Complex conjugation, Modulus, and Inequalities

Let $z = x + iy$ be a complex number (i.e., $z \in \mathbb{C}$ with $x, y \in \mathbb{R}$). We define the real and imaginary parts of z by Re $z = x$ and Im $z = y$. The *modulus* of z, denoted by $|z|$, is a nonnegative real number given by

$$|z| = \sqrt{x^2 + y^2}\,. \tag{1}$$

The modulus of z is often called the absolute value of z and is also called the magnitude of z. These alternative names arise as follows. First, if z is real, then $|z|$ is the absolute value of the real number z. Thus, eq. (1) is simply an extension to the complex numbers of a result well known when applied to real numbers. Second, if one views the complex numbers geometrically as living in the complex plane, where the complex number z is identified by the point (x, y) in the plane, then $|z|$ is the length or magnitude of a two-dimension vector pointing from the origin to (x, y).

Given the complex number $z = x + iy$, the *complex conjugate* of z, denoted by z^*, is defined to be

$$z^* \equiv x - iy\,.$$

Equivalently, if one represents the complex number z in its polar form, $z = re^{i\theta}$, then the complex conjugate of z is given by $z^* = re^{-i\theta}$.

Given two complex numbers z and w, the following identities are noteworthy:

1. $(z^*)^* = z\,,$
2. Re $z = \frac{1}{2}(z + z^*)$ and Im $z = -\frac{1}{2}i(z - z^*)\,,$
3. $(z + w)^* = z^* + w^*\,,$
4. $(zw)^* = z^*w^*\,,$
5. $|z^*| = |z|$
6. $|zw| = |z|\,|w|\,,$
7. $|z|^2 = zz^*\,.$

These identities hold for all $z, w \in \mathbb{C}$. The proofs of the above identities are straightforward and are left for the reader.

The following three inequalities are also quite useful. For all $z, w \in \mathbb{C}$, the following inequalities are satisfied:

1. $|\text{Re}\ z| \leq |z|$ and $|\text{Im}\ z| \leq |z|\,,$
2. $|z + w| \leq |z| + |w|$ (this is the famous *triangle inequality*),
3. $|z + w| \geq \big||z| - |w|\big|\,.$

Note that on the right hand side of the third inequality above, we are taking the absolute value of the real number $|z| - |w|$.

The proof of the first inequality is immediate since $|z|^2 = (\mathrm{Re}\ z)^2 + (\mathrm{Im}\ z)^2$ [as a consequence of eq. (1)] and $|z| \geq 0$. The triangle inequality is proven by employing the identities listed above as follows:

$$\begin{aligned} |z+w|^2 &= (z+w)(z^* + w^*) \\ &= |z|^2 + |w|^2 + zw^* + z^*w \\ &= |z|^2 + |w|^2 + 2\mathrm{Re}\,(zw^*) \\ &\leq |z|^2 + |w|^2 + 2|zw^*|\,, \end{aligned} \tag{2}$$

where the last step is a consequence of the first inequality listed above. Noting that $|z|^2 + |w|^2 + 2|zw^*| = |z|^2 + |w|^2 + 2|z|\,|w| = (|z| + |w|)^2$, it follows that

$$|z+w|^2 \leq (|z| + |w|)^2\,.$$

Since both sides of this inequality are real and nonnegative, we can take the positive square root of both sides to obtain

$$|z+w| \leq |z| + |w|\,, \tag{3}$$

which is the triangle inequality. The triangle inequality is most easily understood in the complex plane by considering the vector addition of two vectors represented by the complex numbers z and w. The vector addition corresponds to a triangle in the complex plane whose three sides have length $|z|$, $|w|$ and $|z+w|$. Hence, the triangle inequality is simply the statement that the length of one side of a triangle cannot be larger than the sum of the lengths of the other two sides.

It is instructive to consider under what conditions the triangle inequality becomes an equality. Reviewing the derivation given by eq. (2), it follows that the triangle inequality becomes an equality when $\mathrm{Re}\ (zw^*) = |zw^*|$, which holds if and only if zw^* is real and nonnegative. To see what this latter condition entails, let us denote $R \equiv zw^* \geq 0$. If $R = 0$, then either $z = 0$ and/or $w = 0$, in which case the triangle inequality trivially reduces to an equality. If $R \neq 0$, then $Rw = zw^*w = z|w|^2$, which implies that $w = z|w|^2 R^{-1}$. That is, z and w are proportional and the proportionality constant is real and positive. Thus, one can conclude that if the complex numbers z and w are proportional and the proportionality constant is real and nonnegative, then the triangle inequality reduces to an equality.*

Finally, the proof of the third inequality is left for the ambitious student (or can be found in the first reference cited below).

References

These notes are based in part on material that appears on pp. 7–8 of H.A. Priestley, *Introduction to Complex Analysis*, 2nd Edition (Oxford University Press, Oxford, UK, 2003). See also Appendix A of Stephan Ramon Garcia and Roger A. Horn, *A Second Course in Linear Algebra* (Cambridge University Press, Cambridge, UK, 2017).

*This last result is again easy to understand in the complex plane by noting that if z and w are proportional and the proportionality constant is real and nonnegative, then the vectors corresponding to z and w are parallel. The triangle that results from the vector addition is then degenerate, i.e., the length of the third side of the degenerate triangle is simply the sum of the lengths of the other two sides.