The Cartan-Killing Form

1. The Cartan metric tensor

We begin by introducing a metric tensor and a scalar product on a real Lie algebra $\mathfrak{g}$. Suppose that $X, Y \in \mathfrak{g}$. We first define a linear operator, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, such that

$$\mathrm{ad}_X(Y) = \big[X, Y\big] \,. \tag{1}$$

Then, the scalar product of two elements of the Lie algebra, called the Killing form (also called the Cartan-Killing form in the literature), is defined as

$$(X, Y) = \mathrm{Tr}\big(\mathrm{ad}_X \; \mathrm{ad}_Y\big) \,. \tag{2}$$

One can evaluate (X, Y) explicitly by choosing a basis for the Lie algebra, $\{\mathcal{A}_i\}$, which satisfies the commutation relations,

$$\big[\mathcal{A}_i, \mathcal{A}_j\big] = f_{ij}^k \mathcal{A}_k \,, \tag{3}$$

where the structure constants f_{ij}^k are real and $i, j, k = 1, 2, \ldots, \dim \mathfrak{g}$.[1] Any element of $X \in \mathfrak{g}$ is a real linear combination of the basis vectors, $\mathcal{A}_k$, i.e., $X = x^i \mathcal{A}_i$ with $x^i \in \mathbb{R}$. With respect to the basis, $\{\mathcal{A}_i\}$, the matrix elements of the linear operator $\mathrm{ad}_{\mathcal{A}_i}$ are easily obtained by noting that,

$$\mathrm{ad}_{\mathcal{A}_i}(\mathcal{A}_j) = \big[\mathcal{A}_i, \mathcal{A}_j\big] = f_{ij}^k \mathcal{A}_k \,, \tag{4}$$

from which it follows that the matrix elements of $\mathrm{ad}_{\mathcal{A}_i}$ are given by,[2]

$$(\mathrm{ad}_{\mathcal{A}_i})^k{}_j = f_{ij}^k \,. \tag{5}$$

Hence, it follows that $\mathrm{ad}_X(\mathcal{A}_j) = [X, \mathcal{A}_j] = x^i \big[\mathcal{A}_i, \mathcal{A}_j\big] = f_{ij}^k x^i \mathcal{A}_k$, which yields

$$(\mathrm{ad}_X)^k{}_j = f_{ij}^k x^i \,. \tag{6}$$

We can not compute the trace in eq. (2),

$$(X, Y) = \sum_{jk} (\mathrm{ad}_X)^k{}_j (\mathrm{ad}_Y)^j{}_k = f_{ij}^k f_{\ell k}^j x^i y^\ell \,. \tag{7}$$

[1] That is, the dimension of the Lie algebra $\mathfrak{g}$ (denoted by $\dim \mathfrak{g}$) is equal to the maximal number of linearly independent basis vectors spanning the Lie algebra.

[2] Recall that using bra and ket notation, one can write $T\,|j\rangle = \sum_k |k\rangle \langle k|\, T\,|j\rangle$. Equivalently, $T\,|j\rangle = T^k{}_j\,|k\rangle$ where the $T^k{}_j \equiv \langle k|\, T\,|j\rangle$ are the matrix elements of the operator T with respect to the basis $\{|i\rangle\}$ and there is an implicit sum over the repeated index k.

We now introduce the Cartan metric tensor of the Lie algebra, $g_{i\ell}$. For $X = x^i\mathcal{A}_i$ and $Y = y^i\mathcal{A}_i$, we define the metric tensor in terms of the scalar product in the usual way,

$$(X, Y) = g_{i\ell}x^i y^\ell \,. \tag{8}$$

Comparing eqs. (7) and (8) then yields an explicit expression for the Cartan metric tensor,

$$g_{i\ell} = f^k_{ij} f^j_{\ell k} \,. \tag{9}$$

Equivalently, one can write,

$$g_{i\ell} = (\mathcal{A}_i, \mathcal{A}_\ell) = \mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_i}, \mathrm{ad}_{\mathcal{A}_\ell}\big) \,, \tag{10}$$

where the $\{\mathcal{A}_k\}$ are a basis for the Lie algebra $\mathfrak{g}$. Note that the metric tensor is a symmetric covariant tensor, since

$$g_{\ell i} = f^k_{\ell j} f^j_{ik} = f^j_{\ell k} f^k_{ij} = g_{i\ell} \,, \tag{11}$$

after relabeling $j \to k$ and $k \to j$.

The expression for the Cartan metric tensor is basis-dependent. Recall that given a Lie group G, the basis vectors of the corresponding Lie algebra $\mathfrak{g}$ are defined by

$$\mathcal{A}_i = \left.\frac{\partial A\big(\vec{\boldsymbol{a}}(t)\big)}{\partial a^i}\right|_{t=0} \,, \tag{12}$$

where the coordinates of the Lie group element, $A \in G$ are specified by $\vec{\boldsymbol{a}}$. The analytic curve, $\vec{\boldsymbol{a}}(t)$, passes through the identity element at $t = 0$, which corresponds to the origin of the coordinate system on the Lie group manifold. If we make a change of coordinates, $\vec{\boldsymbol{a}}' = \vec{\boldsymbol{a}}'(\vec{\boldsymbol{a}})$ on the Lie group manifold, then the basis vectors of $\mathfrak{g}$ are changed to

$$\mathcal{A}'_i = \left.\frac{\partial A\big(\vec{\boldsymbol{a}}'(t)\big)}{\partial a'^{\,i}}\right|_{t=0} \,, \tag{13}$$

Employing the chain rule,

$$\frac{\partial A\big(\vec{\boldsymbol{a}}'\big)}{\partial a'^{\,i}} = \frac{\partial A\big(\vec{\boldsymbol{a}}'(\vec{\boldsymbol{a}})\big)}{\partial a^j}\,\frac{\partial a^j}{\partial a'^{\,i}} \,. \tag{14}$$

Setting $t = 0$, it follows that

$$\mathcal{A}'_i = \left(\frac{\partial a^j}{\partial a'^{\,i}}\right)_{t=0} \mathcal{A}_j \,. \tag{15}$$

Using eq. (3) and the corresponding commutation relations of the transformed basis vectors, $\big[\mathcal{A}'_i, \mathcal{A}'_j\big] = f'^{\,k}_{ij}\mathcal{A}'_k$, it follows that

$$f'^{\,k}_{ij} = \left(\frac{\partial a'^{\,k}}{\partial a^n}\frac{\partial a^\ell}{\partial a'^{\,i}}\frac{\partial a^m}{\partial a'^{\,j}}\right)_{t=0} f^n_{\ell m} \,. \tag{16}$$

where $\partial a^\ell/\partial a'^{\,i}$ is the inverse Jacobian defined by,

$$\frac{\partial a'^{\,i}}{\partial a^n}\frac{\partial a^\ell}{\partial a'^{\,i}} = \delta^\ell_n \,. \tag{17}$$

In particular, the structure constants change if one performs a linear transformation,

$$a'^{j} = M^{j}{}_{k} a^{k} \,, \tag{18}$$

where M is a nonsingular matrix (j labels the rows and k labels the columns). In this case, it follows that

$$\mathcal{A}'_i = (M^{-1})^{j}{}_{i}\, \mathcal{A}_j \,, \qquad f'^{k}_{ij} = M^{k}{}_{n} (M^{-1})^{\ell}{}_{i} (M^{-1})^{m}{}_{j} f^{n}_{\ell m} \,. \tag{19}$$

Using eqs. (9) and (19), it follows that $g'_{i\ell} = f'^{k}_{ij} f'^{j}_{\ell k}$ is related to $g_{i\ell}$ as expected for a rank two covariant tensor,

$$g'_{i\ell} = (M^{-1})^{j}{}_{i} (M^{-1})^{m}{}_{\ell} g_{jm} \,. \tag{20}$$

If we define the matrix $S \equiv M^{-1}$, then we can rewrite eq. (20) in matrix form,

$$\mathcal{G}' = S^{\mathsf{T}} \mathcal{G} S \,, \tag{21}$$

where $\mathcal{G}$ is a real symmetric matrix whose matrix elements are given by g_{jm}. Using Sylvester's theorem (see Appendix A), a real invertible matrix S exists such that,

$$\mathcal{G}' = S^{\mathsf{T}} \mathcal{G} S = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}, \underbrace{0, 0, \ldots, 0}_{t}) \,, \tag{22}$$

where r, s and t are non-negative integers such that $r + s + t = \dim \mathfrak{g}$. Moreover, if $\mathcal{G}$ is positive definite then $s = t = 0$ and $(S^{\mathsf{T}} \mathcal{G} S)_{ij} = \delta_{ij}$. Likewise, if $\mathcal{G}$ is negative definite then $r = t = 0$ and $(S^{\mathsf{T}} \mathcal{G} S)_{ij} = -\delta_{ij}$.

2. Properties of the Killing form

Recall that for $X, Y \in \mathfrak{g}$,[3]

$$(\exp \mathrm{ad}_X) Y = e^{X} Y x^{-X} \in \mathfrak{g} \,. \tag{23}$$

We therefore introduce the notation,

$$\mathrm{Ad}_g \equiv \exp \mathrm{ad}_X \,, \quad \text{where } g \equiv e^{X}. \tag{24}$$

Since $g = e^{X} \in G$, this allows us to rewrite eq. (23) as,

$$\mathrm{Ad}_g(Y) = g Y g^{-1} \,. \tag{25}$$

The following identity, which is equivalent to the Jacobi identity, is notable. For $X, Y \in \mathfrak{g}$,

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y] \,, \tag{26}$$

To prove this result, consider the action of both sides of eq. (26) on $Z \in \mathfrak{g}$.

$$\mathrm{ad}_{[X,Y]}(Z) = \big[[X, Y], Z\big] = \big[Z, [Y, X]\big] \,, \tag{27}$$

[3]See Theorem 1 in the class handout entitled, *Results for Matrix Exponentials and Logarithms.*

and

$$\begin{aligned}[\mathrm{ad}_X, \mathrm{ad}_Y](Z) &= \big(\mathrm{ad}_X\,\mathrm{ad}_Y - \mathrm{ad}_Y\,\mathrm{ad}_X\big)(Z) = \mathrm{ad}_X([Y,Z]) - \mathrm{ad}_Y([X,Z]) \\ &= \big[X,[Y,Z]\big] - \big[Y,[X,Z]\big] \\ &= -\big[X,[Z,Y]\big] - \big[Y,[X,Z]\big]\,, \end{aligned} \tag{28}$$

after using the antisymmetry of the commutator. Hence, eqs. (26)–(28) yields,

$$\big[Z,[Y,X]\big] + \big[X,[Z,Y]\big] + \big[Y,[X,Z]\big] = 0\,, \tag{29}$$

which is the Jacobi identity. Note that if we put $X = \mathcal{A}_i$ and $Y = \mathcal{A}_j$ in eq. (26), we obtain

$$[\mathrm{ad}_{\mathcal{A}_i}, \mathrm{ad}_{\mathcal{A}_j}] = \mathrm{ad}_{[\mathcal{A}_i,\mathcal{A}_j]} = f_{ij}^k\,\mathrm{ad}_{\mathcal{A}_k}\,, \tag{30}$$

after using eq. (3).[4] Hence, $\{\mathrm{ad}_X \mid X \in \mathfrak{g}\}$ constitutes the adjoint representation of $\mathfrak{g}$, with basis vectors $\{\mathrm{ad}_{\mathcal{A}_i}\}$.

Finally, we prove one additional identity,

$$\mathrm{ad}_{\mathrm{Ad}_g(X)} = \mathrm{Ad}_g\,\mathrm{ad}_X(\mathrm{Ad}_g)^{-1}\,. \tag{31}$$

First, we note that $\mathrm{Ad}_g\,\mathrm{Ad}_{g^{-1}}(Y) = Ad_g(g^{-1}Yg) = g(g^{-1}Yg)g^{-1} = Y$, which implies that

$$(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}\,. \tag{32}$$

Then, by using the definitions of the operators ad and Ad, it follows that for $X, Y \in \mathfrak{g}$ and $g \in G$,

$$\begin{aligned}\mathrm{Ad}_g\,\mathrm{ad}_X\,\mathrm{Ad}_{g^{-1}}(Y) &= \mathrm{Ad}_g\,\mathrm{ad}_X(g^{-1}Yg) = Ad_g\big([X,g^{-1}Yg]\big) = g([X,g^{-1}Yg]g^{-1} \\ &= gXg^{-1}Y - YgXg^{-1} = [gXg^{-1},Y] = \mathrm{ad}_{gXg^{-1}}(Y) = \mathrm{ad}_{\mathrm{Ad}_g(X)}(Y)\,. \end{aligned} \tag{33}$$

Combining eqs. (32) and (33) yields eq. (31).

For $X, Y, Z \in \mathfrak{g}$ and $g \in G$, the Killing form, $(X,Y) = \mathrm{Tr}\big(\mathrm{ad}_X\ \mathrm{ad}_Y\big)$, satisfies the following four properties:

$$\begin{aligned} &\text{1. bilinearity:} && (\alpha X + \beta Y, Z) = \alpha(X,Z) + \beta(Y,Z)\,, && (34)\\ &\text{2. symmetry:} && (X,Y) = (Y,X)\,, && (35)\\ &\text{3. antisymmetric:} && \big(\mathrm{ad}_Z(X),Y\big) = -\big(X,\mathrm{ad}_Z(Y)\big)\,, && (36)\\ &\text{4. orthogonal:} && \big(\mathrm{Ad}_g(X),\mathrm{Ad}_g(Y)\big) = (X,Y)\,, && (37)\end{aligned}$$

where α, $\beta \in \mathbb{R}$.

The proofs of eqs. (34)–(37) are straightforward. Bilinearity of the Killing form follows from the linearity of the operator ad and the linearity of the trace. The symmetry of the Killing form follows from the identity $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Eq. (36) is a consequence of the Jacobi

[4] Observe that $\mathrm{ad}_{kX} = k\,\mathrm{ad}_X$ for $k \in \mathbb{R}$, since [cf. eq. (1)], $\mathrm{ad}_{kX}(Y) = [kX,Y] = k[X,Y] = k\,\mathrm{ad}_X(Y)$, for $X, Y \in \mathfrak{g}$ and $k \in \mathbb{R}$.

identity [cf. eqs. (26)–(29)]. To show this, consider the identity, $[A, BC] = [A, B]C + B[A, C]$. it then follows from eq. (26) that

$$\begin{aligned}[\mathrm{ad}_Z, \mathrm{ad}_X\,\mathrm{ad}_Y] &= [\mathrm{ad}_Z, \mathrm{ad}_X]\,\mathrm{ad}_Y + \mathrm{ad}_X[\mathrm{ad}_Z, \mathrm{ad}_Y] \\ &= \mathrm{ad}_{[Z,X]}\,\mathrm{ad}_Y + \mathrm{ad}_X\,\mathrm{ad}_{[Z,Y]}\ . \end{aligned} \tag{38}$$

Taking the trace of both sides of eq. (38) and using the fact that the trace of a commutator vanishes, one obtains

$$\big([Z, X], Y\big) + \big(X, [Z, Y]\big) = 0\,, \tag{39}$$

which is equivalent to eq. (36) in light of eq. (1). Finally, we use eq. (31) to compute,

$$\begin{aligned}\big(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)\big) &= \mathrm{Tr}\big\{\mathrm{ad}_{\mathrm{Ad}_g(X)}\,\mathrm{ad}_{\mathrm{Ad}_g(Y)}\big\} = \mathrm{Tr}\big\{\mathrm{Ad}_g\,\mathrm{ad}_X(\mathrm{Ad}_g)^{-1}\,\mathrm{Ad}_g\,\mathrm{ad}_Y(\mathrm{Ad}_g)^{-1}\big\} \\ &= \mathrm{Tr}\big(\mathrm{ad}_X\,\mathrm{ad}_Y\big) = (X, Y)\,, \end{aligned} \tag{40}$$

using the invariance of a trace under a cyclic permutation of its arguments. Thus, we have confirmed eq. (37).

It is instructive to examine eqs. (36) and (37) by expressing $X = x^k\mathcal{A}_k$ and $y = y^k\mathcal{A}_k$ with respect to a basis $\{\mathcal{A}_k\}$ of the Lie algebra $\mathfrak{g}$. We introduce the matrix elements of the operators Ad_g and ad_X, which are henceforth denoted by $A^j{}_k$ and $B^j{}_k$, respectively [cf. footnote 2],

$$\mathrm{Ad}_g(\mathcal{A}_k) = g\mathcal{A}_k g^{-1} = A^j{}_k\mathcal{A}_j\,, \tag{41}$$

$$\mathrm{ad}_X(\mathcal{A}_k) = [X, A_k] = B^j{}_k\mathcal{A}_j\,. \tag{42}$$

Writing $X = x^k\mathcal{A}_k \in \mathfrak{g}$, it follows that

$$\mathrm{Ad}_g(X) = x^k A^j{}_k\mathcal{A}_j\,, \qquad\qquad \mathrm{ad}(X) = x^k B^j{}_k\mathcal{A}_j\,. \tag{43}$$

Then, eqs. (8) and (37) yield,

$$g_{k\ell}x^k y^\ell = g_{ij}x^k y^\ell A^i{}_k A^j_\ell\,. \tag{44}$$

Since eq. (44) is valid for any choice of $X, Y \in \mathfrak{g}$, it follows that

$$g_{k\ell} = g_{ij}A^i{}_k A^j{}_\ell\,. \tag{45}$$

In matrix notation, where $\mathcal{G}$ is a real symmetric matrix whose matrix elements are given by g_{jm} and $A \equiv \mathrm{Ad}_g$, we can rewrite eq. (45) as

$$\mathcal{G} = A^{\mathsf{T}}\mathcal{G}A\,. \tag{46}$$

Similarly, eqs. (8) and (36) yield,

$$g_{ij}B^i{}_\ell x^\ell y^j = -g_{\ell k}x^\ell B^k{}_j y^j\,. \tag{47}$$

Since eq. (47) is valid for any choice of $X, Y \in \mathfrak{g}$, it follows that

$$g_{ij}B^i{}_\ell = -g_{\ell k}B^k{}_j\,. \tag{48}$$

In matrix form, eq. (48) can be written as $\mathcal{G}B = -(\mathcal{G}B)^{\mathsf{T}} = -B^{\mathsf{T}}\mathcal{G}$, after using $\mathcal{G} = \mathcal{G}^{\mathsf{T}}$. Equivalently, one can write,

$$B = -\mathcal{G}^{-1}B^{\mathsf{T}}\mathcal{G}\,. \tag{49}$$

3. Killing form of a semisimple Lie algebra

In this section, we prove two theorems concerning the Killing form of a semisimple Lie algebra $\mathfrak{g}$.

Theorem 1: If $\mathfrak{g}$ is a semisimple Lie algebra, then the Killing form is nondegenerate and conversely.

Proof: Suppose that $\mathfrak{g}$ is non-semisimple. Then, $\mathfrak{g}$ possesses a nonzero abelian ideal $\mathfrak{a}$. By definition, an ideal $\mathfrak{a}$ satisfies the condition that for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$, we have $[X, Y] \in \mathfrak{a}$. If $\{\mathcal{A}_n\}$ is a basis for $\mathfrak{g}$, then a subset of $\{\mathcal{A}_n\}$ will span the ideal $\mathfrak{a}$. Suppose that $\mathcal{A}_i \in \mathfrak{a}$ and $\mathcal{A}_j \in \mathfrak{g}$. Then, we can conclude that $[\mathcal{A}_i, \mathcal{A}_j] \in \mathfrak{a}$. Since $[\mathcal{A}_i, \mathcal{A}_j] = f_{ij}^k \mathcal{A}_k \in \mathfrak{a}$, it follows that $f_{ij}^k = 0$ unless $\mathcal{A}_k$ is in the set of basis vectors that span $\mathfrak{a}$. Moreover, if both $\mathcal{A}_i$ and $\mathcal{A}_j$ are in the set of basis vectors that span $\mathfrak{a}$ then $[\mathcal{A}_i, \mathcal{A}_j] \in \mathfrak{a}$ must vanish (since $\mathfrak{a}$ is abelian), which implies that $f_{ij}^k = 0$ for all k.

Consider the Cartan metric tensor, $g_{i\ell} = f_{ij}^k f_{\ell k}^j$, in the case where its indices i and ℓ run over values corresponding to the basis vectors of $\mathfrak{a}$. The argument presented above implies that because $\mathfrak{a}$ is an ideal, the index k runs only over values corresponding to the basis vectors of $\mathfrak{a}$ (since otherwise $f_{ij}^k = 0$). Finally, because $\mathfrak{a}$ is an abelian ideal, it follows that when the indices ℓ and k run over values corresponding to the basis vectors of $\mathfrak{a}$, then $f_{\ell k}^j = 0$. Hence, we conclude that $g_{i\ell} = 0$, when the indices i and ℓ run over values corresponding to the basis vectors of $\mathfrak{a}$. Consequently, $\det \mathcal{G} = 0$, or equivalently, the Killing form is degenerate.

Conversely, if the Killing form is degenerate, then $\{X \,|\, (X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$ is a nonzero solvable ideal of $\mathfrak{g}$. Thus, $\mathfrak{g}$ is non-semisimple.[5] Details can be found in the references.

Theorem 2: If $\mathfrak{g}$ is a compact semisimple Lie algebra, then for any nonzero $X \in \mathfrak{g}$, it follows that $(X, X) < 0$ and conversely.

Proof: Since Ad_g (for $g \in G$) is the adjoint representation of the Lie group G, it follows that the matrix elements of the adjoint representation satisfy eq. (46). Moreover, in light of eq. (22), one is always free to choose the coordinates on the Lie group manifold such that

$$\mathcal{G} = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}), \tag{50}$$

where $n = r + s$ is the dimension of the Lie algebra $\mathfrak{g}$. Note that in light of Theorem 1 above, the Killing form is nondegenerate, which implies that $\det \mathcal{G} \neq 0$ and hence $t = 0$ in the notation of eq. (22). Recall that the matrix Lie group $\mathrm{O}(r, s)$ is defined by[6]

$$\mathrm{O}(r, s) = \{A \in M_{r+s}(\mathbb{R}) \,|\, A^{\mathsf{T}} \mathcal{G} = \mathcal{G} A^{-1}\}, \tag{51}$$

[5] Given a Lie algebra $\mathfrak{a}$, one can define a sequence of derived ideals, $\mathfrak{a}^{(1)} = [\mathfrak{a}, \mathfrak{a}]$, $\mathfrak{a}^{(2)} = [\mathfrak{a}^{(1)}, \mathfrak{a}^1]$, etc. By induction and with the aid of the Jacobi identity, one can show that $\mathfrak{a}^{(n)}$ is an ideal of $\mathfrak{a}$ for any value of n. Then $\mathfrak{a}$ is solvable if a positive integer k exists such that $\mathfrak{a}^{(k)} = [\mathfrak{a}^{(k-1)}, \mathfrak{a}^{(k-1)}] = \{\vec{\mathbf{0}}\}$. It follows that $\mathfrak{a}^{(k-1)}$ is an abelian ideal of $\mathfrak{a}$. Hence, if $\mathfrak{g}$ is non-semisimple then it possesses a nonzero solvable ideal, which implies that it also possesses a nonzero abelian ideal.

[6] See, e.g., the class handout providing a table of the real Lie algebras corresponding to the classical matrix Lie groups, taken from J.F. Cornwell, *Group Theory in Physics: An Introduction* (Academic Press Inc., San Diego, CA, 1997).

where $M_{r+s}(\mathbb{R})$ is the set of all $(r+s)\times(r+s)$ real matrices and $\mathcal{G}$ is given in eq. (50). Thus, it follows from eq. (46) that the adjoint group corresponding to the matrices of the adjoint representation of G must be a subgroup of $\mathrm{O}(r,s)$.

Likewise, since ad_X is the adjoint representation of the Lie algebra $\mathfrak{g}$, it follows that the matrix elements of the adjoint representation satisfy eq. (49). Recall that the matrix Lie algebra $\mathfrak{so}(r,s)$ is defined by [cf. footnote 6],

$$\mathfrak{so}(r,s)=\{B\in M_{r+s}(\mathbb{R})\,|\,B^{\mathsf{T}}\mathcal{G}=-\mathcal{G}B\}\,. \tag{52}$$

Thus, it follows from eq. (49) that the matrices of the adjoint representation of $\mathfrak{g}$ must be a subalgebra of $\mathfrak{so}(r,s)$.

Consider the special case where $\mathcal{G}=\pm\mathbf{I}$, where $\mathbf{I}$ is the $n\times n$ identity matrix. In this case, eqs. (46) and (49) yield,

$$A^{\mathsf{T}}A=\mathbf{I}\,,\qquad\qquad B^{\mathsf{T}}=-B\,. \tag{53}$$

In particular, the adjoint group corresponding to the matrices of the adjoint representation of G constitute a subgroup of $\mathrm{O}(n)$ and the matrices of the adjoint representation of $\mathfrak{g}$ constitute a subalgebra of $\mathfrak{so}(n)$. Moreover, for $X\in\mathfrak{g}$,[7]

$$(X,X)=\mathrm{Tr}\big(\mathrm{ad}_X\,\mathrm{ad}_X\big)=\mathrm{Tr}B^2=-\mathrm{Tr}(BB^{\mathsf{T}})=-\sum_{i,j}B_{ij}B_{ij}<0\,, \tag{54}$$

unless $B=0$ (or equivalently, unless $\mathrm{ad}_X=0$). In particular, $(\mathcal{A}_i,\mathcal{A}_i)<0$ [no sum over i], which in light of eq. (10) means that $g_{ii}<0$. Thus, we conclude that in the special case under consideration, $\mathcal{G}=-\mathbf{I}$. Hence, a negative definite Killing form implies that the adjoint group corresponding to the matrices of the adjoint representation of G constitutes a closed subgroup of $\mathrm{O}(n)$, which implies that the Lie group G is compact. Conversely, an indefinite Killing form implies that the adjoint group constitutes a subgroup of $\mathrm{O}(r,s)$ where neither r nor s is zero. In this case, G is a noncompact group.

Note that the analysis presented above is based on the assumption that $\mathfrak{g}$ is a real Lie algebra. In the case of a complex Lie group, we showed in class that any compact complex Lie group is abelian. Thus, any compact semisimple Lie group (and its corresponding Lie algebra) must be real. That is, Theorem 2 is only relevant for real Lie algebras. In contrast, Theorem 1 applies both to real and complex Lie algebras.

4. Complex semisimple Lie algebras and their real forms

In general, the structure constants of a complex Lie algebra are complex. However, in the case of a complex semisimple Lie algebra, one can always transform the original basis to a new basis in which the structure constants are real. Conversely, starting from eq. (3) [where the structure constants are real], one can complexify a real Lie algebra $\mathfrak{g}$ by writing $X=x^i\mathcal{A}_i$ with $x^i\in\mathbb{C}$. This process is called *complexification*. We shall denote the resulting complex Lie algebra by $\mathfrak{g}_{\mathbb{C}}$.

[7]Note that when $g_{ij}\propto\delta_{ij}$, then there is no distinction between covariant and contravariant indices, and one can write all tensor quantities with lowered indices.

Given a complex semisimple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$, with basis vectors satisfying $[\mathcal{A}_i, \mathcal{A}_j] = f_{ij}^k \mathcal{A}_k$ with $f_{ij}^k \in \mathbb{R}$, one can restrict $x^i \in \mathbb{R}$ to obtain a real semisimple Lie algebra, $\mathfrak{g}_{\mathbb{R}}$, with the same commutation relations.[8] The Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is an example of a *real form* of $\mathfrak{g}_{\mathbb{C}}$. More generally, a real form of $\mathfrak{g}_{\mathbb{C}}$ is defined as any real Lie algebra whose complexification yields $\mathfrak{g}_{\mathbb{C}}$. Note that $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = \dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$. That is, given a complex Lie algebra of complex dimension n, the corresponding real forms have real dimension n. In particular, the maximal number of linearly independent basis vectors is n for both $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{R}}$.

If $\mathfrak{g}_{\mathbb{C}}$ is semisimple, then its Killing form is non-degenerate (in light of Theorem 1). Hence, the corresponding real form, which has the same Cartan metric tensor, is also semisimple. Indeed, one can always transform the original basis of $\mathfrak{g}$ to a new basis in which the Cartan metric tensor is of the form given by eq. (50). However, this new basis is not unique. For example, one can always choose to multiply some of the basis vectors by i as long as all the structure constants remain real. In particular, it is always possible to find a basis for a semisimple $\mathfrak{g}_{\mathbb{C}}$ such that $\mathcal{G} = -\mathbf{I}$, in which case the corresponding real form is compact.

Indeed, there exists a simple algorithm to derive various possible real forms of $\mathfrak{g}_{\mathbb{C}}$. The algorithm consists of first choosing a basis for $\mathfrak{g}_{\mathbb{C}}$ such that the Cartan metric tensor is of the form given by eq. (50). Next, we consider all possible ways of multiplying a subset of the basis vectors $\{\mathcal{A}_k\}$ of $\mathfrak{g}_{\mathbb{C}}$ each by i such that $f_{ij}^k \in \mathbb{R}$. The resulting commutation relations will differ in some cases from the original one by some signs. Nevertheless, complex linear combinations of the new basis vectors, $x^i \mathcal{A}_i$ ($x^i \in \mathbb{C}$), still generate $\mathfrak{g}_{\mathbb{C}}$. After all possibilities for the $\{\mathcal{A}_i\}$ are considered, one now restricts the x^i to be real in each case in order to obtain the corresponding real forms. Note that among all possible real forms obtained in this way, only one of them is compact, corresponding to the basis in which $\mathcal{G} = -\mathbf{I}$. Other real forms obtained by the algorithm above necessarily possess an indefinite Killing form, and thus correspond to noncompact real Lie algebras.

A simple example demonstrates the procedure for finding real forms of $\mathfrak{g}_{\mathbb{C}}$. Consider the commutation relations of $\mathfrak{sl}(2,\mathbb{C})$,

$$[\mathcal{A}_i, \mathcal{A}_j] = \epsilon_{ijk} \mathcal{A}_k \,, \tag{55}$$

where the indices i, j, k run over 1,2 and 3. The corresponding Cartan metric tensor is negative definite,[9]

$$g_{i\ell} = \epsilon_{kij}\epsilon_{j\ell k} = -2\delta_{i\ell} \,. \tag{56}$$

If we restrict $x^i \in \mathbb{R}$, then we obtain the real compact Lie algebra, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Consider what happens if we redefine $\mathcal{A}_1 \to i\mathcal{A}_1$ and $\mathcal{A}_2 \to i\mathcal{A}_2$, while leaving $\mathcal{A}_3$ unchanged. The new commutation relations of $\mathfrak{sl}(2,\mathbb{C})$ are now given by,

$$[\mathcal{A}_1, \mathcal{A}_2] = -\mathcal{A}_3 \,, \qquad [\mathcal{A}_2, \mathcal{A}_3] = \mathcal{A}_1 \,, \qquad [\mathcal{A}_3, \mathcal{A}_1] = \mathcal{A}_2 \,. \tag{57}$$

In this case, the Cartan metric is now indefinite,

$$g_{i\ell} = f_{ij}^k f_{\ell k}^j = \mathrm{diag}(2, 2, -2) \,. \tag{58}$$

[8] Not every complex non-semisimple Lie algebra has a real form. See, e.g., Example 5.1.24 on p. 88 of Ref. 7 or Example 5 on p. 19 of Ref. 8.

[9] We can normalize the basis vectors appropriately such that $g_{i\ell} = -\delta_{i\ell}$ (although it is not necessary for this discussion).

Thus, if we now restrict $x^i \in \mathbb{R}$, the resulting real Lie algebra is noncompact. Indeed, the commutation relations given in eq. (57) correspond to those of $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1) \cong \mathfrak{so}(2,1)$. Note that any other redefinition of a subset of the basis vectors via multiplication by i, such that the structure constants remain real, will yield Lie algebras that are isomorphic to one of the two possible classes of real forms identified above.

It is straightforward to show that real forms of a simple Lie algebra are simple. However, the complexification of a real simple Lie algebra can yield either a simple or a semisimple real algebra. For example, the complexification of the real simple Lie algebra $\mathfrak{so}(3,1)$ yields the semisimple complex Lie algebra $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

There is another way of obtaining a real Lie algebra from a complex Lie algebra called *realification*. Starting from the basis vectors, $\{\mathcal{A}_k\}$, of the complex Lie algebra, the elements of $\mathfrak{g}_{\mathbb{C}}$ are of the form $x^k\mathcal{A}_k$, with $x^k \in \mathbb{C}$. One can now construct a real Lie algebra that is given by real linear combinations of the basis vectors $\{\mathcal{A}_k, i\mathcal{A}_\ell\}$. We shall denote this real Lie algebra by $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$.[10] Note that if $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = n$, then $\dim_{\mathbb{R}} (\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}} = 2n$. As an example of this construction, the realification of $\mathfrak{sl}(2,\mathbb{C})$ yields $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \cong \mathfrak{so}(3,1)$. This isomorphism is important in the study of the Lie algebra of the Lorentz group.

One can now consider the complexification of the $2n$-dimensional real Lie algebra, $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$, which will be denoted by $(\mathfrak{g}_{\mathbb{C}})^*_{\mathbb{R}}$. Note that $\dim_{\mathbb{C}} (\mathfrak{g}_{\mathbb{C}})^*_{\mathbb{R}} = 2n$. One can easily verify that,

$$(\mathfrak{g}_{\mathbb{C}})^*_{\mathbb{R}} \cong \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}\,. \tag{59}$$

Thus, in our previous example, the complexification of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. Since the complexification of $\mathfrak{so}(3,1)$ is $\mathfrak{so}(4,\mathbb{C})$, it then follows that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. In particular, $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ [or equivalently, $\mathfrak{so}(3,1)$] is one of real forms of $\mathfrak{so}(4,\mathbb{C})$.

5. A completely antisymmetric third rank tensor

Using the Cartan metric tensor, one can construct a completely antisymmetric third rank tensor that is related to the structure constants of the Lie algebra $\mathfrak{g}$,

$$f_{ijk} \equiv g_{k\ell} f^{\ell}_{ij}\,. \tag{60}$$

To prove that f_{ijk} is a completely antisymmetric third rank tensor, we employ eqs. (9) and (30) to write,

$$f_{ijk} = f^{\ell}_{ij}\mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_\ell}\big) = \mathrm{Tr}\big(f^{\ell}_{ij}\,\mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_\ell}\big) = \mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_k}[\mathrm{ad}_{\mathcal{A}_i},\mathrm{ad}_{\mathcal{A}_j}]\big)\,, \tag{61}$$

after using the linearity of the trace. That is,

$$f_{ijk} = \mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_i}\,\mathrm{ad}_{\mathcal{A}_j} - \mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_j}\,\mathrm{ad}_{\mathcal{A}_i}\big)\,, \tag{62}$$

which is manifestly antisymmetric under the interchange of any pair of indices i, j and k due to the invariance of the trace under a cyclic permutation of its arguments.

Note that in the special case where $\mathfrak{g}$ is a compact semisimple Lie algebra, we can choose a basis in which $g_{ij} = -\delta_{ij}$. For this basis choice, it follows that the structure constants of $\mathfrak{g}$, $f^k_{ij} = -f_{ijk}$, are completely antisymmetric under the interchange of any pair of indices.

[10] In the literature, the subscript $\mathbb{R}$ is sometimes omitted. One then says that the complex Lie algebra $\mathfrak{g}$ of complex dimension n can be regarded as a real Lie algebra of real dimension $2n$.

APPENDIX A: Sylvester's Theorem

In this appendix we prove the following theorem, often called Sylvester's law of inertia.[11]

Sylvester's Theorem: Consider an $n \times n$ hermitian matrix $M = M^\dagger$. Then there exists an invertible matrix S such that

$$S^\dagger M S = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}, \underbrace{0, 0, \ldots, 0}_{t}), \tag{63}$$

where r, s and t are non-negative integers such that $r + s + t = n$. That is, $S^\dagger M S$ is a diagonal matrix whose elements consist of 1 repeated r times, -1 repeated s times and 0 repeated t times along the diagonal. Moreover, if M is positive definite then $s = t = 0$ and $(S^\mathsf{T} M S)_{ij} = \delta_{ij}$. Likewise, if M is negative definite then $r = t = 0$ and $(S^\mathsf{T} M S)_{ij} = -\delta_{ij}$.

Proof: Since M is hermitian, it follows that the eigenvalues of M are real. Moreover, M is unitarily diagonalizable. That is, a unitary matrix U exists such that

$$U^\dagger M U = \mathrm{diag}(\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{r+s}, \underbrace{0, 0, \ldots, 0}_{t}), \tag{64}$$

where the eigenvalues $\lambda_1, \ldots, \lambda_r$ are all positive real numbers, the eigenvalues $\lambda_{r+1}, \ldots, \lambda_{r+s}$ are all negative real numbers, and there are zero eigenvalues with multiplicity t. We can therefore define a diagonal matrix,

$$D \equiv \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}, \sqrt{-\lambda_{r+1}}, \ldots, \sqrt{-\lambda_{r+s}}, \underbrace{1, 1, \ldots, 1}_{t}). \tag{65}$$

Setting $S = UD^{-1}$, it follows that,

$$S^\dagger M S = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}, \underbrace{0, 0, \ldots, 0}_{t}), \tag{66}$$

which completes the proof. Recall that a positive (negative) definite matrix is an hermitian matrix whose eigenvalues are positive (negative). Thus, if M is positive definite, it follows that $s = t = 0$ and $(S^\mathsf{T} M S)_{ij} = \delta_{ij}$. Likewise, if M is negative definite then $r = t = 0$ and $(S^\mathsf{T} M S)_{ij} = -\delta_{ij}$.

Corollary: Consider an $n \times n$ real symmetric matrix $G = G^\mathsf{T}$. Then there exists an invertible real matrix S such that

$$S^\mathsf{T} G S = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}, \underbrace{0, 0, \ldots, 0}_{t}), \tag{67}$$

where r, s and t are non-negative integers such that $r + s + t = n$. Moreover, if G is positive definite then $s = t = 0$ and $(S^\mathsf{T} G S)_{ij} = \delta_{ij}$. Likewise, if G is negative definite then $r = t = 0$ and $(S^\mathsf{T} G S)_{ij} = -\delta_{ij}$.

[11] See, e.g., Howard Eves, *Elementary Matrix Theory* (Dover Publications, Inc., Mineola, NY, 1980) pp. 237–245.

Proof: Since G is a real symmetric matrix, its eigenvalues are all real and it is diagonalizable by a real orthogonal matrix. That is, a real orthogonal matrix Q exists such that

$$Q^{\mathsf{T}} G Q = \mathrm{diag}(\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{r+s}, \underbrace{0, 0, \ldots, 0}_{t}) \,, \tag{68}$$

where the eigenvalues, $\lambda_1, \ldots, \lambda_r$ [$\lambda_{r+1}, \ldots, \lambda_{r+s}$], are all positive [negative] real numbers and the zero eigenvalues of G have multiplicity t. We again define the diagonal matrix D by eq. (65). Setting $S = QD^{-1}$ (which is a real matrix) then establishes eq. (67).

References

I have found the following references useful in preparing these note.

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