T^d]T^b T^c\big) = -i\,\mathrm{Tr}(T^b T^d T^b T^c) + i\,\mathrm{Tr}(T^d T^b T^b T^c)\,, \tag{22}$$

$$\begin{aligned} d_{abd}\,\mathrm{Tr}(T^a T^b T^c) &= \mathrm{Tr}\left[\left(\{T^b\,,\,T^d\} - \frac{1}{N}\delta_{bd}\mathbb{1}\right)T^b T^c\right] \\ &= \mathrm{Tr}(T^b T^d T^b T^c) + \mathrm{Tr}(T^d T^b T^b T^c) - \frac{1}{N}\,\mathrm{Tr}(T^d T^c)\,. \end{aligned} \tag{23}$$

The traces are easily evaluated using eqs. (3)–(5) and (14), and we end up with

$$f_{abd}\,\mathrm{Tr}(T^aT^bT^c)=\tfrac{1}{4}iN\delta_{cd}\,,\tag{24}$$

$$d_{abd}\,\mathrm{Tr}(T^aT^bT^c)=\left(\frac{N^2-4}{4N}\right)\delta_{cd}\,.\tag{25}$$

Comparing eqs. (24) and (25) with eqs. (19) and (20), we conclude that,[3]

$$f_{abc}f_{abd}=N\delta_{cd}\,,\tag{26}$$

$$d_{abc}d_{abd}=\left(\frac{N^2-4}{N}\right)\delta_{cd}\,.\tag{27}$$

Consider a d-dimensional irreducible representation, R^a of the generators of $\mathfrak{su}(N)$. The cubic Casimir operator $C_3\equiv d_{abc}R^aR^bR^c$, commutes with all the $\mathfrak{su}(N)$ generators. Hence in light of Schur's lemma, C_3 is proportional to the $d\times d$ identity matrix. In particular, the cubic Casimir operator in the defining representation of $\mathfrak{su}(N)$ is given by

$$d_{abc}T^aT^bT^c=C_{3F}\mathbb{1}\,.\tag{28}$$

To evaluate C_{3F}, we multiply eq. (15) d_{abd} to obtain

$$d_{abc}T^aT^b=\frac{N^2-4}{2N}T^c\,,\tag{29}$$

after using eqs. (26) and (27). Multiplying the above result by T^c and employing eq. (4) yields

$$d_{abc}T^aT^bT^c=\frac{N^2-4}{2N}C_F\mathbb{1}\,.\tag{30}$$

Hence, using eqs. (5) and (28), we obtain

$$C_{3F}=\frac{(N^2-1)(N^2-4)}{4N^2}\,.$$

For completeness, we note the following result that resembles eq. (29),

$$f_{abc}T^aT^b=\tfrac{1}{2}\big(\{T^a\,,\,T^b\}+[T^a\,,\,T^b]\big)=\tfrac{1}{2}f_{abc}[T^a\,,\,T^b]=\tfrac{1}{2}if_{abc}f_{abd}T^d=\tfrac{1}{2}iNT^c\,,$$

after employing eq. (24). Hence, in light of eqs. (4) and (5) it follows that

$$f_{abc}T^aT^bT^c=\tfrac{1}{2}iNC_F\mathbb{1}=\tfrac{1}{4}i(N^2-1)\mathbb{1}\,.$$

Indeed, in any irreducible representation of $\mathfrak{su}(N)$, a similar analysis yields

$$f_{abc}R^aR^bR^c=\tfrac{1}{2}iNC_2\,,\tag{31}$$

where $C_2\equiv R^aR^a$ is the quadratic Casimir operator in representation R. Hence, $f_{abc}R^aR^bR^c$ is proportional to C_2 and thus is not an independent Casimir operator.[4]

[3]Note that eqs. (24), (25) and (21) are equivalent to eqs. (44) and (45), respectively.

[4]It may seem that eq. (30) implies that the cubic Casimir operator is proportional to the quadratic Casimir operator. However, the derivation of eq. (30) relies on eq. (15), which only applies to the generators of $\mathfrak{su}(N)$ in the defining representation. For an arbitrary d-dimensional irreducible representation of $\mathfrak{su}(N)$, C_2 and C_3 are generically independent.

We now introduce the generators of $\mathfrak{su}(N)$ in the adjoint representation, which will be henceforth denoted by F^a. The F^a are $(N^2-1)\times(N^2-1)$ antisymmetric matrices, since the dimension of the adjoint representation is equal to the number of generators of $\mathfrak{su}(N)$. Explicitly, the matrix elements of the adjoint representation generators are determined by the structure constants,

$$(F^a)_{bc} = -if_{abc}\,. \tag{32}$$

It is also convenient to define a set of $(N^2-1)\times(N^2-1)$ traceless symmetric matrices

$$(D^a)_{bc} = d_{abc}\,, \tag{33}$$

where the d_{abc} is defined by eq. (17). Since $d_{abb}=0$ it follows that $\operatorname{Tr} D^a = 0$. The properties of the F^a and D^a matrices have been examined in Refs. [1,2].

The F^a satisfy the commutation relations of the $\mathfrak{su}(N)$ generators,

$$[F^a\,,\,F^b] = if_{abc}F^c\,, \tag{34}$$

which is equivalent to the Jacobi identity,

$$f_{abe}f_{ecd} + f_{cbe}f_{aed} + f_{dbe}f_{ace} = 0\,. \tag{35}$$

Likewise, there is a second commutation relation of interest,

$$[F^a\,,\,D^b] = [D^a\,,\,F^b] = if_{abc}D^c\,, \tag{36}$$

which is equivalent to the two identities,

$$f_{abe}d_{cde} + f_{ace}d_{bde} + f_{ade}d_{bce} = 0\,, \tag{37}$$

$$f_{abe}d_{cde} + f_{cbe}d_{ade} + f_{dbe}d_{ace} = 0\,. \tag{38}$$

The relations,

$$F^aD^b + F^bD^a = D^aF^b + D^bF^a = d_{abc}F^c\,, \tag{39}$$

are also noteworthy. Combining eqs. (36) and (39) yields,

$$F^aD^b + D^aF^b = d_{abc}F^c + if_{abc}D^c\,. \tag{40}$$

The expression for the commutator $[D^a\,,\,D^b]$ is more complicated,

$$\bigl[D^a\,,\,D^b\bigr]_{cd} = if_{abe}(F^e)_{cd} - \frac{2}{N}\biggl(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\biggr)\,, \tag{41}$$

which is equivalent to the identity,

$$f_{abe}f_{cde} = \frac{2}{N}\biggl(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\biggr) + d_{ace}d_{bde} - d_{bce}d_{ade}\,. \tag{42}$$

Interchanging $b \leftrightarrow c$ and subtracting, the resulting expression can be rewritten as

$$(F^aF^b + D^aD^b)_{cd} = \frac{2}{N}\biggl(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\biggr) + d_{abe}(D^e)_{cd} + if_{abe}(F^e)_{cd}\,. \tag{43}$$

Eq. (43) is equivalent to the identity,

$$f_{ace}f_{bde} - f_{abe}f_{cde} = \frac{2}{N}\Big(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\Big) + d_{abe}d_{cde} - d_{ace}d_{bde}\,.$$

The quadratic Casimir operator in the adjoint representation is

$$F^aF^a = C_A\mathbf{I}\,, \qquad\qquad \text{where } C_A = N, \tag{44}$$

and $\mathbf{I}$ is the $(N^2-1)\times(N^2-1)$ identity matrix, which is equivalent to eq. (26). Two other similar expressions of interest are

$$D^aD^a = \left(\frac{N^2-4}{N}\right)\mathbf{I}\,, \qquad\qquad F^aD^a = 0\,, \tag{45}$$

which are equivalent to eqs. (27) and (21), respectively.

Using the above results, we can derive additional identities of interest. For example,

$$f_{abc}F^bF^c = \tfrac{1}{2}f_{abc}\big[F^b\,,\,F^c\big] = \tfrac{1}{2}if_{abc}f_{bcd}F^d = \tfrac{1}{2}iNF^a\,, \tag{46}$$

$$f_{abc}F^bD^c = \tfrac{1}{2}f_{abc}\big[F^b\,,\,D^c\big] = \tfrac{1}{2}if_{abc}f_{bcd}D^d = \tfrac{1}{2}iND^a\,, \tag{47}$$

$$f_{abc}D^bD^c = \tfrac{1}{2}f_{abc}\big[D^b\,,\,D^c\big] = \tfrac{1}{2}i\left(f_{abc}f_{bcd} - \frac{4}{N}\delta_{ad}\right)F^d = i\left(\frac{N^2-4}{2N}\right)F^a\,. \tag{48}$$

It then follows that

$$f_{abc}F^aF^bF^c = \tfrac{1}{2}iN^2\mathbf{I}\,, \qquad\qquad f_{abc}D^aF^bF^c = 0\,, \tag{49}$$

$$f_{abc}D^aD^bF^c = \tfrac{1}{2}i(N^2-4)\mathbf{I}\,, \qquad\qquad f_{abc}D^aD^bD^c = 0\,. \tag{50}$$

For completeness, we quote the analogous identities with f_{abc} replaced by d_{abc}. These identities are proved in Appendix A of these notes.

$$d_{abc}F^bF^c = \tfrac{1}{2}ND^a\,, \tag{51}$$

$$d_{abc}F^bD^c = \left(\frac{N^2-4}{2N}\right)F^a\,, \tag{52}$$

$$d_{abc}D^bD^c = \left(\frac{N^2-12}{2N}\right)D^a\,. \tag{53}$$

It then follows that

$$d_{abc}F^aF^bF^c = 0\,, \qquad\qquad d_{abc}D^aF^bF^c = \tfrac{1}{2}(N^2-4)\mathbf{I}\,, \tag{54}$$

$$d_{abc}D^aD^bF^c = 0\,, \qquad\qquad d_{abc}D^aD^bD^c = \left(\frac{(N^2-4)(N^2-12)}{2N^2}\right)\mathbf{I}\,. \tag{55}$$

Note that the first equation in eq. (54) implies that the cubic Casimir operator in the adjoint representation vanishes, i.e., $d_{abc}F^aF^bF^c = 0$.

Finally, we quote a number of useful trace identities [1–4].

$$\mathrm{Tr}\, F^a = \mathrm{Tr}\, D^a = 0\,, \qquad \mathrm{Tr}(F^a D^b) = 0\,, \tag{56}$$

$$\mathrm{Tr}(F^a F^b) = N\delta_{ab}\,, \qquad \mathrm{Tr}(D^a D^b) = \left(\frac{N^2-4}{N}\right)\delta_{ab}\,, \tag{57}$$

$$\mathrm{Tr}(F^a F^b F^c) = \tfrac{1}{2} iN f_{abc}\,, \qquad \mathrm{Tr}(D^a F^b F^c) = \tfrac{1}{2} N d_{abc}\,, \tag{58}$$

$$\mathrm{Tr}(D^a D^b F^c) = i\left(\frac{N^2-4}{2N}\right) f_{abc}\,, \qquad \mathrm{Tr}(D^a D^b D^c) = \left(\frac{N^2-12}{2N}\right) d_{abc}\,. \tag{59}$$

Additional identities involving traces of four generators can also be derived. Ref. [4] provides the following results,[5]

$$\mathrm{Tr}(F^a F^b F^c F^d) = \delta_{ad}\delta_{bc} + \tfrac{1}{2}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd}) + \tfrac{1}{4}N(f_{ade}f_{bce} + d_{ade}d_{bce})\,, \tag{60}$$

$$\mathrm{Tr}(F^a F^b F^c D^d) = \tfrac{1}{4} iN(d_{ade}f_{bce} - f_{ade}d_{bce})\,, \tag{61}$$

$$\mathrm{Tr}(F^a F^b D^c D^d) = \tfrac{1}{2}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}) + \left(\frac{N^2-8}{4N}\right) f_{ade}f_{bce} + \tfrac{1}{4}N d_{ade}d_{bce}\,, \tag{62}$$

$$\mathrm{Tr}(F^a D^b F^c D^d) = -\tfrac{1}{2}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}) + \tfrac{1}{4}N(f_{ade}f_{bce} + d_{ade}d_{bce})\,, \tag{63}$$

$$\mathrm{Tr}(F^a D^b D^c D^d) = \frac{2i}{N} f_{ade}d_{bce} + i\left(\frac{N^2-8}{4N}\right) f_{abe}d_{cde} + \tfrac{1}{4} iN d_{abe}f_{cde}\,, \tag{64}$$

$$\begin{aligned}\mathrm{Tr}(D^a D^b D^c D^d) = {} & \left(\frac{N^2-4}{N^2}\right)\delta_{ad}\delta_{bc} + \left(\frac{N^2-8}{2N^2}\right)\delta_{ab}\delta_{cd} + \tfrac{1}{2}\delta_{ac}\delta_{bd} + \tfrac{1}{4}N f_{ade}f_{bce} \\ & + \left(\frac{N^2-16}{4N}\right) d_{ade}d_{bce} - \frac{4}{N} d_{abe}d_{cde}\,. \end{aligned} \tag{65}$$

Alternative expressions for eqs. (61)–(65) are given in Appendix B [3].

As a check of eq. (60), let us set $a = c$ and sum over a. After employing eqs. (26) and (27) and relabeling d by c, we obtain

$$\mathrm{Tr}(F^a F^b F^a F^c) = \tfrac{1}{2} N^2 \delta_{bc}\,. \tag{66}$$

Alternatively, one can obtain the above result directly by using eqs. (26), (44), (57) and (58) to compute

$$\begin{aligned}\mathrm{Tr}(F^a F^b F^a F^c) &= \mathrm{Tr}\big((if_{abd}F^d + F^b F^a)F^a F^c\big) = if_{abd}\,\mathrm{Tr}(F^d F^a F^c) + \mathrm{Tr}(F^b F^a F^a F^c) \\ &= if_{abd}\big(\tfrac{1}{2} iN f_{dac}\big) + N^2\delta_{bc} = \tfrac{1}{2}N^2\delta_{bc}\,, \end{aligned} \tag{67}$$

which confirms the result of eq. (66). Similarly, the results of eqs. (61)–(65) can also be checked by multiplication by either a Kronecker delta, f_{abc} or d_{abc} and then employing the trace formulae previously derived.

Various applications of the identities above can be found in a paper by Roger Cutler and Dennis Sivers [5]. Many of the identities contained in these notes are also reproduced in Appendix B of Ref. [5] (after correcting some obvious typographical errors).

[5] In Ref. [4], the coefficient of $iNd_{abe}f_{cde}$ in eq. (64) is incorrectly given by $\frac{1}{2}$.

For completeness, we note that two additional identities for the F and D matrices can be derived in the case of $N = 3$ that have no analogs for general N. These were first presented in Ref. [6] and are given in Appendix C of these notes.

APPENDIX A: Proof of eqs. (51)–(53)

First, we note that eqs. (51)–(53) are equivalent to the last three trace identifies of eqs. (58) and (59),

$$\mathrm{Tr}(D^a F^b F^c) = d_{ade}(F^d F^e)_{bc}\,, \tag{68}$$
$$\mathrm{Tr}(D^a D^b F^c) = d_{ade}(F^d D^e)_{bc}\,, \tag{69}$$
$$\mathrm{Tr}(D^a D^b D^c) = d_{ade}(D^d D^e)_{bc}\,, \tag{70}$$

after using eqs. (32) and (33). Multiplying eq. (40) on the left by F^e and taking a trace yields

$$\mathrm{Tr}(F^e F^a D^b) = \tfrac{1}{2} N d_{abe}\,, \tag{71}$$

in light of eqs. (56) and (57). Likewise, multiplying eq. (40) on the right by D^e and taking a trace yields

$$\mathrm{Tr}(F^a D^b D^e) = \frac{i(N^2-4)}{2N} f_{abe}\,. \tag{72}$$

Multiplying eq. (43) on the right by $(D^f)_{de}$ and taking the trace (by setting $c = e$ and summing over e) yields,

$$\mathrm{Tr}(F^a F^b D^f + D^a D^b D^f) = \left(\frac{N^2-6}{N}\right) d_{abf}\,. \tag{73}$$

Finally, we use the result of eq. (71) to obtain

$$\mathrm{Tr}(D^a D^b D^f) = \left(\frac{N^2-12}{2N}\right) d_{abf}\,. \tag{74}$$

APPENDIX B: Traces of four generators revisited

The trace of a product of four generators in the defining representation is easily derived. Applying eq. (15) twice, and taking the trace with the help of eq. (3) yields

$$\mathrm{Tr}(T^a T^b T^c T^d) = \frac{1}{4N}\delta_{ab}\delta_{cd} + \tfrac{1}{8}\big(d_{abe}d_{cde} - f_{abe}f_{cde} + i f_{abe}d_{cde} + i f_{cde}d_{abe}\big)\,.$$

It is convenient to employ eqs. (38) and (42) to produce a more symmetric version,

$$\begin{aligned}\mathrm{Tr}(T^a T^b T^c T^d) = \frac{1}{4N}\big(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\big) + \tfrac{1}{8}\big(d_{abe}d_{cde} - d_{ace}d_{bde} + d_{ade}d_{bce}\big) \\ + \tfrac{1}{8} i\big(d_{abe}f_{cde} + d_{ace}f_{bde} + d_{ade}f_{bce}\big)\,.\end{aligned} \tag{75}$$

A nice check of eq. (75) is to rederive eq. (14) by setting $a = c$ and summing over a.

The traces of products of four adjoint matrices (either F^a or D^a) are given in eqs. (60)–(65). It is sometimes convenient to eliminate the product $f_{ade}f_{bce}$ in favor of Kronecker deltas and the d^{abc} by using eq. (42). The following results were obtained in Ref. [3],

$$
\begin{aligned}
\mathrm{Tr}(F^aF^bF^cF^d) &= \delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + \tfrac{1}{4}N\big(d_{abe}d_{cde} - d_{ace}d_{bde} + d_{ade}d_{bce}\big)\,,\\
\mathrm{Tr}(F^aF^bF^cD^d) &= \tfrac{1}{4}iN(d_{abe}f_{cde} + f_{abe}d_{cde})\,,\\
\mathrm{Tr}(F^aF^bD^cD^d) &= \left(\frac{N^2-4}{N^2}\right)\big(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\big) + \left(\frac{N^2-8}{4N}\right)\big(d_{abe}d_{cde} - d_{ace}d_{bde}\big) + \tfrac{1}{4}Nd_{ade}d_{bce}\,,\\
\mathrm{Tr}(F^aD^bF^cD^d) &= \tfrac{1}{4}N\big(d_{abe}d_{cde} - d_{ace}d_{bde} + d_{ade}d_{bce}\big)\,,\\
\mathrm{Tr}(F^aD^bD^cD^d) &= i\left(\frac{N^2-12}{4N}\right)f_{abe}d_{cde} + \frac{i}{N}\big(f_{ade}d_{bce} - f_{ace}d_{bde}\big) + \tfrac{1}{4}iNd_{abe}f_{cde}\,,\\
\mathrm{Tr}(D^aD^bD^cD^d) &= \left(\frac{N^2-4}{N^2}\right)\big(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc}\big) + \left(\frac{N^2-16}{4N}\right)\big(d_{abe}d_{cde} + d_{ade}d_{bce}\big) - \tfrac{1}{4}Nd_{ace}d_{bde}\,.
\end{aligned}
$$

Note that the second equation above is consistent with eq. (61) in light of eq. (36), and the fifth equation above is consistent with eq. (64) in light of eq. (37).

APPENDIX C: Two additional identities for $N = 3$

Two additional identities are special to the case of $N = 3$ and do not generalize to arbitrary N. These identities can be derived from the characteristic equation of a general element of the $\mathfrak{su}(3)$ Lie algebra [2, 6],

$$
\begin{aligned}
\big\{F^a\,,\,F^b\big\}_{cd} &= 3d_{abe}(D^e)_{cd} + \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\,, &\quad (76)\\
\big\{D^a\,,\,D^b\big\}_{cd} &= -d_{abe}(D^e)_{cd} + \tfrac{1}{3}\big(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\big)\,. &\quad (77)
\end{aligned}
$$

These two identities can be rewritten as

$$
\begin{aligned}
3d_{abe}d_{cde} - f_{ace}f_{bde} - f_{ade}f_{bce} &= \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}\,, &\quad (78)\\
d_{abe}d_{cde} + d_{ace}d_{bde} + d_{ade}d_{bce} &= \tfrac{1}{3}\big(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\big)\,. &\quad (79)
\end{aligned}
$$

Combining eqs. (34) and (76) then yields,

$$
(F^aF^b)_{cd} = \tfrac{1}{2}if_{abe}(F^e)_{cd} + \tfrac{3}{2}d_{abe}(D^e)_{cd} + \tfrac{1}{2}\big(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\big)\,. \tag{80}
$$

Likewise, combining eqs. (41) and (77) yields,

$$
(D^aD^b)_{cd} = \tfrac{1}{2}if_{abe}(F^e)_{cd} - \tfrac{1}{2}d_{abe}(D^e)_{cd} + \tfrac{1}{6}\big(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\big) + \tfrac{1}{2}\delta_{ad}\delta_{bc}\,. \tag{81}
$$

Note that the sum of eqs. (80) and (81) yields the $N = 3$ version of eq. (43). Unfortunately, there are no separate analogs of eqs. (80) and (81) for $N \neq 3$.

References

[1] L.M. Kaplan and M. Resnikoff, J. Math. Phys. **8**, 2194 (1967).

[2] A.J. MacFarlane, A. Sudbery and P.H. Weisz, Commun. Math. Phys. **11**, 77 (1968).

[3] J.A. de Azcarraga, A.J. Macfarlane, A.J. Mountain and J.C. Perez Bueno, Nucl. Phys. B **510**, 657 (1998).

[4] See Appendix A of V.S. Fadin and R. Fiore, Phys. Rev. D **72**, 014018 (2005).

[5] R. Cutler and D.W. Sivers, Phys. Rev. D **17**, 196 (1978).

[6] P. Tarjanne, Ann. Acad. Sci. Fenn. Ser. A, VI. Physica, **105** (1962) pp. 3–94.