The Alternating Series Test

An alternating series is defined to be a real series of the form:

$$S = \sum_{n=0}^{\infty} (-1)^n a_n \,, \tag{1}$$

where all the $a_n > 0$. The alternating series test is a set of conditions that, if satisfied, imply that the series is convergent. Here is the general form of the theorem:

Theorem: If the real series $\sum_{n=0}^{\infty} b_n$ respects the following three properties:

1. The signs of the b_n alternate for all $n \geq n_0$, where n_0 is some fixed positive number;
2. The b_n are monotonically decreasing in magnitude for all $n \geq n_0$. That is, $|b_{n+1}| \leq |b_n|$ for all $n \geq n_0$;
3. $\lim_{n\to\infty} b_n = 0$;

then the series is convergent. If property 3 does not hold, then the series diverges. If property 3 is respected but property 1 and/or property 2 do not hold, then the alternating series test is inconclusive.

Note that property 1 corresponds to the statement that after the first n_0 terms, the remaining series is an alternating series. Hence, in this note I will assume that property 1 holds.*

Sketch of proof of the theorem: For simplicity, we assume that $n_0 = 0$. Let $S_N = \sum_{n=0}^{N} (-1)^n a_n$ be the sum of the first $N + 1$ terms of the alternating series. Then, it is easy to check that $S_1 > S_3 > S_5 > \cdots$ and $S_2 < S_4 < S_6 < \cdots$. Consequently, $S_{2n} < S < S_{2n+1}$ for all n. It then follows that for any positive error bound ϵ, there exists an N such that $|S_n - S_m| < \epsilon$ for all $m, n \geq N$. Hence, the alternating sum converges to $S = \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1}$.

An important corollary to the theorem, which follows from the above proof, allows us to estimate the error in the approximation of S by the sum of its first $n + 1$ terms.

*If properties 2 and 3 hold but property 1 does not hold, then the test is inconclusive. For example, the infinite harmonic series diverges whereas the sum of the inverse squares of the positive integers converges. Both these series respect properties 2 and 3 but not property 1.

Corollary: If the alternating sum converges to S, then the error in the approximation of S by the sum of its first $n+1$ terms is bounded by the absolute value of the next term in the series. That is,

$$|S - S_n| < a_{n+1}\,, \qquad \text{for} \quad n \geq n_0\,,$$

where $a_{n+1} > 0$ in the notation of eq. (1).

By the way, it is easy to exhibit a divergent series that satisfies properties 1 and 3 but does not satisfy property 2. Consider the alternating series:

$$S = 1 - \frac{1}{1\cdot 2} + \frac{1}{2} - \frac{1}{2\cdot 3} + \frac{1}{3} - \frac{1}{3\cdot 4} + \frac{1}{4} - \frac{1}{4\cdot 5} + \cdots\,.$$

This is an alternating series that satisfies condition 3 but violates condition 2. However, note that:

$$1 - \frac{1}{1\cdot 2} = \frac{1}{2}\,, \qquad \frac{1}{2} - \frac{1}{2\cdot 3} = \frac{1}{3}\,, \qquad \frac{1}{3} - \frac{1}{3\cdot 4} = \frac{1}{4}\,, \qquad \text{etc.}$$

If we denote the sum of the first N terms of the series S by S_N, it then follows that:

$$S_{2N} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N+1}\,, \qquad\qquad S_{2N+1} = S_{2N} + \frac{1}{N+1}\,.$$

Since the infinite harmonic series diverges, it follows that

$$\lim_{N\to\infty} S_{2N} = \lim_{N\to\infty} S_{2N+1} = \infty\,,$$

and we conclude that the series S diverges.

Moreover, it is also easy to provide an example of a convergent alternating series that respects property 3 but violates property 2. Here is one such example:

$$T = \sum_{k=0}^{\infty} a_k = 1 - \frac{1}{2} + \frac{1}{3^2} - \frac{1}{2^3} + \frac{1}{3^4} - \frac{1}{2^5} + \cdots + \frac{1}{3^{2k}} - \frac{1}{2^{2k+1}} + \cdots\,.$$

Indeed, this series violates property 2, since

$$\left|\frac{a_{2k+1}}{a_{2k}}\right| = \frac{3^{2k}}{2^{2k+1}} = \frac{1}{2}\left(\frac{9}{4}\right)^k > 1\,.$$

More generally, the condition $|a_{n+1}| < |a_n|$ fails to hold for all even n. Nevertheless, one can easily show that this series is convergent by the comparison test. Noting that

$$|a_n| \leq \frac{1}{2^n}\,,$$

it follows that T converges faster than the geometric series $\sum_{n=0}^{\infty} 1/2^n$. In fact it is easy to see that the series T is the difference of two separate geometric series. I challenge you to prove that the sum of this series is $T = 11/24$.