Pointwise and Uniform Convergence

1. Mathematical Definitions

A power series,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

is an example of a sum over a series of functions

$$f(x) = \sum_{n=0}^{\infty} g_n(x) , \tag{1}$$

where $g_n(x) = a_n x^n$. It is useful to consider the more general case. Let us consider a sum of the form given in eq. (1) and ask whether the sum is convergent. If we consider each x separately, then we can determine whether the sum converges at the point x. Suppose that the sum is determined to converge for all points $x \in A$, where A is some interval on the real axis. Typical intervals are: the *open* interval $a < x < b$, which we will denote by (a, b) and the *closed* interval $a \leq x \leq b$, which we will denote by $[a, b]$. Of course, we could also consider half-open, half-closed intervals, such as $a < x \leq b$, denoted by $(a, b]$ and $a \leq x < b$, denoted by $[a, b)$. In this notation, the parenthesis indicates that the endpoint is not included in the interval, whereas the square bracket indicates that the endpoint is included in the interval.

If $f(x)$ converges for all $x \in A$, we say that the sum given by eq. (1) is *pointwise convergent* over the interval $x \in A$. In this case, A is called the *interval of convergence*. A classic example is the infinite geometric series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n , \qquad |x| < 1 . \tag{2}$$

The above sum converges pointwise to the function $(1-x)^{-1}$ inside the open interval $-1 < x < 1$. Using the standard procedures, it is easy to see that the sum diverges for all $|x| \geq 1$.

Let us return to the general case of

$$f(x) = \sum_{n=0}^{\infty} g_n(x) , \qquad x \in A ,$$

where A is the interval of convergence. Suppose that the $g_n(x)$ are continuous functions. Does this imply that $f(x)$ is continuous? The great mathematician

Augustin Louis Cauchy got the answer wrong. In 1821, he claimed to prove that all infinite sums of continuous functions are continuous. It took over 30 years before the error was properly corrected.

It is simple to provide a counterexample to Cauchy's claim. Consider the series:

$$f(x) = \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+nx^2)[1+(n-1)x^2]} \,. \tag{3}$$

Although this series looks complicated, we can simplify it using partial fractions. The following is an algebraic identity:

$$g_n(x) \equiv \frac{x^2}{(1+nx^2)[1+(n-1)x^2]} = \frac{1}{1+(n-1)x^2} - \frac{1}{1+nx^2} \,. \tag{4}$$

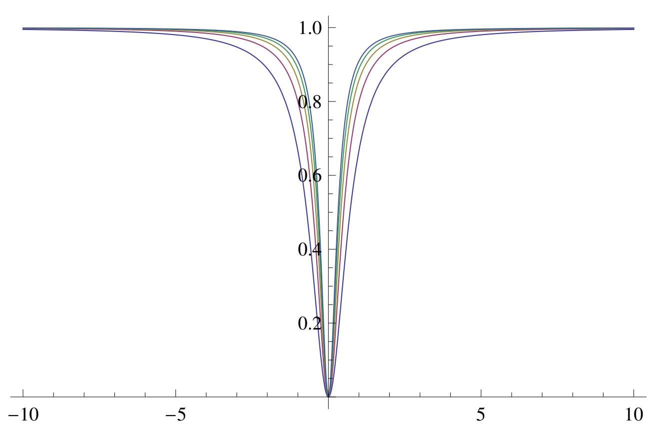
The sum over the first N terms is a telescoping series, which yields

$$\begin{aligned}\sum_{n=1}^{N} \frac{x^2}{(1+nx^2)[1+(n-1)x^2]} &= \left(1 - \frac{1}{1+x^2}\right) + \left(\frac{1}{1+x^2} - \frac{1}{1+2x^2}\right) + \cdots \\ &\qquad + \left(\frac{1}{1+(N-1)x^2} - \frac{1}{1+Nx^2}\right) \\ &= 1 - \frac{1}{1+Nx^2} = \frac{Nx^2}{1+Nx^2} \,. \end{aligned} \tag{5}$$

Thus,

$$f(x) = \lim_{N\to\infty} \frac{Nx^2}{1+Nx^2} = \begin{cases} 1\,, & \text{for } x \neq 0\,, \\ 0\,, & \text{for } x = 0\,. \end{cases}$$

That is, $f(x)$ is discontinuous at $x = 0$. Below, I have plotted $Nx^2/(1+Nx^2)$ for $N = 2$, 4, 6, 8 and 10. As N increases, the curves begin to approach the function $f(x)$, which is equal to 1 for all $x \neq 0$. Of course, $f(0) = 0$, so in the limit of $N \to \infty$, the functions exhibits a discontinuity.



In this case, the interval of convergence is the entire x-axis, i.e., $-\infty < x < \infty$. To formally prove that the sum is pointwise convergent, we define

$$f_N(x) \equiv \sum_{n=1}^{N} \frac{x^2}{(1+nx^2)[1+(n-1)x^2]} = \frac{Nx^2}{1+Nx^2}\,.$$

Then for $x \neq 0$,

$$|f(x) - f_N(x)| = \frac{1}{1+Nx^2} < \epsilon\,, \tag{6}$$

which implies that

$$N > \frac{1-\epsilon}{\epsilon x^2}\,. \tag{7}$$

That is, given an ϵ no matter how small, one can always find an N such that $|f(x) - f_N(x)| < \epsilon$. This implies that for $x \neq 0$, $\lim_{N\to\infty} f_N(x) = f(x)$. Note that as $x \to 0$, the required value of N grows arbitrarily large. This means that the *rate* of convergence is getting slower and slower the closer x is to the origin. This is the source of the ultimate discontinuity at $x = 0$.

These observations motivate the following definition.

Definition: Given $f_N(x) = \sum_{n=1}^{N} f_n(x)$ and $\lim_{N\to\infty} f_N(x) = f(x)$, where $x \in A$, the sum is *uniformly convergent* in the interval $x \in A$ if given a positive error bound ϵ, there always exists a response N such that for any $n \geq N$ we have $|f(x) - f_n(x)| < \epsilon$. The *same* N must work for all $x \in A$.

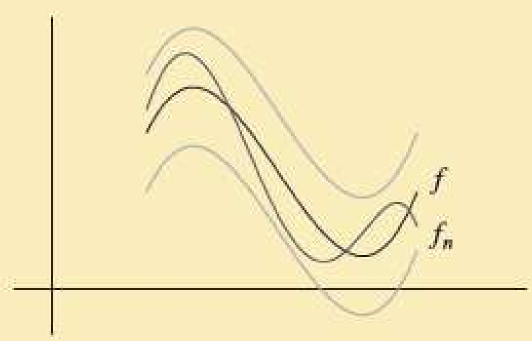


Figure 14.6. Uniform convergence on the whole interval.

The concept of uniform convergence is illustrated in the above figure, taken from ref. 2. We see that the approximation f_n to f lies completely within an ϵ-band over the entire interval.

By comparing the definition of uniform convergence with eqs. (6) and (7), it is clear that the sum given in eq. (3) is pointwise convergent but is *not* uniformly convergent over any interval that contains the point $x = 0$. Indeed, uniform convergence is a more stringent requirement than pointwise convergence. However, the advantage of uniform convergence is that the properties of the functions $g_n(x)$ (such as continuity) are preserved by the infinite sum.

2. Theorems about uniformly converging series

We quote a few key theorems without proofs (for details, see e.g., refs. 1 and 2 given at the end of these notes).

Theorem 1: If $f(x) = \displaystyle\sum_{n=1}^{\infty} g_n(x)$ converges uniformly over the interval $x \in A$, and if the $g_n(x)$ are continuous at every point $x \in A$, then $f(x)$ is also continuous at every point $x \in A$.

Since $f(x)$ defined in eq. (3) is discontinuous at $x = 0$, it follows that $f(x)$ is not uniformly convergent over any interval that includes the point $x = 0$. However, the converse to Theorem 1 is false. In particular, there are many examples of non-uniformly convergent sums that are continuous over the interval of convergence.

Theorem 2: If $f(x) = \displaystyle\sum_{n=1}^{\infty} g_n(x)$ converges uniformly over the open interval $x \in A$, and if the $g_n(x)$ are differentiable at every point in A, then $f(x)$ is also differentiable at every point $x \in A$, and $f'(x) = \displaystyle\sum_{n=1}^{\infty} g'_n(x)$ (where $g' \equiv dg/dx$).

That is, it is legal to interchange the order of summation and differentiation. While this interchange is clearly valid for finite sums, it requires proof if the sum is over an infinite number of terms. For non-uniformly convergent sums, interchanging the order of an infinite summation and differentiation may fail. As an example, consider

$$f(x) = \sin x \sum_{n=1}^{\infty} g_n(x) \,, \tag{8}$$

where $g_n(x)$ is defined in eq. (4). Using eq. (5), it follows that:

$$f_N(x) \equiv \sin x \sum_{n=1}^{N} g_n(x) = \frac{Nx^2 \sin x}{1 + Nx^2} \,.$$

Thus,

$$f(x) = \lim_{N\to\infty} f_N(x) = \sin x \,,$$

which holds for all real values of x including the point $x = 0$. Because of the x^2 in the numerator of eq. (4), it is clear that $g_n'(0) = 0$ for all n. Thus,

$$\sum_{n=1}^{\infty} g_n'(0) = 0\,.$$

However, since $f(x) = \sin x$, we have $f'(x) = \cos x$, and $f'(0) = 1$. Thus,

$$f'(0) \neq \sum_{n=1}^{\infty} g_n'(0)\,. \tag{9}$$

The reason for this odd behavior is that $f(x)$ is not uniformly convergent at $x = 0$.

Theorem 3: If $f(x) = \sum_{n=1}^{\infty} g_n(x)$ converges uniformly over the closed interval $x \in [a, b]$, and if the $g_n(x)$ are integrable over the interval $[a, b]$, then $f(x)$ is also integrable over the interval $[a, b]$, and

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b g_n(x)dx\,.$$

That is, it is mathematically permissible to interchange the order of summation and integration. While this interchange is clearly valid for finite sums, it requires proof if the sum is over an infinite number of terms. For non-uniformly convergent sums, interchanging the order of an infinite summation and integration may fail. As an example, consider

$$f(x) = \sum_{n=1}^{\infty} g_n(x)\,, \qquad -1 < x \leq 1\,, \tag{10}$$

where

$$g_n(x) \equiv x^{n-1}(1-x)\left[n^2 x - (n-1)^2\right]\,. \tag{11}$$

We recognize that the sum of the first N terms of $f(x)$ is a telescoping sum, which is easily computed:

$$f_N(x) = \sum_{n=1}^{N} g_n(x) = N^2 x^N (1-x)\,. \tag{12}$$

Thus, $f(x) = \lim_{N\to\infty} f_N(x) = 0$ for $|x| < 1$ and $f(1) = 0$ due to the factor of $1 - x$ in the $g_n(x)$. Hence,

$$\int_0^1 f(x)\,dx = 0\,. \tag{13}$$

However, notice that

$$\sum_{n=1}^{N} \int_0^1 g_n(x)\,dx = \int_0^1 dx \sum_{n=1}^{N} g_n(x) = \int_0^1 f_N(x)\,dx = N^2 \int_0^1 x^N(1-x)\,dx$$

$$= N^2 \left[\frac{1}{N+1} - \frac{1}{N+2}\right] = \frac{N^2}{(N+1)(N+2)}\,, \tag{14}$$

where we have used the fact that it is always permissible to interchange the order of integration and summation if the latter is a sum over a finite number of terms. Taking the limit of eq. (14) as $N \to \infty$ yields

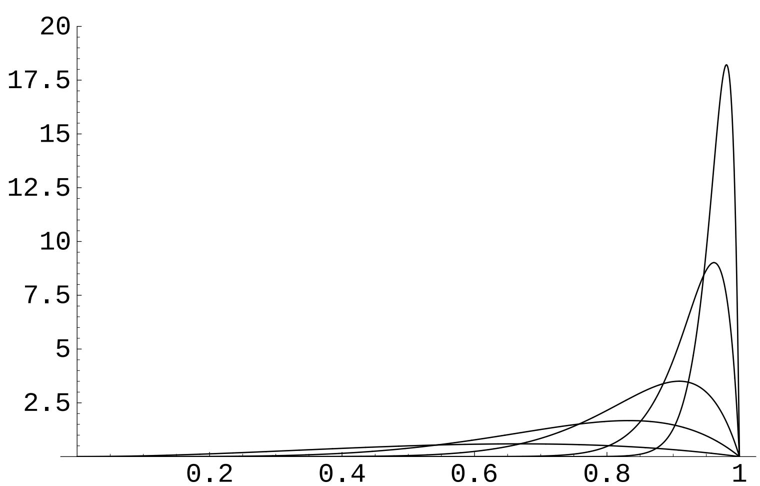
$$\sum_{n=1}^{\infty} \int_0^1 g_n(x)\,dx = 1\,. \tag{15}$$

Comparing eq. (13) with eq. (15), we conclude that

$$\int_0^1 dx \sum_{n=1}^{\infty} g_n(x) \;\neq\; \sum_{n=1}^{\infty} \int_0^1 g_n(x) dx\,. \tag{16}$$

Again, this behavior can be attributed to the fact that the sum is not uniformly convergent at $x = 1$.

To understand what is happening, consider a plot of $f_N(x)$ [eq. (12)] for various values of N. Below, I have plotted $f_N(x)$ for $N = 2$, 5, 10, 25 and 50. As N increases, the hump in the graph of $f_N(x)$ gets higher and narrower and is pushed further to the right.



As $N \to \infty$, $f_N(x) \to 0$ but in a very non-uniform way, since the area under the graph of $f_N(x)$ approaches 1 in the same limit!

3. A test for uniform convergence

The simplest test for a uniformly convergent series is the Weierstrass M-test:

> **The Weierstrass M-test:** If $f(x) = \sum_{n=1}^{\infty} g_n(x)$, and if $|g_n(x)| \leq M_n$ for every n greater than or equal to some fixed integer N, for all $x \in A$, and if $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly to $f(x)$ over the interval $x \in A$. Moreover, at every point $x \in A$, the convergence is absolute.

This is the analog of the comparison test for numerical series. The converse of the Weierstrass M-test is false. Namely, it is possible for a uniformly convergent series to fail the Weierstrass M-test. An example of this is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x^2+n}\,. \tag{17}$$

This sum is only conditionally convergent. Thus, it cannot pass the Weierstrass M-test, since any series that satisfies the Weierstrass M-test must be absolutely convergent. Nevertheless, one can prove that eq. (17) is also uniformly convergent.

4. The uniform convergence of power series

Finally, we can now understand why power series are so nice. As a consequence of the Weierstrass M-test, we have this important result.

Theorem 4: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with a finite radius of convergence $R > 0$, then for any $0 < r < R$, this power series converges uniformly and absolutely over the *closed* interval $[-r, r]$.* If the radius of convergence is infinite, then the power series converges uniformly over the closed interval $[-r, r]$ for any finite positive value of r.

Proof: For any $|x| \leq r$, we have $|a_n x^n| \leq |a_n| r^n$. But, $\sum_{n=0}^{\infty} |a_n| r^n$ converges absolutely (as a consequence of the ratio test). Hence the Weierstrass M-test applies.

As a result, $f(x)$ is continuous over any closed interval $[r, r]$ (for $r < R$), and one can differentiate and integrate power series by differentiating or integrating each term in the series. Returning to the example of the infinite geometric series, we conclude that eq. (2) is uniformly convergent over any closed interval $[-r, r]$, where $0 < r < 1$. However, it would be *incorrect* to claim that eq. (2) is uniformly convergent over the open interval $(-1, 1)$. The problem here is that the rate of convergence is infinitely slow as $x \to \pm 1$. This is not surprising, since eq. (2) diverges at $x = \pm 1$.

For the general power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R, the series converges for $|x| < R$ and diverges for $|x| > R$. However, the convergence properties at $x = \pm R$ must be checked separately. The following theorem is relevant.

*The notation used here for open and closed intervals is introduced below eq. (1).

Theorem 5: Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with a finite radius of convergence $R > 0$ and an interval of convergence A. Then,

1. If $A = [-R, R]$, then the series converges uniformly (but not necessarily absolutely) over the closed interval $[-R, R]$.

2. If $A = (-R, R]$, then the series converges uniformly (but not necessarily absolutely) over any closed interval $[a, R]$ for all $-R < a < R$.

3. If $A = [-R, R)$, then the series converges uniformly (but not necessarily absolutely) over any closed interval $[-R, b]$ for all $-R < b < R$.

4. If $A = (-R, R)$, then the series converges uniformly and absolutely over any closed interval $[a, b]$ for all $-R < a < b < R$.

The infinite geometric series [eq. (2)] is an example of case 4 of Theorem 5. An example of case 2 is

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}\,, \qquad -1 < x \leq 1\,.$$

This sum is conditionally convergent at $x = 1$ and divergent at $x = -1$. Thus, we conclude that the series converges uniformly on any closed interval $a \leq x \leq 1$ where $-1 < a < 1$. In particular, the sum is uniformly convergent at $x = 1$.

Our final example is Euler's dilogarithm, which is defined by its power series

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}\,, \qquad |x| \leq 1\,.$$

The radius of convergence of this power series is 1, and the series converges uniformly and absolutely at $x = \pm 1$. In fact, one can show that:

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}\,, \qquad \mathrm{Li}_2(-1) = -\frac{\pi^2}{12}\,.$$

This is an example of case 1 of Theorem 5.

References:

1. David Bressoud, *A Radical Approach to Real Analysis*, 2nd edition (Mathematical Association of America, Washington, DC, 2007).

2. Brian S. Thomson, Judith B. Bruckner and Andrew M. Bruckner, *Elementary Real Analysis* (Prentice-Hall, Inc., Upper Saddle River, NJ, 2001).