Taylor Series Expansions

In this short note, a list of well-known Taylor series expansions is provided. We focus on Taylor series about the point $x = 0$, the so-called Maclaurin series. In all cases, the interval of convergence is indicated. The variable x is real.

We begin with the infinite geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,, \qquad |x| < 1\,. \tag{1}$$

If we change the sign of x, we obtain $(-x)^n = (-1)^n\,x^n$, which then yields:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n\,, \qquad |x| < 1\,. \tag{2}$$

The two expansions above diverge for all $|x| \geq 1$.

Next we write down the binomial expansion, assuming at first that p is a non-negative integer,

$$(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n\,, \tag{3}$$

where the binomial coefficient is defined as

$$\binom{p}{n} \equiv \frac{p!}{n!(p-n)!} = \begin{cases} 1\,, & \text{if } n = 0\,, \\ \dfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\,, & \text{if } n = 1, 2, 3, \ldots. \end{cases} \tag{4}$$

Inserting eq. (4) into eq. (3), one can obtain an equivalent expression for the binomial series that is valid (assuming the series converges) for any real number p,

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}\, x^n\,, \qquad |x| < 1\,. \tag{5}$$

Of course, if p is a non-negative integer, then the sum in eq. (5) is finite (containing precisely $p + 1$ nonzero terms) and therefore converges trivially for all real values of x, as expected.* Otherwise, the radius of convergence of the binomial series is $R = 1$, by virtue of the ratio test. The convergence properties at the endpoints of

*Strictly speaking, if $p = 0$ and $x = -1$, then $(1+x)^p$ is not well defined. In the mathematics literature, it is common to *define* $(1+x)^0 = 1$ for all real values of x, including the point $x = -1$.

the interval of convergence can be determined by a separate analysis. At $x = 1$, the series converges absolutely for $p \geq 0$, converges conditionally for $-1 < p < 0$ and diverges for $p \leq -1$. At $x = -1$, the series converges absolutely for $p \geq 0$ and diverges for $p < 0$.

We now list the Taylor series for the exponential and logarithmic functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \qquad |x| < \infty\,,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\,, \qquad -1 < x \leq 1\,. \tag{6}$$

Note that the Taylor expansion for $\ln(1+x)$ can be easily derived by integrating eq. (2).

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\,,$$

after shifting the summation index by one unit. The series given in eq. (6) diverges at $x = -1$ and is conditionally convergent at $x = 1$. Likewise, we can obtain the Taylor series for $\ln(1-x)$ by either integrating eq. (1) or by replacing x with $-x$ in eq. (6). Either way,

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\,, \qquad -1 \leq x < 1\,. \tag{7}$$

Subtracting eqs. (6) and (7) yields another Taylor series of interest,

$$\tfrac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}\,, \qquad |x| < 1\,.$$

Next, we examine the Taylor series of the trigonometric functions.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \infty\,,$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \infty\,,$$

$$\tan x = \sum_{n=0}^{\infty} (-1)^n T_{2n+1} \frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,,$$

$$x \cot x = 1 - \sum_{n=1}^{\infty} (-1)^{n-1}\, 2^{2n} B_{2n} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \pi\,,$$

$$\sec x = \sum_{n=0}^{\infty} (-1)^n\, E_{2n} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,,$$

$$x \csc x = 1 + \sum_{n=1}^{\infty} (-1)^{n-1}\, 2(2^{2n-1} - 1) B_{2n} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \pi\,,$$

which defines the tangent numbers T_{2n+1}, the Bernoulli numbers B_{2n}, and the Euler numbers E_{2n}, for all non-negative integers n.† A table of the these numbers for $n \leq 8$ is provided below.

n	T_{2n+1}	B_{2n}	E_{2n}
0	1	1	1
1	-2	$\frac{1}{6}$	-1
2	16	$-\frac{1}{30}$	5
3	-272	$\frac{1}{42}$	-61
4	7936	$-\frac{1}{30}$	1385
5	-353792	$\frac{5}{66}$	-50521
6	22368256	$-\frac{691}{2730}$	2702765
7	-1903757312	$\frac{7}{6}$	-199360981
8	209865342976	$-\frac{3617}{510}$	19391512145

The theory of these numbers is quite interesting. Here, I shall simply summarize a number of simple facts. First, the tangent numbers can be expressed simply in terms of the Bernoulli numbers,‡

$$T_{2k+1} = 2^{2k+2}(2^{2k+2}-1)\frac{B_{2k+2}}{2k+2}\,, \qquad k = 0, 1, 2, 3, \ldots\,. \tag{8}$$

It follows that

$$\tan x = \sum_{n=1}^{\infty} (-1)^{n-1}\, 2^{2n}(2^{2n}-1) B_{2n} \frac{x^{2n-1}}{(2n)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,.$$

You will also notice that whereas tangent numbers and Euler numbers are integers, the Bernoulli numbers (for $n \neq 0$) are non-integer rational numbers. In addition, T_{2n+1} and E_{2n} are positive [negative] for even [odd] n, whereas the Bernoulli numbers B_{2n} (for $n \neq 0$) are positive [negative] for odd [even] n.

†One can generalize the definitions of these numbers for all non-negative subscripts, n, by defining $B_{2n+1} = T_{2n} = E_{2n-1} = 0$ for $n = 1, 2, 3, \ldots$, and $B_1 = -\frac{1}{2}$. But, we will have no need for these more general definitions here.

‡It is common in the mathematical literature (see e.g. Ref. 2) to define the tangent numbers such that the T_{2n+1} are all positive. In this convention, what is denoted in these notes by $(-1)^n T_{2n+1}$ should be replaced by T_{2n+1}. I prefer to define the tangent numbers such that the Taylor series for $\tan x$ explicitly exhibits the factor of $(-1)^n$, which is common to the Taylor expansions of all the other trigonometric functions.

The Taylor series for the hyperbolic functions are closely related to those of the trigonometric functions.

$$
\begin{aligned}
\sinh x &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \infty\,, \\
\cosh x &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \infty\,, \\
\tanh x &= \sum_{n=0}^{\infty} T_{2n+1} \frac{x^{2n+1}}{(2n+1)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,, \\
x \coth x &= 1 + \sum_{n=1}^{\infty} 2^{2n} B_{2n} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \pi\,, \\
\operatorname{sech} x &= \sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \tfrac{1}{2}\pi\,, \\
x \operatorname{csch} x &= 1 - \sum_{n=1}^{\infty} 2(2^{2n-1}-1) B_{2n} \frac{x^{2n}}{(2n)!}\,, \qquad |x| < \pi\,.
\end{aligned}
$$

Finally, we examine the Taylor series of the inverse trigonometric and inverse hyperbolic functions. We list only those functions that possess Taylor series about $x = 0$.

$$
\begin{aligned}
\arcsin x = \tfrac{1}{2}\pi - \arccos x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}\,, \qquad |x| \leq 1\,, \\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}\,, \qquad |x| \leq 1\,, \\
\operatorname{arcsinh} x &= \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}\,, \qquad |x| \leq 1\,, \\
\operatorname{arctanh} x &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}\,, \qquad |x| < 1\,.
\end{aligned}
$$

The Taylor series above for $\arcsin x$, $\arccos x$ and $\arctan x$ correspond to the corresponding principal values of these functions, respectively. In our conventions, $\operatorname{arccot} x \equiv \arctan(1/x)$ is not continuous at $x = 0$ and thus does not possess a Taylor series about $x = 0$. For further details, see the class handout on the inverse trigonometric and hyperbolic functions.

To end these notes, I will provide two simple algorithms for generating the Bernoulli numbers and Euler numbers. Tangent numbers can then be obtained from eq. (8). All Bernoulli numbers can be evaluated starting with B_2 by using

$$
\sum_{n=1}^{N} \binom{2N}{2n-1} \frac{B_{2n}}{2n} = \frac{2N-1}{2(2N+1)}\,, \qquad N = 1, 2, 3, \ldots\,, \tag{9}
$$

and all Euler numbers can be evaluated starting with E_2 by using

$$\sum_{n=1}^{N} \binom{2N}{2n} E_{2n} = -1\,, \qquad N = 1, 2, 3, \ldots\,, \tag{10}$$

where the binomial coefficient above is defined in eq. (4). Eqs. (9) and (10) can be used to build up the table of Bernoulli and Euler numbers presented earlier in this note.

You may wonder why I have not presented a formula for directly computing the Bernoulli and Euler numbers. Such formulae do exist, but they are not simple. Here is one example:

$$B_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} \sum_{j=0}^{k-1} j^n\,.$$

The only formulae that exist for computing a particular Bernoulli or Euler number necessarily involve at least a double sum.

References

1. An excellent reference book for Taylor series of functions and many other properties of mathematical functions can be found in Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1965). This resource is available free on the web and can be either viewed or downloaded from `http://people.math.sfu.ca/~cbm/aands/`.

2. A reference that updates the classic reference book of Abramowitz and Stegun (see Ref. 1) is the *NIST Handbook of Mathematical Functions*, edited by Frank W.J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark (Cambridge University Press, Cambridge, UK, 2010). An electronic form of this book is available free on the web and can be viewed from `http://dlmf.nist.gov/`.