1. Consider the real valued function:

$$f(x)=\frac{1}{\sqrt{1+x^4}}-\cos(x^2)\,.$$

(a) Find the *behavior* of $f(x)$ as $x\to 0$.

Using eqs. (13.2) and (13.5) on p. 26 of Boas,

$$\cos(x^2)=1-\tfrac{1}{2}x^4+\tfrac{1}{24}x^8-\ldots\,,$$
$$(1+x^4)^{-1/2}=1-\tfrac{1}{2}x^4+\tfrac{3}{8}x^8-\ldots\,.$$

Subtracting, we obtain the leading behavior of $f(x)$ as $x\to 0$,

$$f(x)=\frac{1}{\sqrt{1+x^4}}-\cos(x^2)=\tfrac{1}{3}x^8+\mathcal{O}(x^{12})\,. \qquad (1)$$

(b) Determine the value of $f(x)$ at $x=0.01$ to four significant figures.

Using eq. (1),

$$f(0.01)=\tfrac{1}{3}\times 10^{-16}+\mathcal{O}(10^{-24})\,. \qquad (2)$$

Hence, if we want to evaluate $f(0.01)$ to four significant figures, it is sufficient keep only the first term on the right hand side of eq. (57) since the second term is a factor of 10^8 smaller than the first term. Hence, to four significant figures,

$$f(0.01)=3.333\times 10^{-17}\,.$$

2. Evaluate the quantities,

$$\text{(a)}\quad \cos\left[2i\ln\left(\frac{1-i}{1+i}\right)\right]\,, \qquad\qquad \text{(b)}\quad i^{2/3}\,.$$

If either of these quantities is multivalued, you should provide all possible values.

(a) Using the definition of the complex logarithm,

$$\begin{aligned}\ln\left(\frac{1-i}{1+i}\right)&=\mathrm{Ln}\left|\frac{1-i}{1+i}\right|+i\arg\left(\frac{1-i}{1+i}\right)\\ &=\mathrm{Ln}\left|\frac{1-i}{1+i}\right|+i\left(\mathrm{Arg}\left(\frac{1-i}{1+i}\right)+2\pi n\right)\,,\quad \text{where } n=0,\pm1,\pm2,\ldots \end{aligned}\qquad (3)$$

Next, we compute,

$$\left|\frac{1-i}{1+i}\right|^2 = \left(\frac{1-i}{1+i}\right)\left(\frac{1-i}{1+i}\right)^* = \left(\frac{1-i}{1+i}\right)\left(\frac{1+i}{1-i}\right) = 1\,,$$

which implies that

$$\operatorname{Ln}\left|\frac{1-i}{1+i}\right| = \operatorname{Ln} 1 = 0\,.$$

Finally, note that

$$\frac{1-i}{1+i} = \left(\frac{1-i}{1+i}\right)\left(\frac{1-i}{1-i}\right) = \frac{1-2i+i^2}{2} = -i\,.$$

Hence, it follows that

$$\operatorname{Arg} = \left(\frac{1-i}{1+i}\right) = \operatorname{Arg}(-i) = \operatorname{Arg}(e^{-i\pi/2}) = -\tfrac{1}{2}\pi\,,$$

in our convention for the principal value of the argument, where $-\pi < \operatorname{Arg} z \leq \pi$. Hence, eq. (3) yields,

$$\ln\left(\frac{1-i}{1+i}\right) = 2\pi\bigl(n - \tfrac{1}{4}\bigr)i\,.$$

Finally,

$$\cos\left[2i\ln\left(\frac{1-i}{1+i}\right)\right] = \cos\bigl[-2\pi\bigl(2n-\tfrac{1}{2}\bigr)\bigr] = \cos(\pi - 4\pi n) = \cos\pi = -1\,.$$

(b) The definition of the complex power function yields,

$$i^{2/3} = \exp\bigl[\tfrac{2}{3}\ln i\bigr]\,. \tag{4}$$

Then, using the definition of the complex logarithm,

$$\ln i = \operatorname{Ln}|i| + i(\operatorname{Arg}(i) + 2\pi n) = i\pi\bigl(2n + \tfrac{1}{2}\bigr)\,, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

after using $\ln|i| = \operatorname{Ln} 1 = 0$ and $\operatorname{Arg}(i) = \operatorname{Arg}(e^{i\pi/2}) = \frac{1}{2}\pi$. Hence, eq. (4) yields,

$$i^{2/3} = \exp\bigl[\tfrac{2}{3}i\pi(2n+\tfrac{1}{2})\bigr]\,, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots \tag{5}$$

Note that for $n = \ldots, -1, 0, 1, 2, \ldots$,

$$\tfrac{2}{3}(2n+\tfrac{1}{2}) = \ldots, -1, \tfrac{1}{3}, \tfrac{5}{3}, 3, \ldots$$

Since 3 and -1 differ by 4, it follows that as n ranges over all the integers, there are only three distinct possible values for $i^{2/3}$,

$$i^{2/3} = e^{-i\pi}, e^{i\pi/3}, e^{5i\pi/3}\,.$$

Finally, we can evaluate, $e^{-i\pi} = -1$,

$$e^{i\pi/3} = \cos(\pi/3) + i\sin(i\pi/3) = \frac{1+i\sqrt{3}}{2}\,,$$
$$e^{5i\pi/3} = e^{-i\pi/3} = \cos(\pi/3) - i\sin(i\pi/3) = \frac{1-i\sqrt{3}}{2}\,.$$

Hence, we conclude that the possible values of $i^{2/3}$ are:

$$i^{2/3} = -1\,,\ \frac{1+i\sqrt{3}}{2}\,,\ \frac{1-i\sqrt{3}}{2}\,.$$

REMARK: Another approach to this problem is to write:

$$i = \exp\left[\tfrac{1}{2}i\pi + 2\pi i n\right]\,,$$

where n is any integer. Then,

$$i^{2/3} = \exp\left[\tfrac{2}{3}\left(\tfrac{1}{2}i\pi + 2\pi i n\right)\right] = \exp\left[\tfrac{2}{3}i\pi(2n+\tfrac{1}{2})\right]\,,$$

which yields eq. (5). Still another approach is to write

$$i^{2/3} = (i^2)^{1/3} = (-1)^{1/3}\,,$$

which implies that the three possible values of $i^{2/3}$ correspond to the three cube roots of -1. However, one must be careful, since the relation $(z^a)^b = z^{ab}$ is not always true, as discussed in the class handout entitled, *The complex logarithm, exponential and power functions*. In the present application, this relation is satisfied. Indeed, one can see geometrically the possible values of $i^{2/3}$ simply by considering the location of these values on the unit circle in the complex plane.

3. Evaluate the conditionally convergent sum,

$$S \equiv \sum_{n=1}^{\infty} \frac{i^n}{n}\,.$$

Using the series given in eq. (13.4) on p. 26 in Boas, after substituting $-x$ of x, it follows that

$$\mathrm{Ln}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}\,,$$

When z us a complex variable, we interpret the logarithm above as its principal value. Plugging in $z = i$ then yields,

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = -\,\mathrm{Ln}(1-i) = -\,\mathrm{Ln}\,|1-i| - i\,\mathrm{Arg}(1-i)\,.$$

First, we note that $|1-i| = \big[(1-i)(1+i)\big]^{1/2} = \sqrt{2}$. Next, if one plots $1-i$ in the complex plane, one can quickly determine that $\operatorname{Arg}(1-i) = -\frac{1}{4}\pi$. More explicitly, note that

$$e^{-i\pi/4} = \cos(\pi/4) - i\sin(\pi/4) = \tfrac{1}{2}\sqrt{2}(1-i)\,.$$

It follows that

$$1-i = \sqrt{2}e^{-i\pi/4}\,,$$

which implies that $|1-i| = \sqrt{2}$ and $\operatorname{Arg}(1-i) = -\frac{1}{4}\pi$, as asserted above.

Therefore, using $\operatorname{Ln}\sqrt{2} = \frac{1}{2}\operatorname{Ln} 2$, it follows that

$$\boxed{\sum_{n=1}^{\infty}\frac{i^n}{n} = -\tfrac{1}{2}\operatorname{Ln} 2 + \tfrac{1}{4}i\pi\,.}$$

4. Compute the inverse A^{-1} of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

First, we compute the determinant of A using the cofactor expansion,

$$\det\ A = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1(1-0) + 1(0) = 1\,.$$

Next, we compute the adjugate of A, which is given by the transpose of the matrix of cofactors,

$$\operatorname{adj}\ A = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\[2ex] -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\[2ex] \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Using the formula for the matrix inverse,

$$A^{-1} = \frac{1}{\det\ A}\operatorname{adj}\ A\,,$$

it follows that

$$A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} .$$

5. In special relativity, the space-time coordinates of two inertial frames moving at relative constant velocity v are related by

$$x' = \gamma(x - vt) \,, \tag{6}$$
$$t' = \gamma(t - vx/c^2) \,, \tag{7}$$

where

$$\gamma \equiv (1 - v^2/c^2)^{-1/2} \,, \tag{8}$$

and c is the velocity of light.

(a) Rewrite this system of equations in matrix form.

Eqs. (6) and (7) are equivalent to the following matrix equation:

$$\begin{pmatrix} \gamma & -\gamma v/c^2 \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix} .$$

(b) Using Cramer's rule, solve for x and t in terms of x' and t'.

Applying Cramer's rule,

$$t = \frac{\begin{vmatrix} t' & -\gamma v/c^2 \\ x' & \gamma \end{vmatrix}}{\begin{vmatrix} \gamma & -\gamma v/c^2 \\ -\gamma v & \gamma \end{vmatrix}} \,, \qquad\qquad x = \frac{\begin{vmatrix} \gamma & t' \\ -\gamma v & x' \end{vmatrix}}{\begin{vmatrix} \gamma & -\gamma v/c^2 \\ -\gamma v & \gamma \end{vmatrix}} \,.$$

The determinants are easily evaluated, using

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \,.$$

In particular,

$$\begin{vmatrix} \gamma & -\gamma v/c^2 \\ -\gamma v & \gamma \end{vmatrix} = \gamma^2 \left(1 - \frac{v^2}{c^2}\right) = 1 \,,$$

after using the definition of γ given in eq. (8).

Hence, it follow that

$$t = \begin{vmatrix} t' & -\gamma v/c^2 \\ x' & \gamma \end{vmatrix} = \gamma\left(t' + \frac{vx'}{c^2}\right),$$

$$x = \begin{vmatrix} \gamma & t' \\ -\gamma v & x' \end{vmatrix} = \gamma(x' + vt')\,.$$

That is,

$$\begin{aligned} x &= \gamma(x' + vt')\,, \\ t &= \gamma(t' + vx'/c^2)\,. \end{aligned}$$

In particular, the above solution corresponds to interchanging $x \leftrightarrow x'$, $t \leftrightarrow t'$ and $v \leftrightarrow -v$ in eqs. (6) and (7). This is not surprising, since if the reference frame R' is moving at velocity v with respect to R, then the reference frame R is moving at velocity $-v$ with respect to R'.

6. If one of the eigenvalues of the matrix A is $\lambda = 0$, prove that A^{-1} does not exist.

If one of the eigenvalues of the matrix A is $\lambda = 0$, then there exist a nonzero vector $\vec{\boldsymbol{v}} \neq 0$ such that

$$A\vec{\boldsymbol{v}} = 0\,. \tag{9}$$

I will prove that A^{-1} does not exist by contradiction. Suppose that A^{-1} exists. Multiply both sides of eq. (9) by A^{-1} to conclude that $\vec{\boldsymbol{v}} = 0$ is the unique solution. This statement contradicts the assumption that there exist a nonzero vector $\vec{\boldsymbol{v}} \neq 0$ such that $A\vec{\boldsymbol{v}} = 0$. Hence, the only possible conclusion is that A^{-1} does not exist.

Here is a second proof. We showed in class that det A is equal to the product of the eigenvalues of A. Hence, if one of the eigenvalues of the matrix A is $\lambda = 0$, then det $A = 0$. Then, it immediately follows that A^{-1} does not exist.

7. If $A^{\mathsf{T}} = -A$, then we say that A is a skew-symmetric (or antisymmetric) matrix. Prove that if A is antisymmetric and B is symmetric, then $\mathrm{Tr}(AB) = 0$.

If A is antisymmetric and B is symmetric, then

$$A^{\mathsf{T}} = -A \qquad \text{and} \qquad B^{\mathsf{T}} = B\,. \tag{10}$$

Recall that for any matrix M, Tr M^{T} = Tr M. Thus,

$$\mathrm{Tr}[(AB)^{\mathsf{T}}] = \mathrm{Tr}(B^{\mathsf{T}}A^{\mathsf{T}}) = -\,\mathrm{Tr}(BA) = -\,\mathrm{Tr}(AB)\,,$$

where at step 2 we used eq. (10), and at step 3 we used the fact that the trace of a product of matrices is unchanged when the matrices are cyclically permuted. Since the only number that is equal to its negative is zero, it follows that $\mathrm{Tr}(AB) = 0$.

8. Let A be a complex $n \times n$ matrix. Prove that the eigenvalues of $AA^\dagger$ are real and non-negative.

Consider two complex vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$, where $\vec{\boldsymbol{w}} \equiv A^\dagger \vec{\boldsymbol{v}}$. According to eq. (14.4b) on p. 181 of Boas,

$$\langle \vec{\boldsymbol{w}}\,,\,\vec{\boldsymbol{w}}\rangle \geq 0\,, \quad \text{and } \ \langle \vec{\boldsymbol{w}}\,,\,\vec{\boldsymbol{w}}\rangle = 0 \ \text{ if and only if } \ \vec{\boldsymbol{w}} = 0\,. \tag{11}$$

This means that

$$\langle \vec{\boldsymbol{w}}\,,\,\vec{\boldsymbol{w}}\rangle = \langle A^\dagger\vec{\boldsymbol{v}}\,,\,A^\dagger\vec{\boldsymbol{v}}\rangle \geq 0\,. \tag{12}$$

Using the definition of the inner product on a complex vector space [see eq. (10.6) on p. 146 of Boas] and the definition of the hermitian adjoint, $A^\dagger = A^{*\mathsf{T}}$, it follows that:

$$\begin{aligned}
\langle \vec{\boldsymbol{w}}\,,\,\vec{\boldsymbol{w}}\rangle = \langle A^\dagger\vec{\boldsymbol{v}}\,,\,A^\dagger\vec{\boldsymbol{v}}\rangle &= \sum_{i=1}^{n}(A^\dagger\vec{\boldsymbol{v}})_i^*(A^\dagger\vec{\boldsymbol{v}})_i = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(A^\mathsf{T})_{ij}v_j^* A^\dagger_{ik}v_k \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} v_j^* A_{ji}A^\dagger_{ik}v_k = \sum_{j=1}^{n} v_j^*(AA^\dagger\vec{\boldsymbol{v}})_j\,, \\
&= \langle \vec{\boldsymbol{v}}\,,\,AA^\dagger\vec{\boldsymbol{v}}\rangle\,, \qquad (13)
\end{aligned}$$

which must hold for any complex vector $\vec{\boldsymbol{v}}$. In particular, one can choose $\vec{\boldsymbol{v}}$ to be an eigenvector of $AA^\dagger$,

$$AA^\dagger\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}\,.$$

It then follows from eq. (13) that:

$$\langle \vec{\boldsymbol{w}}\,,\,\vec{\boldsymbol{w}}\rangle = \langle A^\dagger\vec{\boldsymbol{v}}\,,\,A^\dagger\vec{\boldsymbol{v}}\rangle = \langle \vec{\boldsymbol{v}}\,,\,AA^\dagger\vec{\boldsymbol{v}}\rangle = \lambda\langle \vec{\boldsymbol{v}}\,,\,\vec{\boldsymbol{v}}\rangle\,, \tag{14}$$

Since $\vec{\boldsymbol{v}}$ is en eigenvector, we know that $\vec{\boldsymbol{v}} \neq 0$, in which case $\langle \vec{\boldsymbol{v}}\,,\,\vec{\boldsymbol{v}}\rangle > 0$. Hence, we can divide eq. (14) by $\langle \vec{\boldsymbol{v}}\,,\,\vec{\boldsymbol{v}}\rangle$ to obtain

$$\lambda = \frac{\langle \vec{\boldsymbol{w}}\,,\,\vec{\boldsymbol{w}}\rangle}{\langle \vec{\boldsymbol{v}}\,,\,\vec{\boldsymbol{v}}\rangle}\,.$$

Using eq. (11), it follows that $\lambda \geq 0$.

9. Suppose that the $n \times n$ matrix M is diagonalizable. Then, one can find an invertible matrix S such that $S^{-1}MS = D$ where D is diagonal. Show that $M^n = SD^nS^{-1}$, where n is a positive integer. This provides a simple way to compute M^n since raising a diagonal matrix to a power is easy. Using these considerations, compute M^{10} where

$$M = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\,. \tag{15}$$

If $S^{-1}MS = D$, then

$$D^n = (S^{-1}MS)^n = \underbrace{S^{-1}MSS^{-1}MSS^{-1}AS\cdots S^{-1}MS}_{n \text{ terms}} = S^{-1}M^nS\,.$$

It follows that

$$\boxed{M^n = SD^nS^{-1}.}$$

We apply this result to M given by eq. (15). First, we compute the eigenvalues by solving the characteristic equation,

$$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0\,.$$

Hence, $3 - \lambda = \pm 1$, which yields two roots: $\lambda = 4$ and $\lambda = 2$. Next, we work out the eigenvectors. Since M is a real symmetric matrix, we know that the eigenvectors will be orthogonal. By normalizing them to unity, the eigenvectors will then be orthonormal. First, we examine:

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}\,,$$

which yields one independent relation, $x = -y$. Hence, the normalized eigenvector is

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=4} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\,.$$

Since M is a real symmetric matrix, eigenvectors corresponding to non-degenerate eigenvalues are orthogonal. Hence, the second normalized eigenvector is orthogonal to the first one, and thus is given by

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$$

One can check this by verifying that,

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$$

The columns of the diagonalizing matrix S are given by the two eigenvectors. Thus,

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\,.$$

We expect that $S^{-1}AS = D$ is a diagonal matrix. Let's check this. First, we note that the columns of C are orthonormal. This implies that S is an orthogonal matrix, as you learned while working out problem 1 on homework set #8. In particular, $S^{-1} = S^{\mathsf{T}}$. Hence,

$$\begin{aligned} S^{-1}MS = S^{\mathsf{T}}MS &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\,. \end{aligned}$$

Note that the order of the eigenvectors appearing as columns in S determines the order of the eigenvalues appearing along the diagonal of

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

Finally, we can compute:

$$M^{10} = SD^{10}S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4^{10} & -4^{10} \\ 2^{10} & 2^{10} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4^{10}+2^{10} & -4^{10}+2^{10} \\ -4^{10}+2^{10} & 4^{10}+2^{10} \end{pmatrix} = \begin{pmatrix} 2^9(2^{10}+1) & -2^9(2^{10}-1) \\ -2^9(2^{10}-1) & 2^9(2^{10}+1) \end{pmatrix}.$$

10. Determine whether the following matrices are diagonalizable. If diagonalizable, indicate whether it is possible to diagonalize the matrix with a unitary similarity transformation.

$$\text{(a) } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \text{(b) } A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}.$$

(a) For the matrix A in part (a) above, we first compute the eigenvalues by solving the characteristic equation,

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0\,.$$

In order for the matrix A to be diagonalizable, there must be two linearly independent eigenvectors corresponding to the degenerate eigenvalue $\lambda = 1$. Thus, we solve for the eigenvectors,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The solution to these equations are: $y = z = 0$, with x arbitrary. Thus, there is only one linearly independent eigenvector corresponding to the eigenvalue $\lambda = 1$. In total, the eigenvectors of A span a two-dimensional subspace of the three-dimensional vector space. Hence, A is *defective*, and is not diagonalizable.

(b) For the matrix $A = [a_{ij}]$ in part (b) above, we observe that it is a real symmetric matrix, i.e. $a_{ij} = a_{ji}$. Thus, it is diagonalizable by an real orthogonal similarity transformation. Since any real orthogonal matrix is unitary, it follows that it is possible to diagonalize A of part (b) with a unitary similarity transformation.

11. A linear transformation T that is represented by a matrix M with respect to the standard basis $\mathcal{B}$ is also represented by the matrix $P^{-1}MP$ with respect to a different basis $\mathcal{B}'$, where P is the invertible matrix that transforms the basis $\mathcal{B}$ into the basis $\mathcal{B}'$.

(a) Show that the characteristic equation for determining the eigenvalues of T does not depend on the choice of basis.

Note that $P^{-1}MP - \lambda\mathbf{I} = P^{-1}(M - \lambda\mathbf{I})P$. The characteristic equation in the basis $\mathcal{B}$ is given by

$$\det(M - \lambda\mathbf{I}) = 0\,.$$

The characteristic equation in the basis $\mathcal{B}'$ is given by

$$\begin{aligned}\det(P^{-1}MP - \lambda\mathbf{I}) &= \det[P^{-1}(M - \lambda\mathbf{I})P] = \det\,P^{-1}\,\det(M - \lambda\mathbf{I})\,\det\,P \\ &= \det\,P^{-1}\,\det\,P\,\det(M - \lambda\mathbf{I}) = \det\,\mathbf{I}\,\,\det(M - \lambda\mathbf{I}) \\ &= \det(M - \lambda\mathbf{I}) = 0\,,\end{aligned}$$

where we have used the fact that $\det(ABC) = \det\,A\,\det\,B\,\det\,C$ and $\det\,\mathbf{I} = 1$. That is, the characteristic equations of $P^{-1}MP$ and M are identical. Hence, the characteristic equation of the matrix representation of T does not depend on the choice of basis.

(b) Show that the eigenvalues of T do not depend on the choice of basis.

The eigenvalues are the roots of the characteristic equation. Since the characteristic equation is the same in all bases, it follows that the eigenvalues are also independent of the basis choice.

12. Suppose that U is a $n\times n$ matrix whose columns comprise an orthonormal set of column vectors. Prove that U is a unitary matrix. Moreover, show that the rows of U must also comprise an orthonormal set of row vectors. (*HINT:* In this problem, orthonormality must be defined with respect to a *complex* vector space.)

Let us denote the n complex orthonormal column vectors of U by:

$$\hat{\boldsymbol{x}}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}\,, \quad \hat{\boldsymbol{x}}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}\,, \ldots\,, \quad \hat{\boldsymbol{x}}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}\,,$$

where x_{ij} indicates the ith component of the jth complex vector. Since the vectors $\hat{\boldsymbol{x}}_j$ all have length equal to one, it follows that:

$$\sum_{i=1}^{n} |x_{ij}|^2 = |x_{1j}|^2 + |x_{2j}|^2 + \cdots + |x_{nj}|^2 = 1\,, \qquad \text{for } j = 1, 2, 3, \ldots n\,. \tag{16}$$

Note that orthonormality in this problem is defined with respect to a complex vector space, so we must employ the complex magnitude in eq. (16). Since the complex vectors $\hat{\boldsymbol{x}}_j$ are orthonormal, it also follows that for $j \neq k$,

$$\langle \hat{\boldsymbol{x}}_{\boldsymbol{j}} \,,\, \hat{\boldsymbol{x}}_{\boldsymbol{k}} \rangle = 0 \quad \Longrightarrow \quad \sum_{i=1}^{n} x_{ij}^* x_{ik} = 0 \,, \quad \text{for} \quad j \neq k \,. \tag{17}$$

The conditions given by eqs. (16) and (17) can be combined to write:

$$\sum_{i=1}^{n} x_{ij}^* x_{ik} = \delta_{jk} \,, \tag{18}$$

where the Kronecker delta is defined by

$$\delta_{jk} = \begin{cases} 1 & \text{for} \quad j = k \,, \\ 0 & \text{for} \quad j \neq k \,. \end{cases}$$

We now define the matrix U as

$$U = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix} .$$

Then, using index notation, the ij element of U is x_{ij} and the ij element of $U^\dagger$ is x_{ji}^*. Hence, the jk element of $U^\dagger U$ is given by

$$(U^\dagger U)_{jk} = \sum_{k=1}^{n} (U^\dagger)_{ji} (U)_{ik} = \sum_{k=1}^{n} x_{ij}^* x_{ik} = \delta_{jk} \,,$$

where we have used eq. (18) to perform the final step. We recognize δ_{jk} as the matrix elements of the $n \times n$ identity matrix, $\mathbf{I}$. Thus we conclude that $U^\dagger U = \mathbf{I}$, or equivalently, $U^\dagger = U^{-1}$. That is, U is a unitary matrix. Hence, we have proven that if U is a matrix whose columns are the components of n orthonormal complex vectors, then U is a unitary matrix.

Note that one can use a similar method to prove that if U is a matrix whose rows are the components of n complex orthonormal vectors, then U is a unitary matrix. To prove this statement, it suffices to prove that $UU^\dagger = \mathbf{I}$. Since a unitary matrix satisfies $U^\dagger = U^{-1}$, it follows that both $UU^\dagger = \mathbf{I}$ and $U^\dagger U = \mathbf{I}$ are satisfied.

13. A linear transformation A is represented by the matrix:

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix},$$

with respect to the standard basis $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$. Consider a new basis, $\mathcal{B}' = \{(2,-2,1), (1,1,0), (-1,1,4)\}$, where the components of the basis vectors of $\mathcal{B}'$ are given with respect to the standard basis $\mathcal{B}$.

(a) What are the components of the basis vectors of $\mathcal{B}$ when expressed relative to the basis $\mathcal{B}'$?

Refer to the class handout entitled, *Vector coordinates, matrix elements and changes of basis.* If we denote $\mathcal{B} = \{\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2, \ldots, \vec{\boldsymbol{b}}_n\}$ and $\mathcal{B}' = \{\vec{\boldsymbol{b}}'_1, \vec{\boldsymbol{b}}'_2, \ldots, \vec{\boldsymbol{b}}'_n\}$, then

$$\vec{\boldsymbol{b}}'_j = \sum_{i=1}^{3} P_{ij} \vec{\boldsymbol{b}}_i \,.$$

We therefore identify the matrix P as:

$$P = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

In particular, the columns of P are the components of the basis vectors of $\mathcal{B}'$ with respect to the standard basis $\mathcal{B}$.

To solve this problem, we need to compute P^{-1}. First we evaluate the determinant. Using the third row to form the expansion in terms of cofactor,

$$\begin{vmatrix} 2 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 4 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = 2 + 16 = 18\,.$$

We now evaluate P^{-1}, using the formula

$$P^{-1} = \frac{\operatorname{adj} P}{\det\, P}\,,$$

where $\operatorname{adj} P$ is the transpose of the matrix of cofactors of P,

$$\operatorname{adj} P = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\[2ex] -\begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} \\[2ex] \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & -4 & 2 \\ 9 & 9 & 0 \\ -1 & 1 & 4 \end{pmatrix}.$$

Hence

$$P^{-1} = \frac{1}{18}\begin{pmatrix} 4 & -4 & 2 \\ 9 & 9 & 0 \\ -1 & 1 & 4 \end{pmatrix}.$$

The components of the basis vectors of $\mathcal{B}$ when expressed relative to the basis $\mathcal{B}'$ are given by

$$\vec{\boldsymbol{b}}_k = \sum_{j=1}^{3} (P^{-1})_{jk} \vec{\boldsymbol{b}}'_j \,.$$

Hence, it follows that with respect to the basis $\mathcal{B}'$,

$$\mathcal{B} = \left\{ \left(\tfrac{2}{9}, \tfrac{1}{2}, -\tfrac{1}{18}\right), \left(-\tfrac{2}{9}, \tfrac{1}{2}, \tfrac{1}{18}\right), \left(\tfrac{1}{9}, 0, \tfrac{2}{9}\right) \right\} .$$

In particular, the columns of P^{-1} are the components of the basis vectors of $\mathcal{B}$ with respect to the standard basis $\mathcal{B}'$.

(b) Determine the matrix representation of A relative to the basis $\mathcal{B}'$.

The matrix representation of A relative to the basis $\mathcal{B}'$ is related by a similarity transformation to the matrix representation of A relative to the basis $\mathcal{B}$,

$$[A]_{\mathcal{B}'} = P^{-1}[A]_{\mathcal{B}} P \,.$$

Using the results of part (a),

$$\begin{aligned}
[A]_{\mathcal{B}'} &= \frac{1}{18}\begin{pmatrix} 4 & -4 & 2 \\ 9 & 9 & 0 \\ -1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 4 \end{pmatrix} \\
&= \frac{1}{18}\begin{pmatrix} 4 & -4 & 2 \\ 9 & 9 & 0 \\ -1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 12 & -3 & 3 \\ -12 & -3 & -3 \\ 6 & 0 & -12 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.
\end{aligned}$$

14. Eq. (11.27) on p. 154 of Boas is imprecisely worded. The correct statement is that a matrix has real eigenvalues and can be diagonalized by an orthogonal similarity transformation if and only if it is a *real* symmetric matrix. (Boas omitted the qualification that the symmetric matrix must be real.) To see that it is important to be precise, consider the following *complex* symmetric matrix,

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} .$$

(a) Determine the eigenvalues and eigenvectors of A.

The characteristic equation is:

$$\det(A - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & i \\ i & -1-\lambda \end{vmatrix} = 0 .$$

Evaluating the determinant yields:

$$-(1\lambda)(1+\lambda) + 1 = 0 \quad \Longrightarrow \quad \lambda^2 = 0 .$$

Thus, the eigenvalue $\lambda = 0$ is degenerate with multiplicity equal to 2. The corresponding eigenvectors are obtained by solving:

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

which is equivalent to:

$$\begin{aligned} x + iy &= 0 , \\ ix - y &= 0 . \end{aligned}$$

Since the second equation above is obtained by multiplying the first equation above by i, it follows that the most general solution to the eigenvalue equation, $A\vec{\boldsymbol{v}} = \mathbf{0}$ is:

$$\vec{\boldsymbol{v}} = x \begin{pmatrix} 1 \\ i \end{pmatrix} , \tag{19}$$

where $x \neq 0$.

(b) How many linearly independent eigenvectors of A exist?

The possible eigenvectors of A were obtained in eq. (19), which implies that there is only one linearly independent eigenvector of A.

(c) Is A diagonalizable?

Since the maximal number of linearly independent eigenvectors of A (which is 1) is strictly less than the dimension of the square matrix A (which is 2), it follows that A is *defective*, or equivalently, A is *not* diagonalizable.

15. Consider the matrix

$$M = \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}\,, \tag{20}$$

where a and b are arbitrary complex numbers.

(a) Compute the eigenvalues of M.

First, we compute the eigenvalues by solving the characteristic equation,

$$\begin{vmatrix} -\lambda & b \\ 0 & a-\lambda \end{vmatrix} = -\lambda(a-\lambda) = 0\,,$$

which yields two roots: $\lambda = 0$ and $\lambda = a$.

(b) Find a matrix C such that $C^{-1}MC$ is diagonal.

Next, we compute the eigenvectors of M. First, we examine:

$$\begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\,,$$

which yields

$$by = 0\,, \qquad ay = 0\,,$$

and implies that $y = 0$. Hence the most general eigenvector corresponding to the $\lambda = 0$ eigenvalue is proportional to

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,.$$

Next, we examine

$$\begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} x \\ y \end{pmatrix}\,,$$

which again yields one independent relation, $by = ax$. Hence the most general eigenvector corresponding to the $\lambda = 0$ eigenvalue is proportional to

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=a} = \begin{pmatrix} b \\ a \end{pmatrix}\,.$$

Note that the two eigenvectors obtained are *not* orthogonal. This is not a surprise, since the eigenvectors of an arbitrary matrix A are orthogonal if and only if A is normal (recall that A is normal if $AA^\dagger = A^\dagger A$). One can quickly verify that the matrix M is *not* normal.

The columns of the diagonalizing matrix C are the eigenvectors of M. Hence,

$$C = \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}\,. \tag{21}$$

As a check, we now show that $C^{-1}MC$ is diagonal. First, we recall that for any invertible 2×2 matrix, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the inverse is given by

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where $\det\ A = ad - bc \neq 0$. Applying this result to eq. (21) yields,

$$C^{-1} = \frac{1}{a}\begin{pmatrix} a & -b \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$C^{-1}MC = \frac{1}{a}\begin{pmatrix} a & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & ab \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

(c) Compute e^M.

Denote $D = C^{-1}MC$ where D is the diagonal matrix obtained in part (b),

$$D = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

Then, it follows that $M = CDC^{-1}$. Using the power series definition of the exponential

$$e^M = e^{CDC^{-1}} = \sum_{n=0}^{\infty} \frac{(CDC^{-1})^n}{n!} = C\left(\sum_{n=0}^{\infty}\frac{D^n}{n!}\right)C^{-1} = Ce^D C^{-1}, \tag{22}$$

where we have used the fact that

$$(CDC^{-1})^n = \underbrace{CDC^{-1}CDC^{-1}CDC^{-1}\cdots CDC^{-1}}_{n\ \text{terms}} = CD^nC^{-1},$$

after noting that $C^{-1}C = \mathbf{I}$. Moreover, for a diagonal matrix,

$$\exp\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

It follows that

$$\exp\begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix} = C\left[\exp\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right]C^{-1} = \frac{1}{a}\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^a \end{pmatrix}\begin{pmatrix} a & -b \\ 0 & 1 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}\begin{pmatrix} a & -b \\ 0 & e^a \end{pmatrix}.$$

Performing the final matrix multiplication yields

$$\boxed{e^M = \begin{pmatrix} 1 & \dfrac{b}{a}(e^a - 1) \\ 0 & e^a \end{pmatrix}} \tag{23}$$

(d) Verify that $\det(e^M) = e^{\mathrm{Tr}\, M}$.

Using eqs. (20) and (23) for M and e^M, respectively, and noting that $\mathrm{Tr}\, M = a$ and $\det e^M = e^a$, it follows that

$$\det(e^M) = e^{\mathrm{Tr}\, M} = e^a\,.$$

(e) In light of the hint given in part (c) and result of part (d), can you see how to prove that $\det e^A = e^{\mathrm{Tr} A}$ is true for any diagonalizable matrix A?

If A is diagonalizable, then there exists an invertible matrix P such that

$$P^{-1}AP = D\,, \tag{24}$$

where D is a diagonal matrix whose diagonal elements are the eigenvalues of A. We shall indicate this diagonal matrix using the notation,

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\,.$$

Using eq. (24),

$$A = PDP^{-1}\,. \tag{25}$$

Then, following the analysis of part (c), it follows that

$$e^A = e^{PDP^{-1}} = Pe^D P^{-1} = P\,\mathrm{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})P^{-1}\,.$$

Using the properties of the determinant,

$$\det e^A = \det(Pe^DP^{-1}) = \det P \det e^D \det P^{-1} = \det e^D\,, \tag{26}$$

after using the property of determinants,[1] $\det P^{-1} = 1/\det P$. Hence eq. (26) yields,

$$\det e^A = \det e^D = \det\bigl[\mathrm{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})\bigr] = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} = e^{\lambda_1+\lambda_2+\ldots+\lambda_n}\,. \tag{27}$$

Next, we take the trace on both sides of eq. (25),

$$\mathrm{Tr}\, A = \mathrm{Tr}(PDP^{-1}) = \mathrm{Tr}\, D = \mathrm{Tr}\bigl[\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\bigr] = \lambda_1 + \lambda_2 + \ldots + \lambda_n\,, \tag{28}$$

after noting that $\mathrm{Tr}(PDP^{-1}) = \mathrm{Tr}(P^{-1}PD) = \mathrm{Tr}(\mathbf{I}D) = \mathrm{Tr}\, D$, since the trace of a product of matrices is unchanged if the matrices inside the trace are cyclically permuted. Exponentiating eq. (28) yields,

$$e^{\mathrm{Tr}\, A} = e^{\lambda_1+\lambda_2+\ldots+\lambda_n}\,. \tag{29}$$

Comparing eqs. (27) and (29) yields,

$$\det e^A = e^{\mathrm{Tr} A}\,. \tag{30}$$

<u>*REMARK:*</u> In fact, $\det e^A = e^{\mathrm{Tr} A}$ is true for *any* matrix A, but if A is not diagonalizable, a more sophisticated technique is required to prove eq. (30). See, e.g., my notes on the matrix exponential at `http://scipp.ucsc.edu/~haber/webpage/MatrixExpLog.pdf`.

[1]Starting from $PP^{-1} = \mathbf{I}$, it follows that $1 = \det \mathbf{I} = \det(PP^{-1}) = \det P \det P^{-1}$, which yields $\det P^{-1} = 1/\det P$.

16. A population of black bears consists of A_k adults and J_k juveniles, where k is the year under consideration. Each year, 40% of the juveniles die and 10% of the juveniles mature into adults. Meanwhile one new cub is born on average for every 2.5 adults in the population thereby adding to the population of the juveniles. In equations,

$$J_{k+1} = 0.5J_k + 0.4A_k\,. \tag{31}$$

Meanwhile, some of the adults die off each year as well, and after one year 80% of the adults remain. That is,

$$A_{k+1} = 0.1J_k + 0.8A_k\,. \tag{32}$$

(a) The equations governing the black bear population can be written in the form

$$\begin{pmatrix} J_{k+1} \\ A_{k+1} \end{pmatrix} = M \begin{pmatrix} J_k \\ A_k \end{pmatrix}\,, \tag{33}$$

where M is a 2×2 matrix. Determine this matrix M.

The matrix equation,

$$\begin{pmatrix} J_{k+1} \\ A_{k+1} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} J_k \\ A_k \end{pmatrix}\,,$$

reproduces eqs. (31) and (32). Hence, we identify,

$$M = \begin{pmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{pmatrix}\,. \tag{34}$$

(b) Assume that at $k = 0$, the population of juvenile and adult black bears is given by J_0 and A_0, respectively. Write a matrix equation that determines J_n and A_n in terms of J_0 and A_0, under the assumption that the dynamics of the black bear population as described by eq. (33) does not change over time.

Inserting $k = 0$ in eq. (33) yields,

$$\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = M \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}\,. \tag{35}$$

Inserting $k = 1$ in eq. (33) yields,

$$\begin{pmatrix} J_2 \\ A_2 \end{pmatrix} = M \begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = M^2 \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}\,, \tag{36}$$

after using the result of eq. (35) Inserting $k = 2$ in eq. (33) yields,

$$\begin{pmatrix} J_3 \\ A_3 \end{pmatrix} = M \begin{pmatrix} J_2 \\ A_2 \end{pmatrix} = M^3 \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}\,, \tag{37}$$

after using the result of eq. (36). Continuing this process iteratively eventually yields,

$$\begin{pmatrix} J_n \\ A_n \end{pmatrix} = M^n \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}\,. \tag{38}$$

(c) By computing $\lim_{n\to\infty} M^n$, determine whether or not black bears will eventually go extinct.

To compute M^n, we first diagonalize the matrix M. The eigenvalues of M are obtained by computing the characteristic equation,

$$\begin{vmatrix} 0.5-\lambda & 0.4 \\ 0.1 & 0.8-\lambda \end{vmatrix} = 0$$

which yields

$$\lambda^2 - 1.3\lambda + 0.36 = (\lambda - 0.9)(\lambda - 0.4) = 0\,.$$

Hence, there exists an invertible matrix P such that

$$P^{-1}MP = D \equiv \begin{pmatrix} 0.9 & 0 \\ 0 & 0.4 \end{pmatrix}\,.$$

It follows that $M = PDP^{-1}$, and hence,

$$M^n = \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{n\ \text{terms}} = PD^nP^{-1} = P\begin{pmatrix} (0.9)^n & 0 \\ 0 & (0.4)^n \end{pmatrix}P^{-1}\,.$$

Noting that $\lim_{n\to\infty} \lambda^n = 0$ for any number that satisfies $|\lambda| < 1$, it follows that

$$\lim_{n\to\infty} M^n = \mathbf{0}\,,$$

where $\mathbf{0}$ is the zero matrix. In this case, eq. (38) implies that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} J_n = 0$, corresponding to the extinction of black bears in the $n \to \infty$ limit.

(d) Based on the result of part (c), can you state a general criterion on the property of M which determines whether or not extinction is inevitable?

If the eigenvalues of M are λ_1 and λ_2, and P is the diagonalizing matrix, then

$$M^n = P\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}P^{-1}\,.$$

Since $\lim_{n\to\infty} \lambda^n = 0$ for any number that satisfies $|\lambda| < 1$, one can conclude that extinction is inevitable if both eigenvalues of the matrix M lie in the range $-1 < \lambda < 1$, since in this case $\lim_{n\to\infty} M^n = \mathbf{0}$.

REMARK: For a comprehensive discussion of the model of the population of juvenile and adult black bears employed in this problem, see Alan Garfinkel, Jane Shevtsov and Yina Guo, *Modeling Life: the mathematics of biological systems* (Springer International Publishing, Cham, Switzerland, 2017).

17. Four equal mass balls lying along the x-axis are attached by three springs. The two outermost balls are fixed, while the two innermost balls are free to oscillate in the x direction. Denote the displacement from equilibrium of the two balls by x_1 and x_2. The total potential energy of the system is:

$$V = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_2 - x_1)^2 \,.$$

(a) Show that the equations of motion for the two displacements can be cast into the matrix equation:

$$\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = A\vec{\boldsymbol{x}}\,,$$

where $\vec{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and A is a 2×2 matrix.

(b) Diagonalize the matrix A and determine the two possible frequencies of vibrations. These are the *normal modes* of the system.

(a) The potential energy is given by

$$V = k(x_1^2 - x_1x_2 + \tfrac{1}{2}x_2^2)\,.$$

The equations of motion are:

$$\begin{cases} m\ddot{x}_1 = -\partial V/\partial x_1 = -2kx_1 + kx_2\,, \\ m\ddot{x}_2 = -\partial V/\partial x_2 = \quad kx_1 - kx_2\,, \end{cases} \tag{39}$$

where $\ddot{x}_1 \equiv d^2x_1/dt^2$ and $\ddot{x}_2 \equiv d^2x_2/dt^2$. Assuming solutions of the form $x_1 = x_{10}e^{i\omega t}$ and $x_2 = x_{20}e^{i\omega t}$, it follows that

$$\ddot{x}_1 = -\omega^2 x_1 \qquad \text{and} \qquad \ddot{x}_2 = -\omega^2 x_2\,.$$

Substituting these results into eq. (39) yields:

$$\begin{cases} -m\omega^2 x_1 = -\partial V/\partial x = -2kx_1 + kx_2\,, \\ -m\omega^2 x_2 = -\partial V/\partial x_2 = \quad kx_1 - kx_2\,. \end{cases}$$

In matrix form, these equations are:

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \text{with} \qquad \lambda \equiv \frac{m\omega^2}{k}\,.$$

(b) Solving the characteristic equation,

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0\,.$$

yields two roots:

$$\lambda = \tfrac{1}{2}\left[3 \pm \sqrt{5}\right]\,.$$

Since $\lambda \equiv m\omega^2/k$, it follows that there are two characteristic frequencies,

$$\omega_1 = \sqrt{\frac{(3+\sqrt{5})k}{2m}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{(3-\sqrt{5})k}{2m}}\,.$$

18. In Example 4 on pp. 441–442 of Boas, the following system of differential equations is solved by the Laplace transform method,

$$y' - 2y + z = 0\,, \qquad y_0 \equiv y(t=0) = 1\,, \tag{40}$$
$$z' - y - 2z = 0\,, \qquad z_0 \equiv z(t=0) = 0\,, \tag{41}$$

where $y' \equiv dy/dt$ and $z' = dz/dt$. In this problem, you will solve this system of equations by another technique.

(a) Show that eqs. (40) and (41) can be rewritten as a matrix differential equation of the form

$$\frac{d\vec{\boldsymbol{x}}}{dt} = A\vec{\boldsymbol{x}}\,, \qquad \vec{\boldsymbol{x}}_0 \equiv \vec{\boldsymbol{x}}(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \tag{42}$$

where

$$\vec{\boldsymbol{x}} = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}\,.$$

Identify the matrix A such that eq. (42) is equivalent to eqs. (40) and (41).

If we rewrite eqs. (40) and (41) as,

$$y' = 2y - z\,, \qquad y_0 \equiv y(t=0) = 1\,, \tag{43}$$
$$z' = y + 2z\,, \qquad z_0 \equiv z(t=0) = 0\,, \tag{44}$$

then in matrix form, eqs. (43) and (44) are equivalent to,

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}\,, \qquad \text{where } \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

That is, we have identified,

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\,, \tag{45}$$

as the matrix that appears in eq. (42).

(b) Show that if A is a matrix of constant coefficients, then the solution to eq. (42) is given by

$$\vec{\boldsymbol{x}}(t) = e^{At}\,\vec{\boldsymbol{x}}_0\,. \tag{46}$$

To show that eq. (46) is a solution to eq. (42), we first prove the following identity,

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A\,. \tag{47}$$

To prove this result, recall that the matrix exponential is defined by the power series,

$$e^{At} = \sum_{n=0}^{\infty}\frac{t^n A^n}{n!} = \mathbf{I} + \sum_{n=1}^{\infty}\frac{t^n A^n}{n!}\,. \tag{48}$$

Taking the derivative of the above result, we can differentiate term by term, since the power series sum is absolutely convergent,

$$\begin{aligned}\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(\mathbf{I} + \sum_{n=1}^{\infty}\frac{t^n A^n}{n!}\right) = \frac{d}{dt}\mathbf{I} + \sum_{n=1}^{\infty}\frac{A^n}{n!}\frac{d}{dt}(t^n) = \sum_{n=1}^{\infty}\frac{A^n}{n!}\,nt^{n-1} \\ &= A\sum_{n=1}^{\infty}\frac{t^{n-1}A^{n-1}}{(n-1)!} = A\sum_{m=0}^{\infty}\frac{t^m A^m}{m!} = Ae^{At}\,.\end{aligned} \tag{49}$$

In the first line of eq. (49), we noted that $(d/dt)\mathbf{I} = 0$, since the identity matrix $\mathbf{I}$ is a constant matrix which does not depend on the parameter t. In obtaining the second line of eq. (49), we used the fact that $n! = n\cdot(n-1)$, and we introduced a new index of summation, $m = n-1$, so that the final summation starts from $m = 0$. Since A commutes with any integer power of A, it follows that $A^n = AA^{n-1} = A^{n-1}A$. Hence, the second line of eq. (49) can also be written as,

$$\frac{d}{dt}e^{At} = \left(\sum_{n=1}^{\infty}\frac{t^{n-1}A^{n-1}}{(n-1)!}\right)A = \left(\sum_{m=0}^{\infty}\frac{t^m A^m}{m!}\right)A = e^{At}A\,.$$

Hence, eq. (47) has been verified.

In light of eq. (47), it follows that if $\vec{\boldsymbol{x}}(t)$ is given by eq. (46), where $\vec{\boldsymbol{x}}_0$ is a constant vector independent of t, then the derivative of $\vec{\boldsymbol{x}}(t)$ with respect to t is given by,

$$\frac{d}{dt}\vec{\boldsymbol{x}}(t) = \left(\frac{d}{dt}e^{At}\right)\vec{\boldsymbol{x}}_0 = Ae^{At}\vec{\boldsymbol{x}}_0 = A\,\vec{\boldsymbol{x}}(t)\,.$$

which coincides with the matrix differential equation given in eq. (42).

The final step is to check that the initial conditions are satisfied by eq. (46). Note that when $t = 0$, $At = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix. Setting $t = 0$ in eq. (48) yields, $e^{\mathbf{0}} = \mathbf{I}$. Hence,

$$\vec{\boldsymbol{x}}(t=0) = \mathbf{I}\vec{\boldsymbol{x}}_0 = \vec{\boldsymbol{x}}_0\,,$$

which coincides with the initial condition specified in eq. (42).

(c) Using the matrix A obtained in part (a), compute e^{At}. Use the method outlined in parts (a)–(c) of problem 15.

To compute e^{At}, we first diagonalize At, where A is given in eq. (45) and reproduced below for the convenience of the reader,

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} .$$

Note that A is a real antisymmetric matrix, $A = -A^{\mathsf{T}}$. It follows that

$$AA^\dagger = AA^{\mathsf{T}} = -A^2 = A^{\mathsf{T}}A = A^\dagger A .$$

That is, A is a normal matrix (which is defined as any matrix A that commutes with its hermitian adjoint $A^\dagger$). Likewise, At is a normal matrix for any value of t, which means that it can be diagonalized by a unitary similarity transformation,

$$U^{-1}At\,U = D . \tag{50}$$

where U is the unitary diagonalizing matrix whose columns are the normalized eigenvectors of At, and D is a diagonal matrix whose diagonal elements are the eigenvalues of At.

Thus, our first task is to work out the eigenvalues and corresponding normalized eigenvectors of At. The eigenvalues of At are the roots of the characteristic equation,

$$\det(At - \lambda\mathbf{I}) = (2t-\lambda)^2 + t^2 = \lambda^2 - 4\lambda t + 5t^2 = 0 .$$

The roots of the characteristic equation are then given by,

$$\lambda = \tfrac{1}{2}\bigl(4t \pm \sqrt{16t^2 - 20t^2}\,\bigr) = (2 \pm i)t .$$

Next, we work out the corresponding eigenvectors. First, for $\lambda = (2+i)t$,

$$\begin{pmatrix} 2t & -t \\ t & 2t \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = (2+i)t\begin{pmatrix} y \\ z \end{pmatrix} ,$$

which yields,

$$\begin{aligned} 2y - z &= (2+i)y , \\ y + 2z &= (2+i)z , \end{aligned}$$

after canceling out the common factor of t. That is $z = -iy$, which is satisfied by both of the above equations. This means that the most general eigenvector is,

$$y\begin{pmatrix} 1 \\ -i \end{pmatrix} .$$

To normalize this vector so that its magnitude is 1, we may take $y = 1/\sqrt{2}$ (up to an overall complex phase that we shall set to 1 for convenience).

Second, for $\lambda = (2 - i)t$,

$$\begin{pmatrix} 2t & -t \\ t & 2t \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = (2+i)t \begin{pmatrix} y \\ z \end{pmatrix},$$

which yields,

$$\begin{aligned} 2y - z &= (2-i)y\,, \\ y + 2z &= (2-i)z\,, \end{aligned}$$

after canceling out the common factor of t. That is $z = iy$, which is satisfied by both of the above equations. This means that the most general eigenvector is,

$$y \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Once again, to normalize this vector so that its magnitude is 1, we may take $y = 1/\sqrt{2}$ (up to an overall complex phase that we shall set to 1 for convenience).

Hence, we conclude that the two normalized eigenvectors corresponding to the eigenvalues $\lambda = 2 + i$ and $\lambda = 2 - i$ are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix},$$

respectively. These two vectors are the columns of the diagonalizing unitary matrix U,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

Since U is unitary, $U^{-1} = U^\dagger$. As a check, we compute,

$$\begin{aligned} U^{-1} At\, U = U^\dagger At\, U &= \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 2t & -t \\ 2t & t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} (2+i)t & (2-i)t \\ (1-2i)t & (1+2i)t \end{pmatrix} \\ &= \begin{pmatrix} (2+i)t & 0 \\ 0 & (2-i)t \end{pmatrix} = D\,. \end{aligned} \tag{51}$$

Indeed, the diagonal elements of D are the eigenvalues of At.

From eq. (50), it follows that

$$At = UDU^{-1}\,.$$

Thus, exponentiating both sides of the above equation yields,

$$e^{At} = e^{UDU^{-1}} = \sum_{n=0}^{\infty} \frac{(UDU^{-1})^n}{n!} = U \left(\sum_{n=0}^{\infty} \frac{D^n}{n!} \right) U^{-1} = U e^D U^{-1} = U e^D U^\dagger\,, \tag{52}$$

where we have used the fact that $U^{-1} = U^\dagger$ for any unitary matrix U. Note that in deriving eq. (52),we have implicitly made use of the following result,

$$(UDU^{-1})^n = \underbrace{UDU^{-1}UDU^{-1}UDU^{-1}\cdots UDU^{-1}}_{n\ \text{terms}} = UD^nU^{-1}\,,$$

after noting that $U^{-1}U = \mathbf{I}$. Moreover, for a diagonal matrix,

$$\exp\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^n = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{pmatrix} = \begin{pmatrix}e^{\lambda_1} & 0\\ 0 & e^{\lambda_2}\end{pmatrix}\,.$$

Hence, we conclude that

$$\begin{aligned}
e^{At} = Ue^DU^\dagger &= \frac{1}{2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}e^{(2+i)t} & 0\\ 0 & e^{(2-i)t}\end{pmatrix}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}e^{(2+i)t} & ie^{(2+i)t}\\ e^{(2-i)t} & -ie^{(2-i)t}\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}e^{(2+i)t}+e^{(2-i)t} & i\big(e^{(2+i)t}-e^{(2-i)t}\big)\\ -i\big(e^{(2+i)t}-e^{(2-i)t}\big) & e^{(2+i)t}+e^{(2-i)t}\end{pmatrix}\,. \qquad (53)
\end{aligned}$$

In light of the relations,

$$\cos t = \tfrac{1}{2}\big(e^{it}+e^{-it}\big)\,, \qquad \sin t = -\tfrac{1}{2}i\big(e^{it}-e^{-it}\big)\,,$$

it then follows that

$$e^{At} = \begin{pmatrix}e^{2t}\cos t & -e^{2t}\sin t\\ e^{2t}\sin t & e^{2t}\cos t\end{pmatrix}\,. \qquad (54)$$

(d) Insert the result of part (c) into eq. (46), and show that the solution you have obtained for eqs. (40) and (41) matches the result obtain by Boas on p. 442.

The last step is to insert the result of eq. (54) into eq. (46),

$$\vec{\boldsymbol{x}}(t) \equiv \begin{pmatrix}y(t)\\ z(t)\end{pmatrix} = e^{At}\,\vec{\boldsymbol{x}}_0\,, \quad \text{where } \vec{\boldsymbol{x}}_0 = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

Hence, it follows that

$$\begin{pmatrix}y(t)\\ z(t)\end{pmatrix} = \begin{pmatrix}e^{2t}\cos t & -e^{2t}\sin t\\ e^{2t}\sin t & e^{2t}\cos t\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}e^{2t}\cos t\\ e^{2t}\sin t\end{pmatrix}\,.$$

That is, the solution to eqs. (40) and (41) is,

$$\begin{aligned}
y(t) &= e^{2t}\cos t\,,\\
z(t) &= e^{2t}\sin t\,,
\end{aligned}$$

in agreement with the results obtained in Example 4 on pp. 441–442 of Boas using the method of Laplace transforms.

19. Radioactive decay is governed by an exponential decay law,

$$\frac{dN}{dt} = -\lambda N\,. \tag{55}$$

(a) Suppose that the number of atoms of an unstable nucleus at time $t = 0$ is equal to $N(t = 0) = N_0$. The half-life, denoted by $t_{1/2}$, is the time it takes for half of the atoms to decay. That is, $N(t_{1/2}) = \frac{1}{2}N_0$. Solve eq. (55) subject to the stated initial condition and find a formula for $t_{1/2}$ in terms of the parameter λ that appears in eq. (55).

Eq. (55) is a separable equation. Hence,

$$\frac{dN}{N} = -\lambda dt\,.$$

After integrating both sides, exponentiating and imposing the initial condition, $N_0 = N(0)$, one obtains the exponential decay law for radioactive decay,

$$N(t) = N_0 e^{-\lambda t}\,.$$

The half-life is defined by:

$$N(t_{1/2}) = \tfrac{1}{2}N_0 = N_0 e^{-\lambda t_{1/2}}\,.$$

That is, $e^{-\lambda t_{1/2}} = \frac{1}{2}$ or

$$\lambda = \frac{\ln 2}{t_{1/2}}\,. \tag{56}$$

(b) An unstable nucleus, A, with a half-life of $t^A_{1/2}$, decays into a nucleus, B. The nucleus B is also unstable, with a half-life of $t^B_{1/2}$, which you may assume is different from $t^A_{1/2}$. The unstable nucleus B decays into a stable nucleus, C.

At time $t = 0$, there are N_0 nuclei of type A. Assume that no nuclei of type B and C are present at time $t = 0$. The exponential decay equations that govern the decays of A and B have been given in Example 2 on pp. 402–403 of Boas. Suppose that $t^A_{1/2} = 30$ minutes and $t^B_{1/2} = 60$ minutes. If $N_0 = 1024$, how many nuclei of types A, B and C remain after four hours?

The exponential decay equations that govern the decays of A and B are:

$$\frac{dN_A}{dt} = -\lambda_A N_A\,, \tag{57}$$

$$\frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A\,, \tag{58}$$

since the population of B is replenished due to the decay of A. The initial conditions for these differential equations are: $N_A(0) = N_0$ and $N_B(0) = 0$. The solution to eq. (57) is immediate:

$$N_A(t) = N_0 e^{-\lambda_A t}\,. \tag{59}$$

To solve eq. (58), we first plug in the solution for $N_A(t)$ given by eq. (59), which yields,

$$\frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_0 e^{-\lambda_A t} \,. \tag{60}$$

This is a linear first order differential equation of the form $dN_B/dt + P(t)N_B = Q(t)$, where we identify $P(t) = \lambda_B$ and $Q(t) = \lambda_A N_0 e^{-\lambda_A t}$. Hence,

$$u(t) \equiv \exp\left\{\int P(t)\,dt\right\} = e^{\lambda_B t} \,.$$

Using eq. (3.9) on p. 401 of Boas, it follows that

$$N_B(t) = e^{-\lambda_B t}\int \lambda_A N_0 e^{(\lambda_B-\lambda_A)t} + Ce^{-\lambda_B t} = \frac{\lambda_A N_0}{\lambda_B - \lambda_A} e^{-\lambda_A t} + Ce^{-\lambda_B t} \,. \tag{61}$$

where C is an integration constant. One can determine C by imposing the initial condition, $N_B(0) = 0$. It then follows that

$$C = \frac{\lambda_A N_0}{\lambda_A - \lambda_B} \,.$$

Inserting this result back into eq. (61) yields,

$$N_B(t) = \frac{\lambda_A N_0}{\lambda_A - \lambda_B}\left(e^{-\lambda_B t} - e^{\lambda_A t}\right) \tag{62}$$

In terms of the half-life, one must use $\lambda = \ln 2/t_{1/2}$. Hence eqs. (59) and (62) yield,

$$N_A(t) = N_0 \left(\frac{1}{2}\right)^{t/t^A_{1/2}} \,, \tag{63}$$

$$N_B(t) = \left(\frac{t^B_{1/2}}{t^B_{1/2} - t^A_{1/2}}\right)\left[\left(\frac{1}{2}\right)^{t/t^B_{1/2}} - \left(\frac{1}{2}\right)^{t/t^A_{1/2}}\right] \,. \tag{64}$$

Let us check two limits. If $t^A_{1/2} \to 0$ then $\lambda_A \to \infty$, and all the nuclei A instantaneously decay into B. Then the problem reduces to a simple exponential decay of B into C, with $N_B(t) = N_0 e^{-\lambda_B t}$. In the limit where $t^B_{1/2} \to \infty$ it follows that $\lambda_B \to 0$, in which case B is stable and $N_B(t) = N_0 - N_A(t) = N_0(1 - e^{-\lambda_A t})$.

(c) Since we start off with N_0 nuclei, we must have

$$N_A(t) + N_B(t) + N_C(t) = N_0 \,, \tag{65}$$

at all times t. Hence,

$$N_C(t) = N_0 - N_A(t) - N_B(t) \,, \tag{66}$$

where $N_A(t)$ and $N_B(t)$ are given by eqs. (63) and (64).

Let us plug in some numbers. $N_0 = 1024$, $t^A_{1/2} = 30$ minutes, $t^B_{1/2} = 60$ minutes, and $t = 4$ hours $= 240$ minutes. Then

$$N_A(t) = N_0/256 = 4 \,, \qquad N_B(t) = 30N_0/256 = 120 \,, \qquad N_C(t) = 1024 - 4 - 120 = 900 \,. \tag{67}$$

20. Solve the equation,

$$y'' + y = 8x \sin x \,, \tag{68}$$

subject to the initial conditions, $y_0 \equiv y(0) = 0$ and $y'_0 \equiv (dy/dx)\big|_{x=0} = 0$. In this problem, you will solve eq. (68) by four different methods.

(a) Solve eq. (68) by using the method of undetermined coefficients [cf. eq. (6.24) on p. 421 of Boas] by first finding the most general solution to

$$y'' + y = 8xe^{ix} \,. \tag{69}$$

Then by taking the imaginary part of the solution to eq. (69), one obtains the general solution to eq. (68) before imposing the two initial conditions stated above, Finally, impose those initial conditions to obtain the final result.

First, we obtain the solution to the homogeneous equation, $y'' + 1 = 0$. The corresponding auxiliary equation is $r^2 + 1 = 0$, with roots $r = \pm i$. Hence, the solution to the homogeneous equation is

$$y_h(x) = Ae^{ix} + Be^{-ix} \,, \tag{70}$$

where A and B are arbitrary complex constants.

To obtain a particular solution to eq. (69), we make use of eq. (6.24) on p. 421 of Boas, which states that if the right hand side of eq. (69) is of the form $e^{cx}P_1(x)$ where c is equal to one of the two roots of the auxiliary equation and $P_1(x)$ is a polynomial of degree 1, then the appropriate form of the particular solution is $y_p(x) = xe^{cx}Q_1(x)$, where $Q_1(x)$ is taken to be a polynomial of degree 1 with undetermined coefficients. That is, we propose the following ansatz,

$$y_p(x) = xe^{ix}(a_1 x + a_0) \,, \tag{71}$$

where a_0 and a_1 are the so-called undetermined coefficients. Taking successive derivatives with respect to x yields,

$$\begin{aligned} y'_p(x) &= e^{ix}\big[a_0 + 2a_1 x + ix(a_0 + a_1 x)\big] \,, \\ y''_p(x) &= e^{ix}\big[2a_1 + 2i(a_0 + 2a_1 x) - x(a_0 + a_1 x)\big] \,. \end{aligned}$$

Plugging the above results back into eq. (69), we end up with,

$$2a_1 + 2i(a_0 + 2a_1 x) = 8x \,, \tag{72}$$

which is an identity that holds for all values of x. Thus, the coefficients of like powers of x on both sides of eq. (72) must be equal. This yields two equations corresponding to the coefficients of the constant term (x^0) and the coefficients of x,

$$2a_1 + 2ia_0 = 0 \,, \qquad 4ia_1 = 8 \,.$$

Hence, it follows that $a_0 = 2$ and $a_1 = -2i$. Plugging these results back into eq. (71) yields,

$$y_p(x) = 2xe^{ix}(1 - ix) \,. \tag{73}$$

Thus, the most general solution to eq. (69) prior to imposing the initial conditions is given by,

$$y(x) = y_p(x) + y_h(x) = 2xe^{ix}(1 - ix) + Ae^{ix} + Be^{-ix}\,. \tag{74}$$

Returning to eq. (69), we can write this equation in an equivalent form,

$$y'' + y = 8x(\cos x + i\sin x)\,. \tag{75}$$

If we also write,

$$y(x) = \operatorname{Re} y(x) + i \operatorname{Im} y(x)\,,$$

and plug this result back into eq. (75), we obtain,

$$\operatorname{Re} y'' + \operatorname{Re} y + i\big[\operatorname{Im} y'' + \operatorname{Im} y\big] = 8x(\cos x + i\sin x)\,. \tag{76}$$

The imaginary part of eq. (76) is,

$$\operatorname{Im} y'' + \operatorname{Im} y = 8x\sin x\,, \tag{77}$$

which precisely corresponds to eq. (68), the differential equation that we initially wanted to solve. Hence, if we make use of eq. (74), then the most general solution to eq. (68) is given by

$$\begin{aligned} \operatorname{Im} y(x) &= 2x\operatorname{Im}\big[(\cos x + i\sin x)(1 - ix)\big] + \operatorname{Im}\big[(A+B)\cos x + i(A-B)\sin x\big] \\ &= 2x(\sin x - x\cos x) + \operatorname{Im}(A+B)\cos x + \operatorname{Re}(A-B)\sin x\,. \end{aligned} \tag{78}$$

Finally, we impose the initial conditions, $y_0 \equiv y(0) = 0$ and $y'_0 \equiv (dy/dx)\big|_{x=0} = 0$. Note that $y_0 = 0$ implies that $\operatorname{Im}(A+B) = 0$. Taking the derivative oe eq. (78),

$$\operatorname{Im} y'(x) = 2(\sin x - x\cos x) + 2x^2\sin x - \operatorname{Im}(A+B)\sin x + \operatorname{Re}(A-B)\cos x\,.$$

Thus, $y'_0 = 0$ implies that $\operatorname{Re}(A-B) = 0$. That is, after plugging $\operatorname{Im}(A+B) = 0$ and $\operatorname{Re}(A-B) = 0$ back into eq. (78), we obtain,

$$\operatorname{Im} y(x) = 2x(\sin x - x\cos x)\,,$$

where $\operatorname{Im} y(x)$ is the solution to eq. (77) after having imposing the two initial conditions, $y_0 = y'_0 = 0$. This is precisely the problem we initially set out to solve, so we are now done.

(b) Use the Laplace transform technique to solve eq. (68).

We first take the Laplace transform of both sides of eq. (68), and employ eqs. (9.1) and (9.2) on p. 440 of Boas, which we repeat here for the reader's convenience,

$$L(y') = pY - y_0\,, \qquad L(y'') = p^2Y - py_0 - y'_0\,,$$

where $Y \equiv L(y)$. The end result is

$$p^2Y + Y = 8L(x\sin x)\,. \tag{79}$$

Next, we consult the table of Laplace transforms given on p. 469 of Boas. Formula L11 indicates that,

$$L(t\sin at) = \frac{2ap}{(p^2+a^2)^2}\,, \qquad \text{for } \operatorname{Re} p > |\operatorname{Im} a|. \tag{80}$$

Choosing $a = 1$ and replacing the variable t with x, eq. (79) yields,

$$(1+p^2)Y = \frac{16p}{(p^2+1)^2}\,.$$

Solving for Y,

$$Y = \frac{16p}{(p^2+1)^3}\,. \tag{81}$$

In order to invert eq. (81) to obtain $y(t)$ from $Y = L(y)$, we need to obtain a Laplace transform of a function that does not yet exist in the Table of Laplace transforms given on p. 469 of Boas. Due to the linearity of the Laplace transform, we can take a partial derivative of eq. (80) with respect to a as follows,

$$\begin{aligned} L(t^2\cos at) &= L\left(\frac{\partial}{\partial a}(t\sin at)\right) = \frac{\partial}{\partial a}L(t\sin at) = \frac{\partial}{\partial a}\left(\frac{2ap}{(p^2+a^2)^2}\right) \\ &= \frac{2p}{(p^2+a^2)^2} - \frac{8a^2p}{(o^2+a^2)^3}\,. \end{aligned} \tag{82}$$

Multiplying eq. (82) by a and subtracting the result from eq. (80) yields,

$$L(t\sin at - at^2\cos at) = \frac{8a^3p}{(p^2+a^2)^3}\,.$$

Comparing this result with $a = 1$ to eq. (81), it follows that

$$y(x) = 2x(\sin x - x\cos x)\,,$$

after replacing the variable t with the variable x. Indeed, $y(x) = 2x(\sin x - x\cos x)$ is the solution to eq. (68) subject to the initial conditions, $y_0 = y_0' = 0$. Not surprisingly, this result is also in agreement with the solution obtained in part (a).

(c) In the class handout, *Applications of the Wronskian to linear differential equations*, it is shown that for the differential equation $y' + a(x)y' + b(x)y = f(x)$, a particular solution is given by,[2]

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx\,, \tag{83}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions to the homogeneous equation, $y' + a(x)y' + b(x)y = 0$, and the Wronskian is given by $W(x) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x)$. Apply the result of eq. (83) to obtain the most general solution of eq. (69). Then, use this solution to solve eq. (68) subject to the initial conditions stated above.

[2] We do not need to include the constant of integration when evaluating the indefinite integrals in eq. (83), since including these constants of integration simply adds a linear combination of $y_1(x)$ and $y_2(x)$ to $y_p(x)$. But, a linear combination of $y_1(x)$ and $y_2(x)$ is the most general solution to the corresponding homogeneous equation, $y' + a(x)y' + b(x)y = 0$, which does not alter the fact that $y_p(x)$ with or without these terms is still a particular solution to $y' + a(x)y' + b(x)y = f(x)$.

We shall apply eq. (83) to obtaining the particular solution of eq. (69). In particular, we identify $f(x) = 8xe^{ix}$. In light of eq. (70), we can also identify $y_1(x) = e^{ix}$ and $y_2(x) = e^{-ix}$. Hence the Wronskian is given by,

$$W(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2 = e^{ix}\big[-ie^{-ix}\big] - e^{-ix}\big[ie^{ix}\big] = -2i\,.$$

Consequently, eq. (83) yields,

$$y_p(x) = -\frac{8}{2i}\left\{-e^{ix}\int x\,dx + e^{-ix}\int xe^{2ix}\,dx\right\} = -2ix^2 e^{ix} + 4ie^{-ix}\int xe^{2ix}\,dx\,. \tag{84}$$

The remaining indefinite integral can be evaluated by integrating by parts. But here is a faster method. Consider the indefinite integral,

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}\,. \tag{85}$$

Taking the partial derivative of eq. (85) with respect to a yields,

$$\frac{\partial}{\partial a}\int e^{ax}\,dx = \frac{\partial}{\partial a}\left(\frac{1}{a}e^{ax}\right) \qquad\Longrightarrow\qquad \int xe^{ax}\,dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right].$$

Setting $a = 2i$, we end up with

$$\int xe^{2ix}\,dx = e^{2ix}\left[\frac{x}{2i} + \frac{1}{4}\right]. \tag{86}$$

Hence, eq. (84) yields,

$$y_p(x) = -2ix^2 e^{ix} + 4ie^{ix}\left[\frac{x}{2i} + \frac{1}{4}\right] = e^{ix}\big[i + 2x(1 - ix)\big]\,.$$

Adding the homogeneous solution given in eq. (70), we conclude that the most general solution to eq. (69) is

$$y(x) = 2xe^{ix}(1 - ix) + A'e^{ix} + Be^{-ix}\,,$$

where $A' \equiv A + i$. Since A and B are arbitrary complex constants, it also follows that A' is an arbitrary complex constant. Thus, we can simply redefine A' and call it A. Having done so, we have reproduced eq. (74). The remainder of the calculation is then the same as the one following eq. (74) in the solution to part (a) above.

(d) This method corresponds to the reduction of order technique described in eqs. (7.21) and (7.22) on p. 434 of Boas. Consider the differential equation.

$$y'' + a(x)y' + b(x)y = f(x)\,, \tag{87}$$

If one solution, $y_1(x)$, of the corresponding homogeneous equation, $y'' + a(x)y' + b(x)y = 0$ is known, then the solution to eq. (87) can be obtained with the change of variables,

$$y(x) = y_1(x)v(x)\,.$$

Then, eq. (87) is converted into a differential equation where the variable y is replaced by v. Since $y_1(x)$ solves the homogeneous equation, you will find that the resulting differential equation in terms of v is of the form,

$$y_1(x)v'' + g(x)v' = f(x)\,, \tag{88}$$

where $g(x)$ is a function that depends on $y_1(x)$ and $a(x)$. Note that there is no term on the left hand side of eq. (88) that is proportional to v. Thus, one can now introduce $w = v'$ and obtain a first order linear differential equation in w, which can be exactly solved.

Apply this technique to solve eq. (69). Show that if $y_1(x) = e^{ix}$, then the resulting equation for w is given by,

$$w' + 2iw = 8ix\,.$$

Solve this equation for w, and then use this result to find the most general solution of eq. (69). Finally, use this solution to solve eq. (68) subject to the initial conditions stated above.

Starting from eq. (69), and using $y_1(x) = e^{ix}$, we define a new variable v such that,

$$y(x) = e^{ix}v(x)\,. \tag{89}$$

It then follows that,

$$y' = ie^{ix}v + e^{ix}v'\,, \qquad\qquad y'' = -e^{ix}v + 2ie^{ix}v' + e^{ix}v''\,.$$

Plugging these results back into $y'' + y = 8ie^{ix}$, we obtain,

$$2ie^{ix}v' + e^{ix}v'' = 8ixe^{ix}\,. \tag{90}$$

If we define $w \equiv v'$, then eq. (90) simplifies to,

$$w'' + 2iw' = 8ix\,. \tag{91}$$

Eq. (91) is a linear first order differential equation of the form $w' + P(x)w = Q(x)$, whose solution is given by eq. (3.9) of p. 401 of Boas. In particular, we identify $P(x) = 2i$ and $Q(x) = 8ix$, in which case,

$$u(x) \equiv \exp\left\{\int P(x)\,dx\right\} = e^{2ix}\,.$$

Hence, eq. (3.9) of p. 401 of Boas yields,

$$w(x) = e^{-2ix}\int 8xe^{2ix}\,dx + Ce^{-2ix}\,,$$

where C is an arbitrary complex constant. Employing eq. (86),

$$w(x) = Ce^{-2ix} + 8\left[\frac{x}{2i} + \frac{1}{4}\right] = Ce^{-2ix} + 4x + 2i\,.$$

Since $w = v'$, one can obtain v by taking the indefinite integral of the above equation.

Then, in light of eq. (89),

$$y(x) = e^{ix}\left\{\int (Ce^{-2ix} + 4x + 2i)dx + C'\right\},$$

where C' is a second integration constant. Performing the integration above yields,

$$y(x) = C'e^{ix} + \tfrac{1}{2}iCe^{-ix} + 2x(1 - ix)e^{ix}.$$

Identifying $A = C'$ and $B \equiv \frac{1}{2}iC$, it follows that

$$y(x) = Ae^{ix} + Be^{-ix} + 2x(1 - ix)e^{ix},$$

where A and B are arbitrary complex constants. Once again, we have reproduced the result of eq. (74). The remainder of the calculation is then the same as the one following eq. (74) in the solution to part (a) above.

21. Solve the following differential equations. Your solution should correspond to the most general form (depending on some number of arbitrary constants), unless initial conditions are specified.

(a) $(2x - y\sin 2x)dx + (2y - \sin^2 x)dy = 0$.

Recall that $M(x, y)dx + N(x, y)dy$ is an exact differential if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Identifying $M(x, y) = 2x - y\sin 2x$ and $N(x, y) = 2y - \sin^2 x$, one can check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sin 2x = -2\sin x\cos x,$$

due to the trigonometric identity noted above. Hence a function $F(x, y)$ exists such that

$$dF = (2x - y\sin 2x)dx + (2y - \sin^2 x)dy = 0, \tag{92}$$

in which case the solution to the differential equation above is

$$F(x, y) = C, \tag{93}$$

where C is an arbitrary constant.

One can determine $F(x, y)$ by comparing eq. (92) with the following identity (which corresponds to the chain rule),

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy. \tag{94}$$

Hence,

$$\frac{\partial F}{\partial x} = 2x - y\sin 2x.$$

Integrating $\partial F/\partial x$ while holding y fixed yields

$$F(x,y) = x^2 + \tfrac{1}{2}y\cos 2x + f(y)\,, \tag{95}$$

where at this stage of the calculation, $f(y)$ is an arbitrary function of y. Differentiating eq. (95) with respect to y yields,

$$\frac{\partial F}{\partial y} = \tfrac{1}{2}\cos 2x + \frac{df}{dy}\,. \tag{96}$$

One can also compute $\partial F/\partial y$ by comparing eqs. (92) and (94), which yields,

$$\frac{\partial F}{\partial y} = 2y - \sin^2 x\,. \tag{97}$$

Recalling the trigonometric identity, $\cos 2x = 1 - 2\sin^2 x$, we can rewrite eq. (96) as,

$$\frac{\partial F}{\partial y} = \tfrac{1}{2} - \sin^2 x + \frac{df}{dy}\,. \tag{98}$$

Comparing eqs. (97) and (98) implies that

$$\frac{df}{dy} = 2y - \tfrac{1}{2}\,.$$

Hence, it follows that

$$f(y) = y^2 - \tfrac{1}{2}y\,.$$

We do not need to keep the integration constant since any constant term in $F(x,y)$ can be reabsorbed into the constant C in eq. (93). Inserting the result for $f(y)$ back into eq. (95) yields,

$$F(x,y) = x^2 + y^2 + \tfrac{1}{2}y(\cos 2x - 1)\,.$$

Since $\cos 2x = 1 - 2\sin^2 x$ as previously noted, we can rewrite this result as

$$F(x,y) = x^2 + y^2 - y\sin^2 x\,.$$

Hence, eq. (93) yields our final solution,

$$x^2 + y^2 - y\sin^2 x = C\,,$$

where C is an arbitrary constant.

(b) $x^2y' - xy = 1/x$.

Assuming $x \neq 0$, we can divide by x to obtain,

$$y' - \frac{y}{x} = \frac{1}{x^3}\,. \tag{99}$$

This is a first order linear differential equation of the form $y' + P(x)y = Q(x)$. The solution is given by eq. (3.9) on p. 401 of Boas. Identifying $P(x) = -1/x$ and $Q(x) = 1/x^3$, we first compute,

$$u(x) \equiv \exp\left\{\int P(x)dx\right\} = \exp\bigl(-\ln|x|\bigr) = \frac{1}{|x|}\,.$$

Writing $|x| = x\,\mathrm{sgn}(x)$, it follows that

$$y(x) = x\int\frac{dx}{x^4} + C'x\,\mathrm{sgn}(x) = -\frac{1}{3x^2} + Cx\,, \qquad (100)$$

where $C \equiv C'\,\mathrm{sgn}(x)$ is an arbitrary constant, and the factors of $\mathrm{sgn}(x)$ originally appearing in the first term on the right hand side of eq. (100) have canceled out.

(c) $y' = \dfrac{y}{x} - \tan\left(\dfrac{y}{x}\right)$.

In light of eq. (4.12) on p. 406 of Boas, we introduce a new variable $v = y/x$, or equivalently,

$$y = xv\,. \qquad (101)$$

Taking a derivative with respect to x yields,

$$y' = v + x\frac{dv}{dx}\,.$$

Hence, the differential equation, $y' = y/x - \tan(y/x)$, is transformed into

$$v + x\frac{dv}{dx} = v - \tan v\,.$$

This results in the separable differential equation,

$$\frac{dv}{\tan v} = -\frac{dx}{x}\,. \qquad (102)$$

Recall that

$$\int\frac{dv}{\tan v} = \int\cot v\,dv = \ln|\sin v|\,.$$

Thus, integrating eq. (102) yields,

$$\ln|\sin v| = -\ln|x| + \ln C'\,,$$

where $\ln C'$ is the integration constant. Exponentiating this result, we end up with

$$\sin v = \frac{C}{x}\,,$$

after absorbing any overall signs into the arbitrary constant C. Finally, using eq. (101), we obtain,

$$x\sin\left(\frac{y}{x}\right) = C\,.$$

(d) $2xy'' + y' = 2x^{5/2}$, assuming $x > 0$.

If we introduce a new variable $v = y'$, then the differential equation above becomes a first order linear differential equation,

$$2xv' + v = 2x^{5/2}\,.$$

Assuming $x \neq 0$, one can divide both sides above by $2x$ to obtain,

$$v' + \frac{v}{2x} = x^{3/2}\,.$$

This is a first order linear differential equation of the form $y' + P(x)y = Q(x)$. The solution is given by eq. (3.9) on p. 401 of Boas. Identifying $P(x) = 1/(2x)$ and $Q(x) = x^{3/2}$, we first compute,

$$u(x) = \exp\left\{\int P(x)dx\right\} = \exp\bigl(\tfrac{1}{2}\ln x\bigr) = x^{1/2}\,,$$

where no absolute value is needed above since by assumption, $x > 0$. It follows that

$$v(x) = \frac{1}{x^{1/2}}\int x^2\,dx + \frac{C}{x^{1/2}} = \tfrac{1}{3}x^{5/2} + \frac{C}{x^{1/2}}\,, \qquad (103)$$

where C is an arbitrary constant.

Finally, using $y' = v$, we integrate to obtain y,

$$y(x) = \tfrac{2}{27}x^{7/2} + 2Cx^{1/2} + C'\,,$$

where C and C' are arbitrary constants.

(e) $yy'' + (y')^2 + 4 = 0$, subject to the initial conditions, $y(1) = 3$ and $(dy/dx)_{x=1} = 0$.

Introducing a new variable $v = y'$ and using the chain rule, we obtain,

$$y'' = v' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = y'\frac{dv}{dy} = v\frac{dv}{dy}\,.$$

Hence, introducing the new variable v produces a first order differential equation in which v becomes the dependent variable and y becomes the independent variable,

$$yv\frac{dv}{dy} + v^2 = -4\,. \qquad (104)$$

Noting that $vdv = \frac{1}{2}dv^2$, we make one more change of variables,[3] $w \equiv v^2$.

Then, eq. (104) is transformed into

$$\tfrac{1}{2}y\frac{dw}{dy} + w = -4\,.$$

[3]If you missed this trick, you can always divide eq. (104) by yv to obtain, a differential equation of the form $dv/dy + P(y)v = Q(y)v^n$, where $P(y) = 1/y$, $Q(y) = -4/y$ and $n = -1$. We recognize this equation as the Bernoulli equation [see eq. (4.1) on p. 404 of Boas]. Following eq. (4.2) on p. 404 of Boas, we introduce $w \equiv v^{1-n} = v^2$, which is precisely the change of variables we are making here.

Assuming that $y \neq 0$, we can divide by $\frac{1}{2}y$ to obtain,

$$\frac{dw}{dy} + \frac{2w}{y} = -\frac{8}{y}\,.$$

This is a first order linear differential equation of the form $dw/dy + P(y)w = Q(y)$. The solution is given by eq. (3.9) on p. 401 of Boas. Identifying $P(y) = 2/y$ and $Q(x) = -8/y$, we first compute,

$$u(y) \equiv \exp\left\{\int P(y)dy\right\} = \exp\bigl(2\ln|y|\bigr) = y^2\,.$$

It follows that,

$$w(y) = -\frac{1}{y^2}\int 8y\,dy + \frac{C}{y^2} = -4 + \frac{C}{y^2}\,,$$

where C is an arbitrary constant.

Using $w = v^2$ and $v = y'$, it follows that,

$$\frac{dy}{dx} = \sqrt{\frac{C}{y^2} - 4}\,. \tag{105}$$

This is a separable first order differential equation. Hence,

$$\frac{y\,dy}{\sqrt{C - 4y^2}} = dx\,.$$

Integrating both sides (using the substitution $u = C - 4y^2$ to perform the integral over y), we end up with

$$-\tfrac{1}{4}\sqrt{C - 4y^2} = x + C'\,, \tag{106}$$

where C and C' are arbitrary constants.

Finally, we impose the initial conditions, $y(1) = 3$ and $(dy/dx)_{x=1} = 0$. Since $y = 3$ when $x = 1$, we insert $y = 3$ into eq. (105) to obtain

$$\frac{dy}{dx} = \sqrt{\frac{C}{9} - 4} = 0\,,$$

which yields $C = 36$. Plugging $C = 36$, $x = 1$ and $y = 3$ into eq. (106) yields $C' = -1$. Hence, eq. (106) yields

$$4(1 - x) = \sqrt{36 - 4y^2}\,.$$

Squaring both sides of this equation, we end up with

$$y^2 + 4(x - 1)^2 = 9\,, \tag{107}$$

By implicit differentiation of the above equation, one can verify that $yy'' + (y')^2 + 4 = 0$ is satisfied. Here is the verification of this assertion. Starting from eq. (107), take one derivative with respect to x, which yields

$$2y\frac{dy}{dx} + 8(x - 1) = 0\,.$$

Taking a second derivative with respect to x, we obtain,

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 8 = 0\,.$$

Dividing this last result by 2 yields the original differential equation.

(f) $x^2y'' - 2xy' + 2y = x\ln x$, assuming that $x > 0$.

This has the form of the Euler-Cauchy differential equation. Using eq. (7.20) on p. 434 of Boas, the change of variable, $x = e^z$ yields,

$$\frac{d^2y}{dz^2} - 3\frac{dz}{dy} + 2y = ze^z\,. \tag{108}$$

The corresponding auxiliary equation is $r^2 - 3r + 2 = (r-2)(r-1) = 0$, which has two roots, $r = 1, 2$. Thus, the solution to the homogeneous corresponding to eq. (108) is

$$y_h(z) = Ae^z + Be^{2z}\,,$$

where A and B are arbitrary constants. To find a particular solution of eq. (108), we employ eq. (6.24) on p. 420 of Boas and propose the ansatz,

$$y_p(z) = ze^z(a_0 + a_1z)\,. \tag{109}$$

Taking successive derivatives with respect to z yields,

$$\frac{dz}{dy} = e^z\bigl[(z+1)(a_0 + a_1z) + a_1z\bigr]\,,$$
$$\frac{d^2}{dy^2} = e^z\bigl[(z+2)(a_0 + a_1z) + 2a_1(z+1)\bigr]\,.$$

Plugging the above results back into eq. (108) yields,

$$\frac{d^2y}{dz^2} - 3\frac{dz}{dy} + 2y = (2a_1 - a_0 - 2a)1z)e^z = ze^z\,.$$

Equating terms with equal powers of z, we obtain

$$2a_1 - a_0 = 0\,, \qquad -2a_1 = 1\,,$$

which yields $a_1 = -\frac{1}{2}$ and $a_0 = -1$. Plugging these results back into eq. (109) yields,

$$y_p(z) = -ze^z\bigl(1 + \tfrac{1}{2}z\bigr)\,.$$

Hence, the solution to eq. (108) is,

$$y(z) = y_p(z) + y_h(z) = -ze^z\bigl(1 + \tfrac{1}{2}z\bigr) + Ae^z + Be^{2z}\,.$$

Since $x = e^z$, the solution to the original differential equation is,

$$y(x) = -x\ln x - \tfrac{1}{2}x\ln^2 x + Ax + Bx^2\,,$$

where A and B are arbitrary constants.

(g) $(x^2+1)y'' - 2xy' + 2y = 0$, subject to the initial conditions $y_0 = y_0' = 1$.

Note that $y(x) = x$ is a solution of $(x^2+1)y'' - 2xy' + 2y = 0$ prior to imposing the initial conditions. Following eqs. (7.21) and (7.22) on p. 434 of Boas, we introduce a new variable $v(x)$ as follows,

$$y = xv(x)\,. \tag{110}$$

Then,

$$y' = v + x\frac{dv}{dx}\,, \qquad\qquad y'' = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\,.$$

Then, the differential equation, $(x^2+1)y'' - 2xy' + 2y = 0$, is transformed into.

$$(x^2+1)\left(2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right) - 2x\left(v + x\frac{dv}{dx}\right) + 2xv = 0\,,$$

which simplifies to,

$$x(x^2+1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0\,. \tag{111}$$

Introducing $w = dv/dx$, eq. (111) yields,

$$x(x^2+1)\frac{dw}{dx} + 2w = 0\,.$$

This is a separable first order differential equation, which can be rewritten as,

$$\frac{dw}{w} = -\frac{2\,dx}{x(x^2+1)}\,. \tag{112}$$

To integrate both sides, we must compute the following indefinite integral,

$$\int \frac{dx}{x(x^2+1)} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln|x| - \tfrac{1}{2}\ln(x^2+1)\,,$$

where the last term was obtained by making the substitution $u = x^2+1$ before integrating. Hence, integrating both sides of eq. (112) yields,

$$\ln|w| = -2\ln|x| + \tfrac{1}{2}\ln(x^2+1) + \ln C'\,,$$

where $\ln C'$ is an integration constant. Exponentiating both sides of the last equation yields,

$$w = C\left(\frac{x^2+1}{x^2}\right)\,,$$

after absorbing the sign of w in a new constant C.

Since $w = v'$, we integrate w to obtain $v(x)$,

$$v(x) = C\int\left(\frac{x^2+1}{x^2}\right)dx + C' = C\int\left(1 + \frac{1}{x^2}\right)dx + C' = C\left(x - \frac{1}{x}\right) + C'\,,$$

where C' is an integration constant. Hence, eq. (110) yields

$$y(x) = C(x^2 + 1) + C'x\,, \tag{113}$$

where C and C' are constants that we shall now fix by imposing the boundary conditions, $y_0 = y'_0 = 1$. Taking the derivative of eq. (113) with respect to x yields,

$$\frac{dy}{dx} = 2xC + C'\,.$$

Imposing $y_0 = y'_0 = 1$ yields $C = C' = 1$. Plugging these results back into eq. (113) yields our final result,

$$y(x) = x^2 + x + 1\,.$$

22. Using the Laplace transform technique, solve the following system of differential equations,

$$y' + z = 2\cos t\,, \tag{114}$$

$$z' - y = 1\,, \tag{115}$$

subject to the initial conditions $y_0 = -1$ and $z_0 = 1$, where $y' \equiv dy/dt$ and $z' \equiv dz/dt$.

The Laplace transforms of y and z are denoted by $Y \equiv L(y)$ and $Z \equiv L(z)$, respectively. Using eq. (9.1) on p. 440 of Boas,

$$L(y') = pY - y_0\,, \qquad\qquad L(z') = pZ - z_0\,.$$

Hence, taking the Laplace transform of the two equations above yield,

$$pY + 1 + Z = 2L(\cos t)\,, \tag{116}$$

$$pZ - 1 - Y = L(1)\,. \tag{117}$$

after imposing the initial conditions $y_0 = -1$ and $z_0 = 1$.

Consulting the table of Laplace transforms on p. 469 of Boas,

$$L(1) = \frac{1}{p}\,, \qquad\qquad \text{for } \operatorname{Re} p > 0\,, \tag{118}$$

$$L(\cos t) = \frac{p}{p^2+1}\,, \quad \text{for } \operatorname{Re} p > 1\,. \tag{119}$$

Inserting these results into eqs. (116) and (117) yields,

$$pY + 1 + Z = \frac{2p}{p^2+1}\,,$$

$$pZ - 1 - Y = \frac{1}{p}\,,$$

which we shall rewrite as

$$pY + Z = -\frac{(p-1)^2}{p^2+1}\,,$$

$$pZ - Y = \frac{p+1}{p}\,,$$

In matrix form,

$$\begin{pmatrix} p & 1 \\ -1 & p \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} -\dfrac{(p-1)^2}{p^2+1} \\[10pt] \dfrac{p+1}{p} \end{pmatrix}.$$

Multiplying both sides above by the inverse matrix,

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \frac{1}{p^2+1}\begin{pmatrix} p & -1 \\ 1 & p \end{pmatrix}\begin{pmatrix} -\dfrac{(p-1)^2}{p^2+1} \\[10pt] \dfrac{p+1}{p} \end{pmatrix}.$$

That is,

$$\begin{aligned} Y &= -\frac{1}{p^2+1}\left[\frac{p(p-1)^2}{p^2+1} + \frac{p+1}{p}\right] = -\frac{1}{p^2+1}\left[p - \frac{2p^2}{p^2+1} + \frac{1}{p} + 1\right]\,, \\ &= -\frac{1}{p^2+1}\left[\frac{p^2+1}{p} - \frac{p^2-1}{p^2+1}\right] = \frac{p^2-1}{(p^2+1)^2} - \frac{1}{p}\,. \end{aligned} \tag{120}$$

$$Z = \frac{1}{p^2+1}\left[p+1-\frac{(p-1)^2}{p^2+1}\right] = \frac{1}{p^2+1}\left[p + \frac{2p}{p^2+1}\right] = \frac{p}{p^2+1} + \frac{2p}{(p+1)^2}\,. \tag{121}$$

In order to invert the above Laplace transforms, we again consult the table of Laplace transforms on p. 469 of Boas. In addition to eqs. (118) and (119), we also will need to employ

$$L(t\sin t) = \frac{2p}{(p^2+1)^2}\,, \quad \text{for } \operatorname{Re} p > 1\,, \tag{122}$$

$$L(t\cos t) = \frac{p^2-1}{(p^2+1)^2}\,, \quad \text{for } \operatorname{Re} p > 1\,. \tag{123}$$

It then follows that

$$L(t\cos t - 1) = \frac{p^2-1}{(p^2+1)^2} - \frac{1}{p}\,, \qquad L(\cos t + t\sin t) = \frac{p}{p^2+1} + \frac{2p}{(p^2+1)^2}\,.$$

Comparing these results with eqs. (120) and (121), it follows that the solution to the system of differential equations given in eqs. (114) and (115) are,

$$y(t) = t\cos t - 1\,,$$

$$z(t) = \cos t + t\sin t\,.$$