Determinant and the Adjugate

In these notes, I shall provide a formal definition of the determinant of an $n \times n$ matrix. I will then introduce the adjugate (also known as the classical adjoint) of an $n \times n$ matrix and show how it is related to the inverse of the matrix (if the inverse exists). Finally, I shall provide a proof of Cramer's rule.

The formulae presented in these notes for the determinant and the inverse of a matrix are mainly of theoretical interest. They often can be used in proofs of other mathematical statements. However, if you are interested in the most efficient methods of numerical computations, the formulae exhibited in these notes become very impractical once n becomes larger than 4. Indeed, the row reduction technique discussed in class is the preferred method for computing both determinants and matrix inverses in practical numerical applications,* as discussed in Appendix A.

1. Even and odd permutations

In order to present the definition of the determinant, one must first understand the concept of even and odd permutations. Consider a set consisting of the first n positive integers, $\{1, 2, \ldots, n\}$. A permutation of this set consists of a reordering of the elements of the set. In all, there are $n!$ possible permutations, where I am including the null permutation which corresponds to the case where no numbers are reordered. For example, starting from $\{1, 2, 3\}$, there are six possible distinct permutations in total, which are listed below:

$$\{1,2,3\}\,,\ \{2,1,3\}\,,\ \{1,3,2\}\,,\ \{2,3,1\}\,,\ \{3,1,2\}\,,\ \{3,2,1\}\,. \tag{1}$$

A *transposition* is defined to be a permutation in which a pair of integers are interchanged. For example, starting from $\{1, 2, 3\}$, one can perform three different transpositions: $1 \leftrightarrow 2$, $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, which results in the permutations corresponding to the following reorderings, $\{2,1,3\}\,, \{3,2,1\}\,, \{1,3,2\}$ listed in eq. (1). Moreover, any permutation is equivalent to a sequence of transpositions. As an example, if I consider the permutation $\{1, 2, 3\} \to \{3, 1, 2\}$, I can accomplish this permutation by the following sequence of transpositions,

$$\{1,2,3\} \to \{1,3,2\} \to \{3,1,2\}\,.$$

The sequence of transpositions is not unique. Another possible sequence of transpositions to describe the permutation $\{1, 2, 3\} \to \{3, 1, 2\}$ is

$$\{1,2,3\} \to \{3,2,1\} \to \{3,1,2\}\,.$$

However, you will notice that in both cases, an even number of transpositions (i.e, two) were used to perform the permutation $\{1, 2, 3\} \to \{3, 1, 2\}$. It is not difficult to prove

*See the class handout entitled, *Elementary Row Operations and Some Applications.*

the following statement:

> A permutation is defined to be even if and only if the number of transpositions needed to carry out the permutation is an even number. Likewise, a permutation is defined to be odd if and only if the number of transpositions needed to carry out the permutation is an odd number. A given permutation must be either even or odd (it cannot be both). In particular, starting from $\{1, 2, \ldots, n\}$, there are precisely $\frac{1}{2}n!$ even permutations and $\frac{1}{2}n!$ odd possible permutations.

The null permutation is even and a transposition is an odd permutation. Thus, in the case of $n = 3$, starting from $\{1, 2, 3\}$, the permutations given in eq. (1) separate out into three even permutations $[\{1, 2, 3\}\,, \{3, 1, 2\}\,, \{2, 3, 1\}]$ and three odd permutations corresponding to the three possible transpositions $[\{2, 1, 3\}\,, \{1, 3, 2\}\,, \{3, 2, 1\}]$.

2. The determinant defined

Given an $n \times n$ matrix $A = [a_{ij}]$, where we are denoting the matrix elements of A by a_{ij}, the determinant of A (denoted by $\det A$) is defined ti be

$$\det A = \sum_P (-1)^P a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}\,, \tag{2}$$

where the sum $\sum_P$ denotes the sum over all possible permutations P of the column indices, $\{1, 2, 3, \ldots, n\} \to \{j_1, j_2, j_3 \ldots, j_n\}$, and the symbol $(-1)^P$ is defined as,

$$(-1)^P = \begin{cases} +1\,, & \text{if the permutation } P \text{ is even,} \\ -1\,, & \text{if the permutation } P \text{ is odd.} \end{cases}$$

To see how this definition works, let us consider the case of a 2×2 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\,.$$

Using eq. (2), we note that there are two possible permutations P: the even (null) permutation $\{1, 2\} \to \{1, 2\}$ and the odd permutation (in this case a transposition), $\{1, 2\} \to \{2, 1\}$. Hence,

$$\det A \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}\,,$$

where I have employed the notation for the determinant of a matrix introduced in eqs. (3.1) and (3.2) on p. 89 of Boas.

Notice that the column indices $\{j_1, j_2, j_3, \ldots, j_n\}$ that appear in eq. (2) are all distinct integers corresponding to a reordering (i.e., permutation) of $\{1, 2, 3, \ldots, n\}$. As a check, you should verify that in the case of $n = 3$, eq. (2) yields,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{11}\,. \tag{3}$$

One can rewrite eq. (2) in another way by introducing the Levi-Civita symbol, $\epsilon_{j_1 j_2 j_3 \cdots j_n}$, which is defined as follows,

$$\epsilon_{j_1 j_2 j_3 \cdots j_n} = \begin{cases} +1\,, & \text{if } \{j_1, j_2, j_3, \ldots j_n\} \text{ is an even permutation of } \{1,2,3,\ldots n\}, \\ -1\,, & \text{if } \{j_1, j_2, j_3, \ldots j_n\} \text{ is an odd permutation of } \{1,2,3,\ldots n\}, \\ \;\;0\,, & \text{if not all the integers } j_1, j_2, j_3, \ldots, j_n \text{ are distinct.} \end{cases} \tag{4}$$

In particular, since $\{1,2,3,\ldots,n\}$, where the integers appear in increasing order, corresponds to the null permutation, it follows that $\epsilon_{123\cdots n} = +1$. Moreover, the Levi-Civita symbol satisfies the following property–it changes sign under the interchange of any pair of indices. Symbolically, this property can be exhibited as follows,

$$\epsilon_{j_1 j_2 \cdots j_k \cdots j_\ell \cdots j_n} = -\epsilon_{j_1 j_2 \cdots j_\ell \cdots j_k \cdots j_n}\,, \tag{5}$$

after interchanging the two indices $j_k \leftrightarrow j_\ell$.

As an example, in the case of $n = 3$, the integers $\{j_1, j_2, j_3\}$ of the Levi-Civita symbol $\epsilon_{j_1 j_2 j_3}$ can take on $3^3 = 27$ possible values. But in only $3! = 6$ cases are the integers $\{j_1, j_2, j_3\}$ distinct. It then follows that

$$\begin{aligned} &\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1\,, \\ &\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1\,, \\ &\epsilon_{111} = \epsilon_{112} = \epsilon_{113} = \epsilon_{121} = \epsilon_{122} = \epsilon_{131} = \epsilon_{133} = \epsilon_{211} = \epsilon_{212} = \epsilon_{221} = \epsilon_{222} \\ &\qquad = \epsilon_{223} = \epsilon_{232} = \epsilon_{233} = \epsilon_{311} = \epsilon_{313} = \epsilon_{322} = \epsilon_{323} = \epsilon_{331} = \epsilon_{332} = \epsilon_{333} = 0\,. \end{aligned} \tag{6}$$

Using the Levi-Civita symbol, the definition of the determinant of the $n \times n$ matrix $A = [a_{ij}]$ can be written as the following n-fold sum,

$$\det A = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \sum_{j_3=1}^{n} \cdots \sum_{j_n=1}^{n} \epsilon_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}\,. \tag{7}$$

Although the sum consists of n^n terms, the Levi-Civita symbol is zero unless the elements of $\{j_1, j_2, j_3, \ldots, j_n\}$ are distinct integers corresponding to a permutation of $\{1,2,3,\ldots,n\}$. Thus, the sum actually consists of $n!$ non-vanishing terms and coincides precisely with our original definition of the determinant given in eq. (2).

Sometimes, an alternate (more symmetrical looking) version of eq. (7) is given,

$$\epsilon_{i_1 i_2 i_3 \cdots i_n} \det A = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \sum_{j_3=1}^{n} \cdots \sum_{j_n=1}^{n} \epsilon_{j_1 j_2 j_3 \cdots j_n} a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n}\,. \tag{8}$$

Since $\epsilon_{123\cdots n} = +1$, it follows that setting $i_1 = 1$, $i_2 = 2$, $i_3 = 3, \ldots, i_n = n$ in eq. (8) simply reproduces eq. (7). The version given by eq. (8) is simply a consequence of the antisymmetry property of the Levi-Civita symbol given in eq. (5). For further details, you may consult Chapter 10, Section 5 on pp. 508–509 of Boas.

Many of the properties of the determinant can be established using one of the definitions of the determinant given in this section. For example, if the matrix A has two identical rows, then its determinant is zero. Here is one way of proving this result. Suppose the first two rows of A are identical. Then, $a_{2j_2} = a_{1j_2}$. Then, in eq. (7), we have

$$\begin{aligned}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \epsilon_{j_1 j_2 j_3\cdots j_n} a_{1j_1}a_{1j_2}a_{3j_3}\cdots a_{nj_n} &= \sum_{j_2=1}^{n}\sum_{j_1=1}^{n} \epsilon_{j_2 j_1 j_3\cdots j_n} a_{1j_2}a_{1j_1}a_{3j_3}\cdots a_{nj_n} \\ &= -\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \epsilon_{j_1 j_2 j_3\cdots j_n} a_{1j_1}a_{1j_2}a_{3j_3}\cdots a_{nj_n} \\ &= 0\,, \end{aligned} \tag{9}$$

In the first step above, we simply performed a relabeling of indices by renaming the index j_1 by j_2 and renaming the index j_2 by j_1. In the penultimate step, we employed the antisymmetry property of the Levi-Civita symbol [cf. eq. (5)] and used the fact that multiplication of two numbers is commutative [i.e., $a_{1j_2}a_{1j_1} = a_{1j_1}a_{1j_2}$]. Moreover, since these are finite sums in eq. (9), changing the order of summation does not modify the end result. Since the only number that is equal to its negative is zero, the final conclusion is established. Inserting the result of eq. (9) back into eq. (7), we conclude that $\det A = 0$. The same type of reasoning works if the ith row and jth row of the matrix A are identical.

It is not too difficult to prove that all of the above results hold if the rows and columns of A are interchanged. For example, the corresponding result analogous to eq. (2) is,

$$\det A = \sum_P (-1)^P a_{i_1 1}a_{i_2 2}a_{i_3 3}\cdots a_{i_n n}\,, \tag{10}$$

where $A = [a_{ij}]$ is an $n\times n$ matrix and the sum is taken over all possible permutations P of the row indices, $\{1, 2, 3, \ldots, n\} \to \{i_1, i_2, i_3 \ldots, i_n\}$. This definition can be rewritten in a form analogous to eq. (7) by employing the Levi-Civita symbol,

$$\det A = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\cdots\sum_{i_n=1}^{n} \epsilon_{i_1 i_2 i_3\cdots i_n} a_{i_1 1}a_{i_2 2}a_{i_3 3}\cdots a_{i_n n}\,. \tag{11}$$

Finally, the result analogous to eq. (8) is,

$$\epsilon_{j_1 j_2 j_3\cdots j_n}\det A = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\cdots\sum_{i_n=1}^{n} \epsilon_{i_1 i_2 i_3\cdots i_n} a_{i_1 j_1}a_{i_2 j_2}a_{i_3 j_3}\cdots a_{i_n j_n}\,. \tag{12}$$

Indeed, it is straightforward to show that eqs. (8) and (12) are consistent with each other. The proof is presented in Appendix B.

The transpose of a matrix A (denoted by A^{T}) is obtained from A by interchanging its rows and columns. That is, if $A = [a_{ij}]$ then $A^{\mathsf{T}} = [a_{ji}]$. In light of the results quoted above, it then follows that

$$\det A = \det A^{\mathsf{T}}\,. \tag{13}$$

In particular, since we have already shown that if a matrix has two identical rows then its determinant is zero, one can also conclude that if a matrix has two identical columns then its determinant is zero.

3. Defining the determinant via the expansion in cofactors

There is another definition of the determinant called the Laplace expansion (or equivalently the cofactor expansion), in which the determinant is defined recursively. One starts with a 1×1 matrix (which is equivalent to a real or complex number), and defines the determinant of this matrix by

$$\det [c] = c\,.$$

The Laplace expansion then defines the determinant of an $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices. In this way, one can build up the formula for the determinant of an $n \times n$ matrix step by step, by first obtaining the expression for the determinant of a 2×2 matrix, then a 3×3 matrix and so on.

To exhibit the relevant formulae, one must first define the minor and the cofactor of the element a_{ij} of the $n \times n$ matrix $A = [a_{ij}]$. The minor of a_{ij}, denoted by M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and the jth column of A. The cofactor of a_{ij}, denoted by C_{ij} is related to M_{ij} by a sign factor as follows,

$$C_{ij} = (-1)^{i+j} M_{ij}\,. \tag{14}$$

Given an $n \times n$ matrix $A = [a_{ij}]$, where the cofactor of a_{ij} is C_{ij}, the determinant of A is given by the following Laplace expansion,

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}\,. \tag{15}$$

The expansion in cofactors exhibited in eq. (15) makes use of the first row of A. However, there is nothing special about the first row. Performing the expansion in cofactors by employing any row of A will yield the same result. Thus, the cofactor expansion about row i is given by,

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}\,, \quad \text{for any fixed choice of } i = 1, 2, \ldots, n. \tag{16}$$

In light of eq. (13), one can also evaluate the determinant of A by using any column in the expansion in cofactors. Thus the cofactor expansion about column j is given by,

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}\,, \quad \text{for any fixed choice of } j = 1, 2, \ldots, n. \tag{17}$$

It is not difficult to derive eqs. (16) and (17) from eqs. (7) and (11), respectively (e.g., see Ref. 1). Here, we will be content to demonstrate the validity of the cofactor expansion about the first row for the determinant of a 3×3 matrix. In light of eq. (3),

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{11} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\,.
\end{aligned} \tag{18}$$

One can now use eq. (16) to establish an important result. Given the $n \times n$ matrix $A = [a_{ij}]$, we wish to evaluate the following sum,

$$\sum_{j=1}^{n} a_{kj} C_{ij} \,, \tag{19}$$

where the indices k and i can take on any integer value between 1 and n. If $k = i$, then the above sum is equal to $\det A$ [cf. eq. (16)]. If $k \neq i$, consider the matrix $A' = [a'_{ij}]$ obtained from A by replacing the ith row of A by its kth row. That is, the ith and kth rows of A' are identical, with

$$a'_{ij} = a'_{kj} = a_{kj} \,.$$

Then, it follows that $\det A' = 0$. But, we can use the cofactor expansion about the ith row of A' to write,

$$0 = \det A' = \sum_{j=1}^{n} a'_{ij} C'_{ij} = \sum_{j=1}^{n} a_{kj} C_{ij} \,. \tag{20}$$

Note that for a fixed value of i, the cofactors C'_{ij} of the ij matrix element of A' are equal to the cofactors C_{ij} of the ij matrix element of A, since C'_{ij} and C_{ij} computed by deleting the ith row (and jth column) of A' and A, respectively, and computing the determinant of the resulting matrix. But the matrices A and A' differ only in their ith row, so the cofactors of each element of the ith row of A and A' must be the same.

Combining eqs. (16) and (20), we can write one equation by employing the Kronecker delta,

$$\delta_{ik} \det A = \sum_{j=1}^{n} a_{kj} C_{ij} \,, \tag{21}$$

where

$$\delta_{ik} = \begin{cases} 1\,, & \text{for } i = k, \\ 0\,, & \text{for } i \neq k. \end{cases} \tag{22}$$

A similar result can be obtained by using the cofactor expansion about the jth column,

$$\delta_{jk} \det A = \sum_{i=1}^{n} a_{ik} C_{ij} \,, \tag{23}$$

4. The adjugate of a matrix and its relation to the matrix inverse

Given an $n \times n$ matrix $A = [a_{ij}]$, the cofactor of a_{ij}, which is denoted by C_{ij}, was defined in eq. (14). One can now introduce the matrix of cofactors, $C = [C_{ij}]$. That is, the matrix elements of the matrix of cofactors are given by the cofactors of the corresponding matrix elements of A. The adjugate (or classical adjoint) of A, denoted by $\operatorname{adj} A$, is defined as the transpose of the matrix of cofactors,

$$\operatorname{adj} A = C^{\mathsf{T}} \,. \tag{24}$$

That is, the matrix elements of $\operatorname{adj} A$ are given by $(\operatorname{adj} A)_{ij} = C_{ji}$.

Using the definition of the adjugate, one can rewrite eq. (21) as

$$\delta_{ik} \det A = \sum_{j=1}^{n} a_{kj} (\operatorname{adj} A)_{ji} \,. \tag{25}$$

Recalling the definition of matrix multiplication, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ matrices, the matrix elements of the product AB are given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \,. \tag{26}$$

Hence, eq. (25) is equivalent to

$$\delta_{ik} \det A = [A(\operatorname{adj} A)]_{ki} \,. \tag{27}$$

Likewise, a similar argument based on eq. (23) yields,

$$\delta_{jk} \det A = [(\operatorname{adj} A)A]_{jk} \,. \tag{28}$$

Introducing the $n \times n$ identity matrix $\mathbf{I}$, whose matrix elements are $\mathbf{I} = [\delta_{ij}]$, eqs. (27) and (28) are equivalent to the matrix equations,

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)\mathbf{I} \,. \tag{29}$$

By definition, the matrix inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = \mathbf{I}$. Hence, it follows from eq. (29) that if $\det A \neq 0$, then

$$\boxed{A^{-1} = \frac{\operatorname{adj} A}{\det A} \,.} \tag{30}$$

If $\det A = 0$, then the matrix inverse of A does not exist.

We end this section by proving a formula discovered originally by Cauchy,

$$\det(\operatorname{adj} A) = \big[\det A\big]^{n-1} \,, \quad \text{where } A \text{ is an } n \times n \text{ matrix.} \tag{31}$$

The first step in the proof makes use of the observation that for any constant c and $n \times n$ matrix A, it follows from eq. (2) that $\det(cA) = c^n \det A$. Hence eq. (30) yields

$$\det(\operatorname{adj} A) = (\det A)^n \det A^{-1} \,. \tag{32}$$

The final step makes use of the following property of determinants,

$$\det(AB) = \det A \det B \,, \tag{33}$$

which was proven in class with the help of row reduction techniques. Taking $B = A^{-1}$ and noting that $\det(AA^{-1}) = \det \mathbf{I} = 1$, it follows from eq. (33) that $\det A \det A^{-1} = 1$. Hence, assuming that $\det A \neq 0$ (which is required if A^{-1} exists), one can conclude that

$$\det A^{-1} = \frac{1}{\det A} \,,$$

Inserting this last result into eq. (32), we end up with eq. (31) as advertised.

5. Proof of Cramer's rule

Cramer's rule: Consider a set of n equations and n unknowns,

$$A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}, \tag{34}$$

where A is an $n \times n$ coefficient matrix, under the assumption that $\det A \neq 0$, the w_j are known numbers and the x_j are the unknowns whose values we wish to compute. Then, Cramer's rule states that the unique solution to this set of equations is given by

$$x_j = \frac{\det\ A^{(j)}}{\det\ A}\,, \qquad\qquad i = 1, 2, 3\ldots, n\,, \tag{35}$$

where the matrix $A^{(j)}$ is obtained by replacing the jth column of A with the right hand side of eq. (34).

<u>**Proof**</u>:

Eq. (34) can be rewritten symbolically as:

$$A\boldsymbol{x} = \boldsymbol{w}\,, \tag{36}$$

where A is an $n \times n$ matrix and $\boldsymbol{x}$ and $\boldsymbol{w}$ are n-component vectors, whose components are explicitly exhibited in eq. (34). Multiplying eq. (36) on the left by A^{-1} (which exists under the assumption that $\det A \neq 0$) yields the unique solution,

$$\boldsymbol{x} = A^{-1}\boldsymbol{w}\,. \tag{37}$$

Employing eq. (30),

$$\boldsymbol{x} = \frac{1}{\det A}(\text{adj}\,A)\boldsymbol{w}\,. \tag{38}$$

Eq. (38) can be written in terms of the components,

$$x_j = \frac{1}{\det A}\sum_{k=1}^{n}(\text{adj}\,A)_{jk}w_k\,. \tag{39}$$

Consider now the matrix $A^{(j)}$ obtained by replacing the jth column of A with the right hand side of eq. (34). We can compute its determinant by employing the cofactor expansion about column j by using eq. (17),

$$\det A^{(j)} = \sum_{k=1}^{n}[A^{(j)}]_{kj}C_{kj}\,, \quad \text{for any fixed choice of } j = 1, 2, \ldots, n. \tag{40}$$

where C_{ij} is the cofactor of the matrix element $[A^{(j)}]_{ij}$. Since the cofactor C_{ij} is computed by deleting the jth column (and the ith row) of the matrix $A^{(j)}$, it follows that C_{ij} is also the cofactor of the matrix element a_{ij} of the matrix $A = [a_{ij}]$ (since the latter differs from $A^{(j)}$ only in the elements that appear in the jth column). Thus, in light of eq. (24), $(\mathrm{adj}\, A)_{jk} = C_{kj}$, and one can rewrite eq. (40) as,

$$\det A^{(j)} = \sum_{k=1}^{n} (\mathrm{adj}\, A)_{jk} [A^{(j)}]_{kj} \,, \quad \text{for any fixed choice of } j = 1, 2, \ldots, n. \tag{41}$$

By definition, the matrix element of $A^{(j)}$ in the kth row and jth column is w_k. That is, $[A^{(j)}]_{kj} = w_k$, and eq. (41) yields,

$$\det A^{(j)} = \sum_{k=1}^{n} (\mathrm{adj}\, A)_{jk} w_k \,, \quad \text{for any fixed choice of } j = 1, 2, \ldots, n. \tag{42}$$

Comparing eqs. (39) and (42), it follows that

$$x_j = \frac{\det\ A^{(j)}}{\det\ A} \,, \quad \text{for any fixed choice of } j = 1, 2, \ldots, n. \tag{43}$$

Hence, eq. (35) is proven.

APPENDIX A: How not to numerically evaluate a determinant

The formulae for the determinant of an $n \times n$ matrix given in these notes are of theoretical interest but of little practical use if n is large. The following results quoted in Ref. 2 may be of interest in this regard. If eq. (2) is used to evaluate the determinant of an $n \times n$ matrix, then $(n-1) \cdot n!$ multiplications are required. Assuming that a computer can perform 10^6 multiplications and neglecting all other operations, the determinant of an 11×11 matrix would take roughly one hour of computing time to evaluate. Under the same assumptions, to evaluate the determinant of a 100×100 matrix would take about 3×10^{146} years (to be compared with the age of the universe which is approximately 14 billion years).

Can we do better by employing the cofactor expansion [e.g. eq. (16)] to compute the determinant? To evaluate the determinant of an $n \times n$ matrix, one is required to perform approximately $(e-1)n!$ multiplications, where $e \simeq 2.718$ is Napier's constant. So, under the same assumptions as before, to evaluate the determinant of a 100×100 matrix would take about 5×10^{144} years. Not much of an improvement.

This is why for any sizable matrix, the most efficient way to evaluate the determinant is to employ the row reduction technique to convert the original matrix into an upper (or lower triangular form). The determinant of the resulting upper (or lower) triangular matrix is then given by the product of its diagonal elements. Given an $n \times n$ matrix, it can be shown that the number of multiplications required to evaluate a determinant using the row reduction technique is of $\mathcal{O}(n^3)$ when n is large (e.g., see Ref. 3). Thus under the assumptions made above, the determinant of a 100×100 matrix can be done using a computer algorithm based on row reduction in about 1 second!

Similar remarks also apply to the numerical evaluation of a matrix inverse.

APPENDIX B: Proof that eqs. (8) and (12) are consistent

It is straightforward to show that eqs. (8) and (12) are consistent. To verify this claim, one multiplies both sides of eq. (8) by $\epsilon_{i_1 i_2 i_3 \cdots i_n}$ and then sums both sides over the n indices i_1, i_2, $i_3, \ldots, i_n$. One can perform the n-fold sum on the left hand side of the resulting equation by making use of the identity,

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\cdots\sum_{i_n=1}^{n}\epsilon_{i_1 i_2 i_3\cdots i_n}\epsilon_{i_1 i_2 i_3\cdots i_n} = n!\,. \tag{44}$$

This identity follows immediately from the definition of the Levi-Civita symbol [eq. (4)], since of the n^n terms in the n-fold sum, only $n!$ terms are nonzero (corresponding to values of $\{i_1, i_2, i_3, \ldots, i_n\}$ that are distinct and permutations of $\{1, 2, 3, \ldots, n\}$). Each of the $n!$ non-vanishing terms of the n-fold sum is equal to either $(+1)(+1) = 1$ or $(-1)(-1) = 1$. Hence, the sum exhibited in eq. (44) is equal to $n!$ as indicated.

Thus, after multiplying both sides of eq. (12) by $\epsilon_{i_1 i_2 i_3 \cdots i_n}$ and performing the n fold sum using eq. (44), it follows that,

$$\det A = \frac{1}{n!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\cdots\sum_{i_n=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n}\cdots\sum_{j_n=1}^{n}\epsilon_{i_1 i_2 i_3\cdots i_n}\epsilon_{j_1 j_2 j_3\cdots j_n} a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3}\cdots a_{i_n j_n}\,. \tag{45}$$

Likewise, after multiplying both sides of eq. (12) by $\epsilon_{j_1 j_2 j_3 \cdots j_n}$ and performing the n fold sum with an identity analogous to eq. (44), one also obtsains eq. (45). Hence, eqs. (8) and (12) are consistent as claimed above.

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