1. Power Series Expansion of the Fermi-Dirac Integral

The Fermi-Dirac integral is defined as:

$$f_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{z^{-1}e^x + 1}\,,$$

where $x \equiv \epsilon/kT$ and $z \equiv e^{\mu/kT}$. We wish to derive a power series expansion for $f_n(z)$ that is useful in the limit of $z \to 0$. We manipulate $f_n(z)$ as follows:

$$\begin{aligned} f_n(z) &= \frac{z}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{e^x + z} \\ &= \frac{z}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{e^x(1 + ze^{-x})} \\ &= \frac{z}{\Gamma(n)} \int_0^\infty e^{-x} x^{n-1}\,dx \sum_{m=0}^\infty (-1)^m (ze^{-x})^m \\ &= \frac{z}{\Gamma(n)} \sum_{m=0}^\infty (-1)^m z^m \int_0^\infty e^{-(m+1)x} x^{n-1}\,dx\,. \end{aligned}$$

Using the well known integral:

$$\int_0^\infty e^{-Ax} x^{n-1}\,dx = \frac{\Gamma(n)}{A^n}\,,$$

and changing the summation variable to $p = m + 1$, we end up with:

$$f_n(z) = \sum_{p=1}^\infty \frac{(-1)^{p-1} z^p}{p^n}\,. \tag{1}$$

which is the desired power series expansion. This result is also useful for deriving additional properties of the $f_n(z)$. For example, it is a simple matter use eq. (1) to verify:

$$f_{n-1}(z) = z\frac{\partial}{\partial z} f_n(z)\,.$$

Note that the classical limit is obtained for $z \ll 1$. In this limit, which also corresponds to the high temperature limit, $f_n(z) \simeq z$. One additional special case is noteworthy. If

$z = 1$, we obtain

$$f_n(1) = \sum_{p=0}^{\infty} \frac{(-1)^{p-1}}{p^n}\,.$$

To evaluate this sum, we rewrite it as follows:

$$\begin{aligned} f_n(1) &= \sum_{p\ \mathrm{odd}} \frac{1}{p^n} - \sum_{p\ \mathrm{even}} \frac{1}{p^n} \\ &= \sum_{p\ \mathrm{odd}} \frac{1}{p^n} + \sum_{p\ \mathrm{even}} \frac{1}{p^n} - 2\sum_{p\ \mathrm{even}} \frac{1}{p^n} \\ &= \sum_{p=1}^{\infty} \frac{1}{p^n} - 2\sum_{p=1}^{\infty} \frac{1}{(2p)^n} \\ &= \left(1 - \frac{1}{2^{n-1}}\right) \sum_{p=1}^{\infty} \frac{1}{p^n}\,. \end{aligned}$$

where p even [odd] above means we sum over positive even [odd] integers. The Riemann zeta-function is defined as:

$$\zeta(n) \equiv \sum_{p=1}^{\infty} \frac{1}{p^n}\,.$$

Thus, we conclude that

$$f_n(1) = \sum_{p=0}^{\infty} \frac{(-1)^{p-1}}{p^n} = \left(1 - \frac{1}{2^{n-1}}\right) \zeta(n)\,. \qquad (2)$$

2. Asymptotic Expansion of the Fermi-Dirac Integral

We again consider the Fermi-Dirac integral, which we rewrite in the following form:

$$f_n(y) \equiv \frac{1}{\Gamma(n)} \int_0^{\infty} \frac{x^{n-1}\,dx}{e^{x-y}+1}\,,$$

where $y \equiv \mu/kT = \ln z$. We now wish to derive an asymptotic expansion for $f_n(y)$ which is useful in the limit of $y \to \infty$.

In the limit of $y \to \infty$, which corresponds to the low temperature limit,

$$\frac{1}{e^{x-y}+1} = \begin{cases} 1, & x < y\,, \\ 0, & x > y\,. \end{cases} \tag{3}$$

This is just the statement that all the energy levels of the Fermi gas are singly occupied (*i.e.*, $n_\epsilon = 1$) below the Fermi energy ($\epsilon_F = \mu$) and are unoccupied (*i.e.*, $n_\epsilon = 0$) above the Fermi energy.⋆ Thus, in this limit,

$$f_n(y) \simeq \frac{1}{\Gamma(n)} \int_0^y x^{n-1}\,dx = \frac{y^n}{n\Gamma(n)} = \frac{y^n}{\Gamma(n+1)}\,.$$

We now want to improve on this approximation. To do so, we subtract this approximate result from the exact integral and study the remainder.

First, it is convenient to introduce some notation. The step function is defined as:

$$\Theta(x) = \begin{cases} 1, & x > 0\,, \\ 0, & x < 0\,. \end{cases}$$

Using this result, eq. (3) implies that:

$$\lim_{y\to\infty} \frac{1}{e^{x-y}+1} = \Theta(y-x)\,.$$

Thus, we examine the following integral:

$$h_n(y) \equiv \int_0^\infty \left[\frac{1}{e^{x-y}+1} - \Theta(y-x)\right] x^{n-1}\,dx\,.$$

Change variables to $w = x - y$. Then,

$$\begin{aligned} h_n(y) &= \int_{-y}^\infty \left[\frac{1}{e^w+1} - \Theta(-w)\right](w+y)^{n-1}\,dw \\ &\simeq \int_{-\infty}^\infty \left[\frac{1}{e^w+1} - \Theta(-w)\right](w+y)^{n-1}\,dw \\ &\simeq y^{n-1}\int_{-\infty}^\infty \left[\frac{1}{e^w+1} - \Theta(-w)\right]\left(1+\frac{w}{y}\right)^{n-1}\,dw\,. \end{aligned}$$

In the second step above, we replaced the lower limit $-y$ with $-\infty$. Since we seek an approximation valid for large y, this should be a good approximation. One can check that

⋆ This behavior is somewhat easier to see by considering the $T \to 0$ limit of $n_\epsilon = 1/[e^{(\epsilon-\mu)/kT}+1]$. Then, change variables back to $x = \epsilon/kT$ and $y = \mu/kT$ to recover eq. (3).

the error we make at this step is exponentially small. In the limit of large y we expand $(1+w/y)^{n-1}$ in a power series using the binomial theorem:

$$\begin{aligned}\left(1+\frac{w}{y}\right)^{n-1} &= \sum_{p=0}^{\infty} C(n-1,p)\left(\frac{w}{y}\right)^p \\ &= \frac{1}{n}\sum_{p=0}^{\infty}\frac{n(n-1)(n-2)\cdots(n-p)}{p!}\left(\frac{w}{y}\right)^p .\end{aligned}$$

Note that this formula is valid even when n is not a positive integer. (For positive integer n, the sum stops after $p = n-1$.) If n is not a positive integer, the above expansion converges only when $|w/y| < 1$. This raises a concern, since we are about to insert this expansion into an integral where w varies from $-\infty$ to $+\infty$. No matter how large y is, this step appears to be illegal in the range of w-integration where $|w| > |y|$. Nevertheless, we shall close our eyes to this problem and proceed. One can show that this step cannot result in a convergent expansion of the integral (when n is not a positive integer). However, the resulting expression is a valid *asymptotic* expansion. Although the proof of this statement lies beyond the scope of this note, it should be noted that the phenomenon just described is quite common in the derivation of asymptotic expansions.

Thus, we insert the expansion of $(1+w/y)^{n-1}$ into the integral, and we interchange the sum and integral (this step also needs mathematical justification). The result is:

$$h_n(y) \simeq \frac{y^{n-1}}{n}\sum_{p=0}^{\infty}\frac{n(n-1)(n-2)\cdots(n-p)}{y^p\, p!}\int_{-\infty}^{\infty}\left[\frac{1}{e^w+1}-\Theta(-w)\right]w^p\,dw\,. \qquad (4)$$

At this point, focus on the integrand. Define $F(w) \equiv (e^w+1)^{-1} - \Theta(-w)$. For positive values of w, $\Theta(-w) = 0$ so that $F(w) = 1/(e^w+1)$ while

$$F(-w) = \frac{1}{e^{-w}+1} - 1 = \frac{-e^{-w}}{e^{-w}+1} = \frac{-1}{e^w+1} = -F(w)\,.$$

That is, $F(w)$ is an odd function of w. Thus, for even p, the integral in eq. (4) vanishes. For odd p, since $w^pF(w)$ is an even function, we can set the lower limit of integration to 0 and multiply by 2 without changing the value of the integral. Thus, setting $p = 2j-1$, with $j = 1, 2, 3, \ldots$, we end up with

$$h_n(y) \simeq \frac{2y^{n-1}}{n}\sum_{j=1}^{\infty}\frac{n(n-1)(n-2)\cdots(n-2j+1)}{y^{2j-1}(2j-1)!}\int_0^{\infty}\frac{w^{2j-1}}{e^w+1}\,dw\,.$$

The integral above is related to the Riemann zeta-function; using eq. (2),

$$\begin{aligned}\int_0^\infty \frac{x^{n-1}}{e^x+1}\,dx &= \Gamma(n) f_n(z=1) \\ &= \Gamma(n)\zeta(n)\left(1-\frac{1}{2^{n-1}}\right).\end{aligned}$$

For positive integer n, $\Gamma(n)=(n-1)!$, so we end up with

$$h_n(y) \simeq \frac{2y^n}{n}\sum_{j=1}^\infty \frac{n(n-1)(n-2)\cdots(n-2j+1)}{y^{2j}}\left(1-\frac{1}{2^{2j-1}}\right)\zeta(2j)\,.$$

The last step is to note the relation between $f_n(y)$ and $h_n(y)$:

$$f_n(y) = \frac{y^n}{\Gamma(n+1)} + \frac{1}{\Gamma(n)} h_n(y)\,.$$

Inserting our result for $h_n(y)$ and using the fact that $\Gamma(n+1)=n\Gamma(n)$, we arrive at the desired asymptotic expansion for $f_n(y)$ for $y \gg 1$:

$$f_n(y) \simeq \frac{y^n}{\Gamma(n+1)}\left[1+2\sum_{j=1}^\infty \frac{n(n-1)(n-2)\cdots(n-2j+1)}{y^{2j}}\left(1-\frac{1}{2^{2j-1}}\right)\zeta(2j)\right].$$

Note that the values of $\zeta(2j)$ are known for positive integer j:

$$\zeta(2j) = \frac{(-1)^{j+1}(2\pi)^{2j}}{2(2j)!}B_{2j}\,,$$

where the B_{2j} are Bernoulli numbers.⋆

For convenience, we display the first two terms of the asymptotic expansion of $f_n(y)$:

$$f_n(y) \simeq \frac{y^n}{\Gamma(n+1)}\left[1+\frac{\pi^2 n(n-1)}{6y^2}+\mathcal{O}\left(\frac{1}{y^4}\right)\right].$$

⋆ Bernoulli numbers are defined via the Taylor series of the function $x/(e^x-1)$. Specifically,

$$\frac{x}{e^x-1} = \sum_{n=0}^\infty \frac{B_n x^n}{n!}\,.$$

By explicitly computing this Taylor series, one finds that $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30,\ldots$ In general, for $n=1,2,3,\ldots$, $B_{2n+1}=0$.

3. Power Series Expansion of the Bose-Einstein Integral

The Bose-Einstein integral is defined as:

$$g_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{z^{-1}e^x - 1}\,,$$

where $x \equiv \epsilon/kT$ and $z \equiv e^{\mu/kT}$. For a Bose gas with a conserved particle number, one must have $\mu < 0$ which means that $0 \leq z < 1$. The photon gas possesses no conserved particle number; hence, $\mu = 0$ or $z = 1$.

We wish to derive a power series expansion for $g_n(z)$ that is useful in the limit of $z \to 0$. The steps are very similar to those of section 1. We manipulate $g_n(z)$ as follows:

$$\begin{aligned} g_n(z) &= \frac{z}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{e^x - z} \\ &= \frac{z}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{e^x(1 - ze^{-x})} \\ &= \frac{z}{\Gamma(n)} \int_0^\infty e^{-x} x^{n-1}\,dx \sum_{m=0}^\infty (ze^{-x})^m \\ &= \frac{z}{\Gamma(n)} \sum_{m=0}^\infty z^m \int_0^\infty e^{-(m+1)x} x^{n-1}\,dx\,. \end{aligned}$$

Evaluating the remaining integral and putting $p = m + 1$ yields the desired power series expansion for the Bose-Einstein integral:

$$g_n(z) = \sum_{p=1}^\infty \frac{z^p}{p^n}\,. \tag{5}$$

This result is also useful for deriving additional properties of the $g_n(z)$. For example, it is a simple matter use eq. (5) to verify:

$$g_{n-1}(z) = z\frac{\partial}{\partial z} g_n(z)\,. \tag{6}$$

As noted above, the allowed region of z corresponds to $0 \leq z \leq 1$ (where $z = 1$ corresponds to the case of a photon gas). The power series for $g_n(z)$ converges for all allowed values of z. As in the Fermi-Dirac case, in the classical limit (*i.e.*, for $z \ll 1$), which also corresponds to the high temperature limit, $g_n(z) \simeq z$. In the low temperature limit, $\mu \to 0$ from below or equivalently, $z \to 1$. This is also relevant for the photon gas. The $z = 1$ result is easily obtained from eq. (5):

$$g_n(1) = \sum_{p=0}^{\infty} \frac{1}{p^n} = \zeta(n)\,,$$

a result that we found in the discussion of the thermodynamics of blackbody radiation.

4. Relations to Special Functions

It turns out that $f_n(z)$ and $g_n(z)$ belong to a class of special functions known as polylogarithms. In particular,

$$\begin{aligned} g_n(z) &\equiv \mathrm{Li}_n(z)\,, \\ f_n(z) &= -\mathrm{Li}_n(-z)\,. \end{aligned}$$

These functions are defined for arbitrary index n (not necessarily an integer) in the region $|z| < 1$ by the corresponding power series [eqs. (5) and (1), respectively]. Analytic continuation can be used to define the polylogarithm in other regions of the complex plane. The polylogarithm cannot be expressed in terms of elementary functions except in some special cases. For example, one immediately recognizes from the series definition [eq. (5)] that

$$\mathrm{Li}_1(z) = -\ln(1 - z)\,.$$

Using eq. (6), it follows that the $\mathrm{Li}_n(z)$ can be expressed in terms of elementary functions only if $n = 1, 0, -1, -2, \ldots$.

The polylogarithms get very little attention in most books of special functions and mathematical physics. This is really a shame, since there is some beautiful mathematics associated with these functions. There are many intriguing identities involving these functions. As an example, I shall quote one such relation, called the duplication formula:

$$\mathrm{Li}_n(z) + \mathrm{Li}_n(-z) = 2^{1-n}\,\mathrm{Li}_n(z^2)\,.$$

You can check that this relation is satisfied for $z = 1$.

Polylogarithms arise in other physics applications as well. Most notable is the appearance of the dilogarithm, $\mathrm{Li}_2(z)$, in the calculations of quantum field theory. In particular,

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1 - x)}{x}\,dx\,,$$

a result easily derived by integrating once the recusion relation [eq. (6)].