The Euler-Cauchy differential equation

A linear differential equation of the form,

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0\,, \tag{1}$$

with a_0, $a_1, \ldots, a_n$ constants is called the homogeneous Euler-Cauchy equation of order n.[1] As shown in Appendices A and C, by introducing a new variable,

$$z = \ln|x|\,, \tag{2}$$

we may convert eq. (1) into an nth order linear differential equation with constant coefficients. Solutions of the latter are well known. Nevertheless, in these notes, we will show how to directly obtain the solutions to the Euler-Cauchy differential equation without introducing the change of variables indicated in eq. (2).

One can also solve the inhomogeneous Euler-Cauchy differential equation, where the right hand side of eq. (1) is replaced by a known function of x,

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x)\,. \tag{3}$$

As in the case of a linear differential equation with constant coefficients, the method of undetermined coefficients is especially useful for certain cases of $f(x)$ that may appear on the right hand side of eq. (3).

For simplicity, these notes will focus primarily on the second order Euler-Cauchy differential equation. Generalizing to the case of the nth order Euler-Cauchy differential equation is straightforward (see Appendix C).

1. The second order homogeneous Euler-Cauchy differential equation

In this section, we examine the solutions to

$$ax^2 y'' + bxy' + cy = 0\,, \tag{4}$$

where $y' \equiv dy/dx$, $y'' \equiv d^2y/dx^2$ and a, b, and c are constants. The general solution to eq. (4) consists of a linear combination of two linearly independent solutions. In Appendix B, we provide a formal derivation of the solutions to eq. (4). However, in light of eq. (2), we generically expect solutions of the form $e^{pz} = e^{p\ln|x|} = |x|^p$.

Thus, we take as an ansatz,[2]

$$y(x) = |x|^p\,, \quad \text{assuming } x \neq 0 \tag{5}$$

[1] Most books simply refer to this equation as the Euler differential equation.

[2] The behavior of the solution at $x = 0$ must be separately assessed.

for a solution to eq. (4). Note that using the chain rule,

$$\frac{d}{dx}|x|^p = p|x|^{p-1}\frac{d}{dx}|x| \,.$$

Since the slope of the line $y = |x|$ is $+1$ for $x > 0$ and -1 for $x < 0$, it follows that

$$\frac{d}{dx}|x| = \operatorname{sgn} x \equiv \begin{cases} +1\,, & \text{for } x > 0, \\ -1\,, & \text{for } x < 0. \end{cases} \tag{6}$$

Since we are excluding $x = 0$ in eq. (5), we do not need to worry that eq. (6) is not defined at $x = 0$. Finally, note that

$$x = |x| \operatorname{sgn} x \,, \quad \text{for all } x \neq 0. \tag{7}$$

Hence, it follows that

$$x\frac{d}{dx}|x|^p = p|x|^p \,.$$

Likewise,

$$x^2\frac{d}{dx}|x|^p = p(p-1)|x|^p \,.$$

Hence, plugging eq. (5) into eq. (4) results in,

$$\bigl[ap(p-1) + bp + c\bigr]|x|^p = 0 \,.$$

Thus, eq. (5) is a solution to eq. (4) if p is a solution to the following quadratic equation,

$$ap(p-1) + bp + c = 0 \,. \tag{8}$$

Eq. (5) is called *the indicial equation.* It is the analog of the auxiliary equation that arises in solving the second order linear differential equation with constant coefficients.

Eq. (5) is equivalent to,

$$ap^2 + (b-a)p + c = 0 \,, \tag{9}$$

and its solutions are given by,

$$p = p_\pm \equiv \frac{1}{2a}\bigl[a - b \pm \sqrt{(a-b)^2 - 4ac}\,\bigr] \,. \tag{10}$$

If $p_+ \neq p_-$, the we can conclude that the most general solution to eq. (4) is[3]

$$y(x) = A|x|^{p_+} + B|x|^{p_-} \,, \tag{11}$$

where A and B are arbitrary constants.

[3]Note that if we require good behavior at $x = 0$, then we can only accept solutions of the form $|x|^p$ in which $\operatorname{Re} p > 0$.

If a, b and c are real constants[4] and $|a-b| < 2\sqrt{ac}$, then p_+ and p_- are complex numbers and $p_- = (p_+)^*$. In this case,

$$p_\pm \equiv \frac{1}{2a}\left[a - b \pm i\sqrt{4ac - (a-b)^2}\,\right].$$

If we denote,

$$\beta \equiv \frac{1}{2a}\sqrt{4ac - (a-b)^2}\,,$$

then

$$|x|^{p_\pm} = |x|^{(a-b)/(2a)}|x|^{i\beta} = |x|^{(a-b)/(2a)}e^{i\beta\ln|x|} = |x|^{(a-b)/(2a)}\big[\cos(\beta\ln|x|) \pm i\sin(\beta\ln|x|)\big]\,. \tag{12}$$

In this case, one may write the most general solution to eq. (4) as,

$$y(x) = |x|^{(a-b)/(2a)}\big[A\sin(\beta\ln|x|) + B\cos(\beta\ln|x|)\big]\,.$$

Finally, we must address the degenerate case, where $p_+ = p_-$. This case arises when $(a-b)^2 = 4ac$, which implies that

$$p_+ = p_- = \frac{a-b}{2a}\,. \tag{13}$$

In this case, the above analysis succeeds only in finding one of the two solutions to eq. (4), namely,

$$y_1(x) = |x|^{(a-b)/(2a)}\,.$$

The second solution is derived in Appendix B. However, here we shall follow an alternate procedure that makes use of eqs. (9) and (10) of the class handout entitled, *Applications of the Wronskian to linear differential equations*. First, we divide eq. (4) by ax^2 and evaluate the Wronskian of eq. (4),

$$W(x) = C\exp\left\{-\frac{b}{a}\int\frac{dx}{x}\right\} = C\exp\left\{-\frac{b}{a}\ln|x|\right\} = C|x|^{-b/a}\,.$$

Then, the second solution is then given by,

$$y_2(x) = y_1(x)\int\frac{W(x)\,dx}{[y_1(x)]^2} = C'|x|^{(a-b)/(2a)}\int\frac{dx}{x} = C'|x|^{(a-b)/(2a)}\ln|x|\,,$$

where $C' = C\,\mathrm{sgn}(x)$ absorbs a minus sign if $x < 0$.

Thus, we conclude that the most general solution to eq. (4) in the case of degenerate roots of the indicial equation [eq. (8)], where $(a-b)^2 = 4ac$, is given by,

$$y(x) = |x|^{(a-b)/(2a)}\big(A + B\ln|x|\big)\,. \tag{14}$$

[4]More generally, a, b and c can be complex numbers. In this case, p_+ and p_- are generically complex with $p_- \neq (p_+)^*$. Of course, one can still employ eq. (11) by interpreting,

$$|x|^p = |x|^{p_R + ip_I} = |x|^{p_R}\big[\cos(p_I\ln|x|) + i\sin(p_I\ln|x|)\big]\,.$$

in the case of $p_R \equiv \mathrm{Re}\,p$ and $p_I \equiv \mathrm{Im}\,p$.

The results of Section 1 are easily summarized. To find the most general solution of the homogeneous second order Euler-Cauchy differential equation [given by eq. (4)], where the coefficients a, b, and c are real ($a \neq 0$), one first computes the two roots of the corresponding indicial equation, $ap(p-1)^2 + bp + c = 0$. Denoting these roots by p_+ and p_-, one can then immediately write down the two linearly independent solutions to eq. (4), which are valid for all $x \neq 0$,

$$y_h(x) = \begin{cases} A|x|^{p_+} + B|x|^{p_-}\,, & \text{for real roots, } p_+ \neq p_-, \\ |x|^p(A + B\ln|x|)\,, & \text{for degenerate (real) roots, } p \equiv p_+ = p_-, \\ |x|^\alpha(A\sin(\beta\ln|x|)x + B\cos(\beta\ln|x|)\,, & \text{for complex roots, } p_\pm \equiv \alpha \pm i\beta, \end{cases} \tag{15}$$

where A and B are arbitrary constants.

2. The second order inhomogeneous Euler-Cauchy differential equation

In this section, we examine the solutions to

$$ax^2y'' + bxy' + cy = f(x)\,. \tag{16}$$

The general solution to eq. (16) is of the form

$$y(x) = y_p(x) + y_h(x)\,, \tag{17}$$

where $y_h(x)$ is given by eq. (15), where the form of the solution depends on the values of the roots of the indicial equation [eq. (8)].

2.1 The method of undetermined coefficients

We begin by considering unctions $f(x)$ that mirror the choices of the functions examined in Section 6 of Chapter 8 on pp. 420–421 of Boas. Indeed, in light of eq. (2), the exponential function e^{px} in the case of Boas is transformed into the power function $|x|^p$, sines and cosines are transformed into $|x|^p\cos(\ln|x|)$ and $|x|^p\sin(\ln|x|)$, and polynomials in x are transformed into polynomials of $\ln|x|$.

In analogy with eq. (6.24) in Section 6 of Chapter 8 on p. 421 of Boas, if the function $f(x)$ in eq. (16) is taken to be

$$f(x) = |x|^c P_n(\ln|x|)\,, \tag{18}$$

where P_n is a polynomial that involves sums of powers of $\ln|x|$ of degree n, then by the method of undetermined coefficients, the appropriate ansatz for particular solution $y_p(x)$ is given by

$$y_p(x) = \begin{cases} |x|^c\, Q_n(\ln|x|)\,, & \text{if } c \text{ is not equal to either } p_+ \text{ or } p_-, \\ |x|^c\ln|x|\, Q_n(\ln|x|)\,, & \text{if } c \text{ equals } p_+ \text{ or } p_- \text{ and } p_+ \neq p_-, \\ |x|^c\ln^2|x|\, Q_n(\ln|x|)\,, & \text{if } c = p_+ = p_-, \end{cases} \tag{19}$$

where $p_\pm$ are the roots of the indicial equation and Q_n is a polynomial of the same degree as P_n with undetermined coefficients. By plugging in this ansatz back into eq. (16), the undetermined coefficients are then fixed. If $P_n = 1$, then Q_n simply become a single undetermined coefficient to be determined. If c is a complex number, then we interpret $|x|^c$ as indicated in footnote 4.

As an example, we shall solve problem 7–20 on page 436 of Boas,

$$x^2y'' - 3xy' + 4y = 6x^2 \ln|x| \,, \tag{20}$$

where I have written $\ln|x|$ (rather than $\ln x$ as Boas does) so that we can solve the differential equation for both positive and negative values of $x \neq 0$.

We first solve the homogeneous equation,

$$x^2y'' - 3xy' + 4y = 0 \,.$$

The corresponding indicial equation is

$$p(p-1) - 3p + 4 = (p-2)^2 = 0 \,.$$

Thus, there is a degenerate root, $p_\pm = 2$, in which case the solution to the homogeneous equation is given by,

$$y_h(x) = x^2(A + B\ln|x|) \,,$$

for $x \neq 0$, where A and B are arbitrary constants.

Next, the particular solution of eq. (20) is obtained by employing eq. (19), where we identify $P_1(\ln|x|) = \ln|x|$, which is a polynomial in $\ln|x|$ of degree 1. This suggests the following ansatz in which we introduce a polynomial, $Q_1(\ln|x|) = a_0 + a_1\ln|x|$, with two undetermined coefficients,

$$y_p(x) = x^2 \ln^2|x|(a_0 + a_1\ln|x|) \,. \tag{21}$$

It follows that[5]

$$\begin{aligned}
y_p'(x) &= x\bigl(2a_1 \ln^3|x| + (2a_0 + 3a_1)\ln^2|x| + 2a_0\ln|x|\bigr) \,, \\
y_p''(x) &= 2a_1\ln^3|x| + (2a_0 + 9a_1)\ln^2|x| + (6a_0 + 6a_1)\ln|x| + 2a_0 \,.
\end{aligned}$$

Plugging eq. (21) into eq. (20),

$$\begin{aligned}
y_p'' - 3xy_p' + 4y_p = x^2\Bigl\{&\ln^3|x|\bigl(2a_1 - 6a_1 + 4a_1\bigr) + \ln^2|x|\bigl(2a_0 - 9a_1 - 6a_0 - 9a_1 + 4a_0\bigr) \\
&+ \ln|x|\bigl(6a_0 + 6a_1 - 6a_0\bigr) + 2a_0\Bigr\} = 2x^2\bigl(3a_1\ln|x| + a_0\bigr) \,.
\end{aligned}$$

[5]Note that by the chain rule,

$$\frac{d}{dx}\ln^p|x| = p\ln^{p-1}|x|\left(\frac{d}{dx}\ln|x|\right) = p\ln^{p-1}|x|\left(\frac{1}{|x|}\frac{d}{dx}|x|\right) = \frac{\operatorname{sgn} x}{|x|}p\ln^{p-1}|x| = \frac{p}{x}\ln^{p-1}|x| \,.$$

Hence, in light of eq. (20), it follows that

$$2x^2\big(3a_1\ln|x| + a_0\big) = 6x^2\ln|x|\,.$$

We conclude that $a_1 = 1$ and $a_0 = 0$. That is,

$$y_p(x) = x^2\ln^3|x|\,. \tag{22}$$

Hence, the most general solution to eq. (20) is (for $x \neq 0$),

$$y(x) = x^2\big(A + B\ln|x| + \ln^3|x|\big)\,.$$

One can use the method of undetermined coefficients presented above to solve problems 7–17 through 7–22 on p. 436 of Boas. Alternatively, one can employ the method outlined in Appendix A, where one first changes variables as indicated in eq. (2) and converts the second order inhomogeneous Euler-Cauchy differential equation into a second order inhomogeneous linear differential equation with constant coefficients. The latter can be solved using the methods outlined in Section 6 of Chapter 8 on pp. 417–422 of Boas. It is instructive to solve eq. (20) using this alternative technique. This is an exercise that is left for the student to complete.

2.2 An explicit formula for the particular solution of the inhomogeneous second order Euler-Cauchy differential equation

Although the method of undetermined coefficients works well if $f(x)$ is of the form given in eq. (18), the calculations tend to be somewhat tedious. A more general method was provided in eq. (12) of the class handout entitled, *Applications of the Wronskian to linear differential equations*. Once the two linearly independent solutions of eq. (4), denoted by $y_1(x)$ and $y_2(x)$, are determined, one can compute the corresponding Wronskian,

$$W(x) = y_1y_2' - y_2y_1'\,. \tag{23}$$

Then, a particular solution of eq. (16), denoted by $y_p(x)$, can be computed directly using the following formula,[6]

$$y_p(x) = -y_1(x)\int\frac{y_2(x)f(x)}{ax^2W(x)}\,dx + y_2(x)\int\frac{y_1(x)f(x)}{ax^2W(x)}\,dx\,. \tag{24}$$

We now consider separately the three cases exhibited in eq. (15).

First, consider the case of real and nondegenerate roots of the indicial equation. In this case, $y_1(x) = |x|^{p_+}$ and $y_2(x) = |x|^{p_-}$. Using eq. (23), it follows that,

$$W(x) = (p_- - p_+)\frac{|x|^{p_+ + p_-}}{x}\,. \tag{25}$$

[6] Note that eq. (12) of the cited handout was derived assuming that the coefficient of y'' in the differential equation is 1. Thus, in the present application, one must replace $f(x)$ by $f(x)/(ax^2)$.

Eq. (24) then yields,

$$y_p(x) = \frac{1}{a(p_+ - p_-)}\left\{|x|^{p_+}\int \frac{f(x)}{|x|^{p_+}}\,\frac{dx}{x} - |x|^{p_-}\int \frac{f(x)}{|x|^{p_-}}\,\frac{dx}{x}\right\}. \tag{26}$$

Second, in the case of degenerate roots, $y_1(x) = |x|^p$ and $y_2(x) = |x|^p \ln|x|$, where $p = (a-b)/(2a)$ [cf. eq. (13)]. In this case, $W(x) = |x|^{2p}/x$, and eq. (24) yields,

$$y_p(x) = \frac{|x|^p}{a}\left\{\ln|x|\int \frac{xf(x)}{|x|^p}\,\frac{dx}{x} - \int \frac{x\ln|x|f(x)}{|x|^p}\,\frac{dx}{x}\right\}. \tag{27}$$

Finally, in the case of complex roots $p_\pm = \alpha \pm i\beta$, $y_1(x) = |x|^\alpha \sin(\beta\ln|x|)$ and $y_2(x) = |x|^\alpha\cos(\beta\ln|x|)$. In this case, $W(x) = -\beta|x|^{2\alpha}/x$, and eq. (24) yields,

$$y_p(x) = \frac{\beta|x|^\alpha}{a}\left\{\sin(\beta\ln|x|)\int\frac{\cos(\beta\ln|x|)f(x)}{|x|^\alpha}\,\frac{dx}{x} - \cos(\beta\ln|x|)\int\frac{\sin(\beta\ln|x|)f(x)}{|x|^\alpha}\,\frac{dx}{x}\right\}. \tag{28}$$

In summary, the solution to eq. (16) is given by eq. (17), where $y_h(x)$ is given by eq. (15) in the three cases of real nondegenerate, real degenerate and complex roots of the indicial equation, and $y_p(x)$ is given in the three corresponding cases by eqs. (26), (27) and (28), respectively.

2.3 Examples

We now provide two examples from Boas that illustrate the results of the previous subsection. First, consider problem 7–18 of Boas,

$$x^2y'' + xy' - y = x - \frac{1}{x}. \tag{29}$$

The indicial equation is $p^2 - 1 = 0$, which has two roots, $p_+ = 1$ and $p_- = -1$. Hence, $y_1(x) = |x|$ and $y_2(x) = |x|^{-1}$. From eq. (25), it follows that $W(x) = -2/x$. Then from eq. (26), we find that for $x \neq 0$,

$$\begin{aligned} y_p(x) &= \frac{1}{2}\left\{x\int\frac{1}{x^2}\left(x - \frac{1}{x}\right)dx - \frac{1}{x}\int\left(x - \frac{1}{x}\right)dx\right\} \\ &= \frac{1}{2}\left\{x\ln|x| + \frac{1}{2x} - \frac{x}{2} + \frac{1}{x}\ln|x|\right\}. \end{aligned} \tag{30}$$

Notice that there are terms in eq. (30) that are proportional to x and x^{-1}, which are solutions to the homogeneous equation, $x^2y'' + xy' - y = 0$. Hence, we can drop these terms since they already appear in $y_h(x) = A|x| + B|x|^{-1}$, where A and B are arbitrary constants. That is,

$$y_p(x) = \tfrac{1}{2}\ln|x|\left(x + \frac{1}{x}\right),$$

is also a solution to eq. (29). Thus, the general solution to eq. (29) for $x \neq 0$ is then,

$$y(x) = A|x| + B|x|^{-1} + \tfrac{1}{2}\ln|x|\left(x + \frac{1}{x}\right). \tag{31}$$

Second, let us apply eq. (27) to obtain the particular solution of eq. (20). In this example, $a = 1$, $f(x) = 6x^2 \ln|x|$ and the roots of the indicial equation are degenerate, with $p = 2$. Then eq. (27) yields,

$$y_p(x) = 6x^2 \left\{ \ln|x| \int \frac{\ln|x|}{x}\, dx - \int \frac{\ln^2|x|}{x}\, dx \right\} .$$

By writing $x^{-1}dx = d\ln|x|$, the integrals above are elementary, and we end up with,

$$y_p(x) = 6x^2 \ln^3|x| \left(\tfrac{1}{2} - \tfrac{1}{3}\right) = x^2 \ln^3|x| \,,$$

for $x \neq 0$, in agreement with the result previously obtained in eq. (22). At least in the two examples above, the method based on eq. (24) is much faster than the method of undetermined coefficients based on eq. (19). In principle, eq. (24) is applicable for any function $f(x)$ [in contrast to the method of undetermined coefficients employed in Section 2.1], although in some cases it may not be possible to evaluate the indefinite integrals appearing in eq. (24) analytically.

APPENDIX A: Transforming the Euler-Cauchy differential equation into a linear differential equation with constant coefficients

In the grand tradition of mathematics, we will show how to solve a second order Euler-Cauchy differential equation by transforming it into a differential equation that has been previously solved. This is the strategy adopted by Boas in Section 7 of Chapter 8 of Boas [cf. Case (d) and eqs. (7.17)–(7.19) on p. 434 of Boas]. The treatment of Boas implicitly assumes that $x > 0$. In this Appendix, the cases of $x > 0$ and $x < 0$ will be treated simultaneously. As noted in footnote 2, the solution at $x = 0$ must be separately assessed.

For simplicity, the derivation below is given for the second order Euler-Cauchy differential equation, although the derivation is easily extended to the corresponding nth order differential equation (see Appendix C). Consider the left hand side of the following differential equation,

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x) \,. \tag{32}$$

We introduce a new variable,

$$z = \ln|x| \,, \tag{33}$$

under the assumption that $x \neq 0$. Inverting this transformation yields,

$$|x| = e^z \,. \tag{34}$$

We wish to rewrite eq. (32) in terms of y as a function of z. To accomplish this, we employ the chain rule. First, we make use of eq. (34) to obtain,

$$\frac{dx}{dz} = \frac{d|x|}{dz}\frac{dx}{d|x|} = e^z \operatorname{sgn} x = |x| \operatorname{sgn} x = x \,, \tag{35}$$

after noting that $dx/d|x| = (d|x|/dx)^{-1} = (\operatorname{sgn} x)^{-1} = \operatorname{sgn} x$ in light of eq. (6). In the final step above, we used $x = |x| \operatorname{sgn} x$. Employing the chain rule again with the help of eq. (35), it then follows that,

$$\frac{dy}{dz} = \frac{dx}{dz}\frac{dy}{dx} = x\frac{dy}{dx}\,, \tag{36}$$

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{dx}{dz}\frac{d}{dx}\left(\frac{dy}{dz}\right) = x\frac{d}{dx}\left(\frac{dy}{dz}\right)\,. \tag{37}$$

Combining the results of eqs. (36) and (37),

$$\frac{d^2y}{dz^2} = x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}. \tag{38}$$

After subtracting eq. (36) from eq. (38), we end up with

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}\,, \tag{39}$$

$$x\frac{dy}{dx} = \frac{dy}{dz}\,. \tag{40}$$

Employing eqs. (39) and (40) in eq. (32), it follows that,

$$a\frac{d^2y}{dz^2} + (b-a)\frac{dy}{dz} + cy = f\big(e^z \operatorname{sgn} x\big)\,. \tag{41}$$

We have succeeded in transforming the Euler-Cauchy differential equation into a linear differential equation with constant coefficients. The sign of the argument of the function on the right hand side of eq. (41) depends on whether $x > 0$ or $x < 0$.

The auxiliary equation corresponding to eq. (41) is

$$ar^2 + (b-a)r + c = 0\,,$$

which precisely matches the indicial equation given in eq. (9). Hence, the roots of the auxiliary equation are p_+ and p_- [cf. eq. (10)].

If p_+ and p_- are nondegenerate, then the solution to the homogeneous equation corresponding to eq. (41) is [cf. eq. (5.11) on p. 440 of Boas] is given by,

$$y_h = Ae^{zp_+} + Be^{zp_-} = A|x|^{p_+} + B|x|^{p_-}\,,$$

after employing $|x| = e^z$ [eq. (34)]. Thus, we have confirmed eq. (11).

If $p \equiv p_+ = p_- = (a-b)/(2a)$ are degenerate roots, then the solution to the homogeneous equation corresponding to eq. (41) is [cf. eq. (5.15) on p. 440 of Boas],

$$y_h = (A + Bz)e^{zp} = |x|^p(A + B\ln|x|)\,,$$

after using eqs. (33) and (34). Thus, we have confirmed eq. (14).

As an example, we shall find the solution of,

$$x^2y'' + xy' - y = x - \frac{1}{x}\,. \tag{42}$$

which was previously obtained in Section 2.3.

First, we consider the case of $x > 0$. Then, eq. (41) yields,

$$\frac{d^2y}{dz^2} - z = e^z - e^{-z} \,. \tag{43}$$

The auxiliary equation, $r^2 - 1 = 0$, yields two nondegenerate roots, $p_\pm = \pm 1$. Hence,

$$y_h(z) = Ae^z + Be^{-z} \,,$$

where A and B are arbitrary constants. To obtain the particular solution $y_p(x)$, we employ the method of undetermined coefficients [cf. eq. (6.18) on p. 420 of Boas] and the principle of superposition [discussed on p. 425 of Boas], which yields the following ansatz,

$$y_p(z) = z(c_1 e^z + c_2 e^{-z}) \,, \tag{44}$$

where the coefficients c_1 and c_2 are to be determined. Differentiating eq. (44) yields,

$$\frac{dy_p}{dz} = c_1 e^z (1+z) - c_2 e^{-z}(z-1) \,,$$

$$\frac{d^2y_p}{dz^2} = c_1 e^z (2+z) - c_2 e^{-z}(2-z) \,.$$

Inserting the above results into eq. (43) yields $c_1 = c_2 = \frac{1}{2}$. Hence, it follows that,

$$y_p(z) = \tfrac{1}{2} z(e^z + e^{-z}) \,,$$

and the most general solution to eq. (43) in the case of $x > 0$ is given by,

$$y(z) = (A + \tfrac{1}{2}z)e^z + (B + \tfrac{1}{2}z)e^{-z} \,.$$

Finally, putting $z = \ln x$ yields,

$$y(x) = x(A + \tfrac{1}{2}\ln x) + \frac{1}{x}(B + \tfrac{1}{2}\ln x)x \,, \quad \text{for } x > 0. \tag{45}$$

Second, we consider the case of $x < 0$. Then, eq. (41) yields,

$$\frac{d^2y}{dz^2} - z = -e^z + e^{-z} \,. \tag{46}$$

The rest of the calculation is nearly identical. Inserting eq. (44) into eq. (46) yields $c_1 = c_2 = -\frac{1}{2}$. Hence, the particular solution is given by,

$$y_p(z) = -\tfrac{1}{2} z(e^z + e^{-z}) \,,$$

and the most general solution to eq. (43) in the case of $x < 0$ is given by,

$$y(z) = (A - \tfrac{1}{2}z)e^z + (B - \tfrac{1}{2}z)e^{-z} \,.$$

Finally, putting $z = \ln(-x)$ yields,

$$y(x) = -x\big[A - \tfrac{1}{2}\ln(-x)\big] - \frac{1}{x}\big[B - \tfrac{1}{2}\ln(-x)\big] \,, \quad \text{for } x < 0. \tag{47}$$

The solutions in two cases, $x > 0$ and $x < 0$ given by eqs. (45) and (47), respectively, can be expressed by the following single equation,

$$y(x) = A|x| + B|x|^{-1} + \tfrac{1}{2}\ln|x| \left(x + \frac{1}{x} \right) \,, \quad \text{for } x \neq 0,$$

in agreement with our previous result obtained in eq. (31).

APPENDIX B: Differential operators and the Euler-Cauchy differential equation

Consider the homogeneous second order Euler-Cauchy equation,[7]

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0\,,\quad \text{with } a\neq 0, \tag{48}$$

which we can rewritten in operator form,

$$(ax^2D^2+bxD+c)y=0\,. \tag{49}$$

where

$$D\equiv\frac{d}{dx}\,.$$

The corresponding indicial equation was given in eq. (9),

$$ap^2+(b-a)p+c=a(p-p_+)(p-p_-)=0\,, \tag{50}$$

whose roots, $p_\pm$, are given by eq. (10)

$$p_\pm\equiv\frac{1}{2a}\big[a-b\pm\sqrt{(a-b)^2-4ac}\,\big]\,. \tag{51}$$

By multiplying out the two factors $(p-p_+)(p-p_-)$ in eq. (50), it immediately follows that

$$p_+ + p_- = 1-\frac{b}{a}\,,\qquad p_+p_- = \frac{c}{a}\,, \tag{52}$$

which can also be confirmed using the explicit form for $p_\pm$ given in eq. (51).

I now claim that the following operator identity holds,

$$ax^2D^2+bxD+c=a\left(x\frac{d}{dx}-p_+\right)\left(x\frac{d}{dx}-p_-\right)\,. \tag{53}$$

To derive eq. (53), we operate with the right hand side of eq. (53) on an arbitrary well-behaved function $f(x)$. Then it follows that,

$$\begin{aligned}
a\left(x\frac{d}{dx}-p_+\right)\left(x\frac{d}{dx}-p_-\right)f(x) &= a\left(x\frac{d}{dx}-p_+\right)\left(x\frac{df}{dx}-p_-f\right)\\
&= a\left[x\frac{d}{dx}\left(x\frac{df}{dx}\right)-x(p_+ + p_-)x\frac{df}{dx}+p_+p_-f\right]\\
&= a\left[x^2\frac{d^2f}{dx^2}+(1-p_+ - p_-)x\frac{df}{dx}+p_+p_-f\right]\\
&= ax^2\frac{d^2f}{dx^2}+bx\frac{df}{dx}+cf\,,\\
&= (ax^2D^2+bxD+c)f(x)\,.
\end{aligned} \tag{54}$$

where we have used eq. (52) in the penultimate step above. Since eq. (54) is valid for any well-behaved function $f(x)$, eq. (54) is equivalent to the operator equation given in eq. (53), and our proof is complete.

[7]See Appendix C for the generalization to the nth order Euler-Cauchy differential equation.

Moreover, the third line of eq. (54) demonstrates that the same result would be obtained if p_+ and p_- are interchanged. This means that the following result holds as a operator equation,

$$\left(x\frac{d}{dx} - p_+\right)\left(x\frac{d}{dx} - p_-\right) = \left(x\frac{d}{dx} - p_-\right)\left(x\frac{d}{dx} - p_+\right) . \tag{55}$$

That is, the two first order differential operators above commute.

Note that the above results, which are relevant for the Euler-Cauchy differential equation, are the analogues of the operator relation,

$$aD^2 + bD + c = a(D - r_1)(D - r_2) = a(D - r_2)(D - r_1) , \tag{56}$$

where r_1 and r_2 are the roots of the auxiliary equation, $ar^2 + br + c = 0$, of a second order linear differential operator with constant coefficients. Eq. (56) was used by Boas in Section 5 of Chapter 8 on pp. 408–411 in deriving the solution of a general second order linear differential equation with constant coefficients.

We may use eq. (53) to solve the homogeneous second order Euler-Cauchy differential equation given in eq. (48). Simply rewrite eq. (49) as,

$$\left(x\frac{d}{dx} - p_+\right)\left(x\frac{d}{dx} - p_-\right) y = 0 . \tag{57}$$

One possible solution of eq. (57) is,

$$\left(x\frac{d}{dx} - p_-\right) y = 0 . \tag{58}$$

This is a separable first order differential equation, which can be rewritten as

$$\frac{dy}{y} = p_-\left(\frac{dx}{x}\right) .$$

Integrating this equation yields

$$\ln|y| = p_- \ln|x| + \ln C ,$$

where $\ln C$ is an integration constant. Hence, integrating the above equation yields,

$$|y| = C|x|^{p_-} .$$

By writing $|y'| = y \operatorname{sgn} y$ and defining $B \equiv C \operatorname{sgn} y$, we end up with

$$y(x) = B|x|^{p_-} . \tag{59}$$

Alternatively, we can interchange the order of the two commuting operators in eq. (57), which yields

$$\left(x\frac{d}{dx} - p_-\right)\left(x\frac{d}{dx} - p_+\right) y = 0 . \tag{60}$$

Here, another possible solution of eq. (60) is

$$\left(x\frac{d}{dx} - p_+\right) y = 0\,.$$

Following the previous analysis that resulted in eq. (59), we find a solution,

$$y(x) = A|x|^{p_+}\,. \tag{61}$$

If $p_+ \neq p_-$, then the the most general solution of eq. (48) is an arbitrary linear combination of eqs. (59) and (61),

$$y(x) = A|x|^{p_+} + B|x|^{p_-}\,,$$

where A and B are arbitrary constants, in agreement with the result of eq. (11).

In the case of $p \equiv p_+ = p_-$, eq. (51) implies that $(a-b)^2 = 4ac$ and

$$p \equiv \frac{a-b}{2a}\,. \tag{62}$$

In this case eq. (57) reduces to,

$$\left(x\frac{d}{dx} - p\right)^2 y = 0\,, \tag{63}$$

Here, we follow the method used by Boas on p. 410 in the case of degenerate roots of the auxiliary equation. We define,

$$u = \left(x\frac{d}{dx} - p\right) y\,. \tag{64}$$

Then, eq. (63) becomes,

$$\left(x\frac{d}{dx} - p\right) u = 0\,. \tag{65}$$

We have already solved this equation above [see eqs. (58) and (59)]. Thus, the solution to eq. (65) is

$$u(x) = B|x|^p\,.$$

Plugging this result back into eq. (64),

$$\left(x\frac{d}{dx} - p\right) y = B|x|^p\,. \tag{66}$$

This is a first order linear differential equation that can be solved by using eq. (3.9) on p. 401 of Boas. Rewriting eq. (66) in the form, $y' + P(x)y = Q(x)$, we can identify $P(x) = -p/x$ and $Q(x) = B|x|^p/x$. Hence,

$$I \equiv \int P(x)\,dx = -p\int\frac{dx}{x} = -p\ln|x| = \ln|x|^{-p}\,,$$

and the solution to eq. (66) is,

$$y(x) = B|x|^p\int\frac{dx}{x} + A|x|^p = |x|^p\big(A + B\ln|x|\big)\,, \tag{67}$$

where p is given by eq. (62), in agreement with eq. (14).

APPENDIX C: The nth order Euler-Cauchy differential equation

The generalization of the results of Appendices A and B to the nth order homogeneous Euler-Cauchy differential equation is straightforward. As in eq. (5), we take as an ansatz,

$$y(x) = |x|^p \,, \quad \text{assuming } x \neq 0 \tag{68}$$

for a solution to eq. (1). Plugging in eq. (68) into eq. (1) yields,

$$\bigl[a_n p(p-1)(p-2)\cdots(p-n+1) + a_{n-1}p(p-1)(p-2)\cdots(p-n+2) + \cdots + a_1 p + a_0\bigr]|x|^p = 0\,.$$

Thus, eq. (68) is a solution to eq. (1) if p is a solution to the following nth order polynomial equation called the indicial equation,

$$\begin{aligned} a_n p(p-1)(p-2)\cdots(p-n+1) + a_{n-1}p(p-1)(p-2)\cdots(p-n+2) + \cdots + a_1 p + a_0 \\ = a_n(p-p_1)(p-p_2)\cdots(p-p_n) = 0\,, \end{aligned} \tag{69}$$

where the n roots of the indicial equation [some of which may be degenerate] have been denoted by $p_1, p_2, \ldots, p_n$. As expected, eq. (69) reduces to eq. (8) in the case of $n = 2$.

The general solution of eq. (1) is a linear combination of the $|x|^{p_i}$ if no degeneracies are present. If there is a k-fold degenerate set of roots, e.g. $p = p_1 = p_2, \cdots = p_k$ (with $k \geq 2$), then the corresponding functions that can appear in the general solution of eq. (1) are linear combinations of the following functions,

$$\bigl\{|x|^p\,,\ |x|^p \ln|x|\,,\ \ldots\,,\ |x|^p \ln^{k-1}|x|\bigr\}\,.$$

One can easily establish this result by generalizing the derivation in the case of $k = 2$ given by eqs. (63)–(67). For example, the final step will involve solving the equation,

$$\left(x\frac{d}{dx} - p\right) y = C|x|^p \ln^{k-2}|x|\,,$$

where C is an arbitrary constant. This will yield,

$$y(x) = C|x|^p \int \ln^{k-2}|x| \frac{dx}{x} = C|x|^p \int \ln^{k-2}|x|\, d\bigl(\ln|x|\bigr) = C'|x|^p \ln^{k-1}|x|\,,$$

where $C' = C/(k-1)$. The student is encouraged to fill in the details.

Returning to eq. (1), we can rewrite this equation in operator form as

$$(a_n x^n D^n + a_{n-1} x^{n-1} D^{n-1} + \cdots + a_1 x D_1 + a_0)y = 0\,, \tag{70}$$

where $D \equiv d/dx$. Following Appendix A, we can again introduce a new variable, $z = \ln|x|$ or equivalently, $|x| = e^z$. Then, eqs. (39) and (40) imply that

$$xD = D_z\,, \qquad x^2 D^2 = D_z^2 - D_z = D_z\,(D_z - 1)\,, \tag{71}$$

where $D_z \equiv d/dz$. The generalization of eq. (71) to the nth derivative involves the Stirling numbers of the first kind, $s(n,k)$, which appear as the absolute values of the coefficients of the expansion of the quantity, $p(p-1)(p-2)\cdots(p-n+1)$, in powers of p,

$$p(p-1)(p-2)\cdots(p-n+1) = \sum_{k=1}^{n}(-1)^{n-k}s(n,k)p^k\,, \qquad \text{for } n=1,2,3,\ldots. \tag{72}$$

In particular, eq. (72) yields simple expressions for the $s(n,k)$ in three cases: $s(n,n)=1$, $s(n,n-1) = 1+2+\ldots+(n-1) = \frac{1}{2}n(n-1)$ and $s(n,1) = (n-1)!$ for all positive integers n. The rest of the Stirling numbers of the first kind can be obtained by employing the recursion relation,[8]

$$s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1)\,, \qquad \text{for } n \geq k \geq 1, \tag{73}$$

subject to the initial conditions, $s(0,0)\equiv 1$ and $s(n,0)=0$, for all positive integers n.

To derive the general result that relates x^nD^n to powers of D_z, one first notes the following two identities,

$$x^n\bigl(D^n x^p\bigr) = p(p-1)(p-2)\cdots(p-n+1)x^p\,, \tag{74}$$

$$(xD)^n\, x^p = p^n x^p\,. \tag{75}$$

In light of eq. (72), it immediately follows from eqs. (74) and (75) that

$$x^n\bigl(D^n x^p\bigr) = \sum_{k=1}^{n}(-1)^{n-k}s(n,k)(xD)^k x^p\,, \qquad \text{for } n=1,2,3,\ldots \tag{76}$$

Since $\{x^p\}$, for $p=0,1,2,3,\ldots$, constitutes a linearly independent set of functions, it follows that

$$x^nD^n = \sum_{k=1}^{n}(-1)^{n-k}s(n,k)(xD)^k\,, \qquad \text{for } n=1,2,3,\ldots \tag{77}$$

holds as an operator identity.[9] Finally, using $xD = D_z$ [cf. eq. (71)] and replacing p with D_z in eq. (72), one sees that eq. (77) is equivalent to the operator identity,

$$x^nD^n = \sum_{k=1}^{n}(-1)^{n-k}s(n,k)D_z^k = D_z(D_z-1)(D_z-2)\cdots(D_z-n+1)\,, \qquad \text{for } n=1,2,3,\ldots \tag{78}$$

Indeed, eq. (78) is the generalization of eq. (71) that we were seeking. Hence,

$$a_nx^nD^n + a_{n-1}x^{n-1}D^{n-1} + \cdots + a_1xD_1 + a_0 = b_nD_z^n + b_{n-1}x^{n-1}D_z^{n-1} + \cdots + b_1D_z + b_0\,, \tag{79}$$

where the b_k can be expressed as linear combinations of the a_k. Thus, we have succeeded in transforming the nth order Euler-Cauchy differential equation into an nth order linear differential equation with constant coefficients.

[8] For further details, see Chapter 6, Section 1 of Ronald L. Graham, Donald E. Knuth and Oren Patashnik, *Concrete Mathematics*, 2nd edition (Addison-Wesley, Reading, MA, 1994). My notation for the Stirling numbers of the first kind follows Chapter 5 of Louis Comtet, *Advanced Combinatorics* (D. Reidel Publishing Company, Dordrecht, Holland, 1974).

[9] Eq. (77) is the subject of exercise 13 on p. 310 of *Concrete Mathematics*, op. cit. (see footnote 8).

Employing eqs. (78) and (79), the b_k are expressed in terms of the a_k as follows,

$$\begin{aligned}
b_n &= a_n\,, \\
b_{n-1} &= a_{n-1} - s(n,n-1)a_n\,, \\
b_{n-2} &= a_{n-2} - s(n-1,n-2)a_{n-1} + s(n,n-2)a_n\,, \\
&\;\vdots \\
b_2 &= a_2 - s(3,2)a_3 + s(4,2)a_4 - \cdots + (-1)^{n-2}s(n,2)a_n\,, \\
b_1 &= a_1 - s(2,1)a_2 + s(3,1)a_3 - \cdots + (-1)^{n-1}s(n,1)a_n\,, \\
b_0 &= a_0\,,
\end{aligned} \tag{80}$$

after noting that $s(k,k) = 1$ for $k = 1, 2, \ldots n$.

It is instructive to check the above expressions in the cases of $n = 2$ and $n = 3$, respectively. In the case of $n = 2$, it follows that $b_2 = a_2$, $b_1 = a_1 - a_2$ and $b_0 = a_0$, which corresponds to the result previously obtained in eq. (41). In the case of $n = 3$,

$$a_3x^3D^3 + a_2x^2D^2 + a_1xD + a_0 = a_3D_z^3 + (a_2 - 3a_3)D_z^2 + (a_1 - a_2 + 2a_3)D_z + a_0\,,$$

where we have used $s(3,2) = 3$, $s(3,1) = 2$ and $s(2,1) = 1$ [these values were obtained from either eq. (72) or eq. (73)].

One can also use eq. (78) to obtain an alternate form of eqs. (79) and (80),

$$\begin{aligned}
&a_nx^nD^n + a_{n-1}x^{n-1}D^{n-1} + \cdots + a_1xD_1 + a_0 \\
&\quad = a_nD_z(D_z-1)(D_z-2)\cdots(D_z-n+1) + a_{n-1}D_z(D_z-1)(D_z-2)\cdots(D_z-n+2) \\
&\qquad\qquad + \cdots + a_2D_z(D_z-1) + a_1D_z + a_0
\end{aligned} \tag{81}$$

If we now employ eq. (69) with p replaced by the operator D_z, then eq. (81) yields

$$a_nx^nD^n + a_{n-1}x^{n-1}D^{n-1} + \cdots + a_1xD + a_0 = a_n(D_z-p_1)(D_z-p_2)\cdots(D_z-p_n)\,, \tag{82}$$

where $p_1,\ p_2, \ldots, p_n$ are the roots of the indicial equation. Consequently, the auxiliary equation corresponding to $a_n(D_z-p_1)(D_z-p_2)\cdots(D_z-p_n)$ is precisely the same as the indicial equation of the corresponding Euler-Cauchy differential equation.

Comparing eqs. (79) and (82) leads to alternative expressions for the b_k,

$$\begin{aligned}
b_n &= a_n\,, \\
b_{n-1} &= -a_n(p_1 + p_2 + \ldots + p_n)\,, \\
b_{n-2} &= a_n(p_1p_2 + p_1p_3 + \ldots + p_{n-1}p_n)\,, \\
&\;\vdots \\
b_0 &= a_0 = a_n(-1)^n p_1p_2\cdots p_n\,.
\end{aligned} \tag{83}$$

The various coefficients of a_n on the right hand sides of eq. (83) involving the roots of the indicial equation are simply related to the coefficients of the indicial polynomial given in eq. (69). Working out these relations will ultimately yield the results given in eq. (80).

Since $D_z = xD$, the operator identity obtained in eq. (82) is equivalent to

$$a_n x^n D^n + a_{n-1}x^{n-1}D^{n-1} + \cdots + a_1 xD + a_0 = a_n(xD - p_1)(xD - p_2)\cdots(xD - p_n)\,, \quad (84)$$

which generalizes eq. (53). In light of eq. (55), all the operators, $xD - p_i$, on the right hand side of eq. (84) commute.

A proof of eq. (84) in the case of $n = 2$ was given in eq. (54). It is instructive to provide an explicit proof in the case of $n = 3$ that follows the method used to derive eq. (54).

$$\begin{aligned}
a_3\left(x\frac{d}{dx} - p_3\right)\left(x\frac{d}{dx} - p_2\right)\left(x\frac{d}{dx} - p_1\right)f(x) &= a_3\left(x\frac{d}{dx} - p_3\right)\left(x\frac{d}{dx} - p_2\right)\left(x\frac{df}{dx} - p_1 f\right) \\
&= a_3\left(x\frac{d}{dx} - p_3\right)\left[x\frac{d}{dx}\left(x\frac{df}{dx}\right) - x(p_1+p_2)x\frac{df}{dx} + p_1p_2 f\right] \\
&= a_3\left(x\frac{d}{dx} - p_3\right)\left[x^2\frac{d^2f}{dx^2} + (1-p_1-p_2)x\frac{df}{dx} + p_1p_2 f\right] \\
&= a_3\left[x\frac{d}{dx}\left(x^2\frac{d^2f}{dx^2}\right) + (1-p_1-p_2)x\frac{d}{dx}\left(x\frac{df}{dx}\right) + p_1p_2 x\frac{df}{dx} - p_3x^2\frac{d^2f}{dx^2}\right. \\
&\qquad\left. -p_3(1-p_1-p_2)x\frac{df}{dx} - p_1p_2p_3 f\right] \\
&= a_3\left[x^3\frac{d^3f}{dx^3} + 2x^2\frac{d^2f}{dx^2} + (1-p_1-p_2)\left[x^2\frac{d^2f}{dx^2} + x\frac{df}{dx}\right]\right. \\
&\qquad\left. +p_1p_2x\frac{df}{dx} - p_3x^2\frac{d^2f}{dx^2} - p_3(1-p_1-p_2)x\frac{df}{dx} - p_1p_2p_3 f\right] \\
&= a_3\left[x^3\frac{d^3f}{dx^3} + x^2(3-p_1-p_2-p_3)\frac{d^2f}{dx^2}\right. \\
&\qquad\left. +x(1-p_1-p_2-p_3+p_1p_2+p_1p_3+p_2p_3)\frac{df}{dx} - p_1p_2p_3 f\right]. \quad (85)
\end{aligned}$$

Next, we consider the indicial polynomial [cf. eq. (69)] in the case of $n = 3$,

$$a_3p(p-1)(p-2) + a_2p(p-1) + a_1p + a_0 = a_3(p-p_1)(p-p_2)(p-p_3)\,. \quad (86)$$

Comparing like powers of p on both sides of eq. (86) yields,

$$a_2 - 3a_3 = -a_3(p_1+p_2+p_3)\,, \qquad a_1 - a_2 + 2a_3 = a_3(p_1p_2+p_1p_3+p_2p_3)\,, \qquad a_0 = -a_3p_1p_2p_3\,.$$

Plugging these results back into eq. (85) yields,

$$a_3\left(x\frac{d}{dx} - p_3\right)\left(x\frac{d}{dx} - p_2\right)\left(x\frac{d}{dx} - p_1\right)f(x) = a_3x^3\frac{d^3f}{dx^3} + a_2x^2\frac{d^2f}{dx^2} + a_1\frac{df}{dx} + a_0\,,$$

which completes the proof.

Repeating such a proof for larger values of n is not practical. You should be impressed with the simplicity of the general proof of eq. (84) previously given, which follows almost immediately from eqs. (69), (74) and (75).