DUE: TUESDAY, MAY 7, 2019

<u>ALERT</u>: You should be ready with an initial choice for a term project topic on Monday April 29. Feel free to consult with me on possible choices. A short written proposal (one paragraph would suffice plus references) is due on Tuesday May 7.

1. The matrix group O(n) consists of real orthogonal $n \times n$ matrices (n is a positive integer), and SO(n) consists of the subgroup of O(n) matrices with determinant equal to one.

(a) Show that SO(n) is a normal subgroup of O(n).

(b) If n is odd, show that $\mathbb{Z}_2 \cong \{\boldsymbol{I}_n\,,\,-\boldsymbol{I}_n\}$ is a normal subgroup of O(n), where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. Prove that O(n) can be written as an internal direct product, $\mathrm{O}(n) \cong \mathrm{SO}(n) \otimes \mathbb{Z}_2$.

(c) Explain why the results of part (b) do not apply to the case of even n. Show that if n is even then O(n) can be written as a semidirect product, $\mathrm{O}(n) \cong \mathrm{SO}(n) \rtimes \mathbb{Z}_2$. Identify explicitly the subgroup of O(n) appearing in this semidirect product that is isomorphic to $\mathbb{Z}_2$.

2. A finite group G can be decomposed into conjugacy classes $\mathcal{C}_k$.

(a) Construct the set $\mathcal{C}'_k \equiv g\mathcal{C}_k g^{-1}$, which is obtained by replacing each element $x \in \mathcal{C}_k$ by gxg^{-1}. Prove that $\mathcal{C}'_k = \mathcal{C}_k$.

(b) Suppose that $D^{(i)}(g)$ is the ith irreducible (finite-dimensional) matrix representation of the finite group G. For a fixed class $\mathcal{C}_k$, prove that

$$\sum_{g \in \mathcal{C}_k} D^{(i)}_{j\ell}(g) = \frac{N_k}{n_i}\chi^{(i)}(\mathcal{C}_k)\delta_{j\ell}\,, \tag{1}$$

where n_i is the dimension of the ith irreducible representation of G, N_k is the number of elements in the kth conjugacy class and $\chi^{(i)}(\mathcal{C}_k)$ is the irreducible character corresponding to the kth conjugacy class.

HINT: Denoting the sum on the left hand side of eq. (1) by A_k and using the result of part (a), prove that $D^{(i)}(g)A_k = A_k D^{(i)}(g)$ for all $g \in G$. Then use Schur's second lemma.

(c) Starting from the completeness result that is satisfied by the matrix elements of the irreducible matrix representations of G and using the result of part (b), derive the completeness relation for the irreducible characters,

$$\frac{N_k}{O(G)}\sum_i \chi^{(i)}(\mathcal{C}_k)[\chi^{(i)}(\mathcal{C}_\ell)]^* = \delta_{k\ell}\,,$$

where $O(G)$ is the order of the group G (i.e. the number of elements of G), and the sum is taken over all inequivalent finite-dimensional irreducible representations.

(d) Using the orthogonality and the completeness relations satisfied by the irreducible characters, prove that the number of inequivalent irreducible representations of G is equal to the number of conjugacy classes.

3. Consider the transformations of the triangle that make up the dihedral group D_3. The elements of this group are $D_3 = \{e, r, r^2, d, rd, r^2d\}$, with the group multiplication law determined by the relations $r^3 = e$, $d^2 = e$ and $dr = r^2 d$, where e is the identity element. In class, the following two-dimensional representation matrices for $r, d \in D_n$ were given,

$$r = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} , \qquad\qquad d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2}$$

Setting $n = 3$, one can construct a two-dimensional matrix representation of D_3.

(a) Consider the six-dimensional function space W consisting of polynomials of degree 2 in two real variables (x, y):

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + h\,, \tag{3}$$

where $a, b, \ldots, h$ are complex constants. We can view $(a, b, \ldots, h)$ as a six-dimensional vector that lives in a vector space which is isomorphic to W. If we perform a transformation of (x, y) under D_3 according to the two-dimensional representation obtained from eq (2) with $n = 3$, then the polynomial $f(x, y)$ given by eq. (3) transforms into another polynomial. That is, the vector $(a, b, \ldots, h)$ transforms under D_3 according to a six-dimensional representation. Compute the 6×6 matrices that represent the elements of D_3. Determine which irreducible representations of D_3 are contained in this six-dimensional representation and their corresponding multiplicities.

(b) Identify the irreducible invariant subspaces of W under D_3. Check that your result is consistent with the results of part (a).

4. Consider the dihedral group D_4 treated in problem 3 of Problem Set 1. The elements of this group are $D_4 = \{e, r, r^2, r^3, d, rd, r^2d, r^3d\}$ with the group multiplication law determined by the relations $r^4 = e$, $d^2 = e$ and $dr = r^3 d$, where e is the identity element.

(a) Write out the conjugacy class multiplication table.

(b) Determine explicitly the matrices of the regular representation.

(c) Using the two dimensional matrix representation given in eq. (2) with $n = 4$, verify that the group multiplication table of D_4 is preserved. Prove that this representation is irreducible.

(d) Construct the character table for the irreducible representations of D_4.

5. Suppose that D is an irreducible n-dimensional representation of a finite group G, and $D^{(1)}$ is a (nontrivial) one-dimensional representation of G. Prove that the direct product $D \otimes D^{(1)}$ is an irreducible representation of G. [*HINT*: Show that for any $g \in G$, $|D^{(1)}(g)| = 1$.]

6. (a) Display all the standard Young tableaux of the permutation group S_4. From this result, enumerate the inequivalent irreducible representations of S_4 and specify their dimensions.

(b) Show that the normal subgroup $\{e, (12)(34), (13)(24), (14)(23)\}$ of S_4 is isomorphic to D_2. Using this result, prove that $D_3 \cong S_4/D_2$.

(c) Using the two-dimensional irreducible representation of D_3 given in class and the result of part (b), construct a two-dimensional representation of S_4 and determine its characters. Is the latter an *irreducible* representation of S_4?

HINT: Show that given a normal subgroup N of a group G and a representation $D^{G/N}$ of the factor group G/N, one can construct a representation D^G of the group G by defining $D^G(g) \equiv D^{G/N}(gN)$ for all $g \in G$.

(d) Using the known one-dimensional representations of S_4 and the results of parts (a) and (c), construct the character table for the group S_4. Determine any unknown entries in the character table by using the orthonormality and completeness relations for the irreducible characters. Using this technique, all entries of the character table can be uniquely determined up to a sign ambiguity in some of the entries.

(e) [*EXTRA CREDIT*] Resolve the sign ambiguity of part (d). For example, determine the matrix representative of the transposition (1 2) in the three-dimensional irreducible representation of S_4. By taking the trace of this matrix, complete the character table of S_4.

7. (a) Verify the following properties of the Pauli matrices $\vec{\boldsymbol{\sigma}} \equiv (\sigma_1\,,\,\sigma_2\,,\,\sigma_3)$:

$$(i)\ \sigma_i\sigma_j = I\delta_{ij} + i\epsilon_{ijk}\sigma_k\,,$$

$$(ii)\ \sigma_2\,\vec{\boldsymbol{\sigma}}\,\sigma_2 = -\vec{\boldsymbol{\sigma}}^{\,*}\,,$$

$$(iii)\ \exp(-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2) = I\cos(\theta/2) - i\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}\sin(\theta/2)\,,$$

where I is the 2×2 identity matrix.

(b) In the angle-and-axis parameterization of SO(3), a rotation by an angle θ about an axis that points along the unit vector $\hat{\boldsymbol{n}}$ is represented by an SO(3) matrix given by $R_{ij}(\hat{\boldsymbol{n}},\theta) = \exp(-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}})_{ij}$, with $(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}})_{ij} \equiv -i\epsilon_{ijk}n_k$. By convention, we assume that $0 \le \theta \le \pi$, and the axis $\hat{\boldsymbol{n}}$ can point in any direction. Evaluate R_{ij} explicitly by summing the Taylor series of the exponential, and show that

$$R_{ij}(\hat{\boldsymbol{n}},\theta) = n_in_j + (\delta_{ij} - n_in_j)\cos\theta - \epsilon_{ijk}n_k\sin\theta\,.$$

(c) Verify the formula: $e^{-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2} = R_{ij}(\hat{\boldsymbol{n}},\theta)\,\sigma_i\,.$

(d) The set of matrices $\exp(-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2)$ constitutes the defining representation of SU(2). Prove that this representation is pseudoreal. [*HINT:* Property (ii) of part (a) is useful here.]