Various kinds of Fourier series

1. Fourier series on the interval $-\ell \le x \le \ell$

Consider the expansion of the function $f(x)$ in a Fourier series, which is defined on the interval $-\ell \le x \le \ell$. Using the results of Chapter 7, section 8 of Boas on pp. 360–362, the Fourier series of $f(x)$ is given by[1]

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right), \qquad \text{for } -\ell \le x \le \ell} \tag{1}$$

The coefficients a_n and b_n are determined by the following formulae [cf. eq. (8.3) on p. 362 of Boas, but note the slightly different convention that is mentioned in footnote 1]:

$$a_n = \frac{1}{\ell(1+\delta_{n0})} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots \tag{2}$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx\,, \qquad \text{for } n = 1, 2, 3, 4, \ldots \tag{3}$$

where the factor of δ_{n0} is a Kronecker delta that is defined as

$$\delta_{n0} = \begin{cases} 1\,, & \text{if } n = 0\,, \\ 0\,, & \text{if } n = 1, 2, 3, \ldots. \end{cases}$$

Eqs. (2) and (3) are a consequence of the orthogonality and completeness of the set of functions $\bigl\{\cos(n\pi x/\ell)\,, \sin(n\pi x/\ell)\bigr\}$ for $n = 0, 1, 2, 3, \ldots$, on the interval $-\ell \le x \le \ell$. Note that $n = 0$ is omitted for the sine function, since the zero function is not relevant for specifying a complete set of functions. The corresponding orthogonality relations are:

$$\int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) dx = \ell(1+\delta_{n0})\,\delta_{mn}\,, \qquad \text{for } m, n = 0, 1, 2, 3, \ldots, \tag{4}$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) dx = \ell\,\delta_{mn}\,, \qquad \text{for } m, n = 1, 2, 3, \ldots, \tag{5}$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) dx = 0\,, \qquad \text{for } m = 1, 2, 3, \ldots \text{ and } n = 0, 1, 2, 3, \ldots \tag{6}$$

[1] Warning: the coefficient a_0 in eq. (1) is denoted by $\frac{1}{2}a_0$ in eq. (8.2) on p. 361 of Boas.

Eqs. (4)–(6) are easily derived by making use of the trigonometric identities,

$$\cos A\cos B = \tfrac{1}{2}\big[\cos(A-B)+\cos(A+B)\big]\,,$$
$$\sin A\sin B = \tfrac{1}{2}\big[\cos(A-B)-\cos(A+B)\big]\,,$$
$$\sin A\cos B = \tfrac{1}{2}\big[\sin(A-B)+\sin(A+B)\big]\,.$$

It follows that if $n\neq m$, then

$$\int_{-\ell}^{\ell}\cos\left(\frac{m\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\left\{\cos\left(\frac{(m-n)\pi x}{\ell}\right)+\cos\left(\frac{(m+n)\pi x}{\ell}\right)\right\}\,dx$$
$$= \frac{\ell}{2(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{\ell}\right)\bigg|_{-\ell}^{\ell} + \frac{\ell}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{\ell}\right)\bigg|_{-\ell}^{\ell} = 0\,,$$

$$\int_{-\ell}^{\ell}\sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\left\{\cos\left(\frac{(m-n)\pi x}{\ell}\right)-\cos\left(\frac{(m+n)\pi x}{\ell}\right)\right\}\,dx$$
$$= \frac{\ell}{2(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{\ell}\right)\bigg|_{-\ell}^{\ell} - \frac{\ell}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{\ell}\right)\bigg|_{-\ell}^{\ell} = 0\,,$$

$$\int_{-\ell}^{\ell}\sin\left(\frac{m\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\left\{\sin\left(\frac{(m-n)\pi x}{\ell}\right)+\sin\left(\frac{(m+n)\pi x}{\ell}\right)\right\}\,dx = 0\,.$$

The last result follows from the fact that $\sin(cx)$ is an odd function of x for any non-zero value of c. In the case of $n=m$, we make use of the trigonometric identities,

$$\cos^2 A = \tfrac{1}{2}(1+\cos 2A)\,,\qquad \sin^2 A = \tfrac{1}{2}(1-\cos 2A)\,,\qquad \cos A\sin A = \tfrac{1}{2}\sin 2A\,.$$

It follows that

$$\int_{-\ell}^{\ell}\cos^2\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\left\{1+\cos\left(\frac{2n\pi x}{\ell}\right)\right\}\,dx = \ell(1+\delta_{m0})\,,\qquad \text{for } n=0,1,2,3,\ldots,$$

$$\int_{-\ell}^{\ell}\sin^2\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\left\{1-\cos\left(\frac{2n\pi x}{\ell}\right)\right\}\,dx = \ell\,,\qquad \text{for } n=1,2,3,\ldots,$$

$$\int_{-\ell}^{\ell}\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\sin\left(\frac{2n\pi x}{\ell}\right)\,dx = 0\,,\qquad \text{for } n=1,2,3,\ldots,$$

and eqs. (4)–(6) are thus established.

Using the orthogonality relations of eqs. (4)–(6), one can easily derive eqs. (2) and (3) for the coefficients a_n and b_n of the Fourier series. First, we multiply both sides of eq. (1) by $\cos(m\pi x/\ell)$ and then integrate both sides of the resulting equation from $-\ell$ to ℓ. Using eqs. (4) and (6), we obtain

$$\int_{-\ell}^{\ell} f(x)\cos\left(\frac{m\pi x}{\ell}\right)\,dx = \sum_{n=0}^{\infty} a_n\ell(1+\delta_{n0})\,\delta_{mn} = a_m\ell(1+\delta_{m0})\,.$$

Hence,

$$a_m = \frac{1}{\ell(1+\delta_{m0})} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{m\pi x}{\ell}\right)\, dx\,,$$

which confirms eq. (2). Likewise, we multiply both sides of eq. (1) by $\sin(m\pi x/\ell)$ and then integrate both sides of the resulting equation from $-\ell$ to ℓ. Using eqs. (5) and (6), we obtain

$$\int_{-\ell}^{\ell} f(x) \sin\left(\frac{m\pi x}{\ell}\right)\, dx = \sum_{n=0}^{\infty} b_n \ell\, \delta_{mn} = b_m \ell\,.$$

Hence,

$$b_m = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{m\pi x}{\ell}\right)\, dx\,,$$

which confirms eq. (3).

2. Fourier sine series on the interval $0 \leq x \leq \ell$

In solving second-order linear partial differential equations subject to boundary conditions, one often encounters a Fourier series, defined on the interval $0 \leq x \leq \ell$, that is composed only of sine functions,

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)\,, \qquad \text{for } 0 \leq x \leq \ell} \tag{7}$$

This arises in Dirichlet problems, where the solutions to the partial differential equation are required to vanish at $x = 0$. Then, it follows that

$$\boxed{b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx\,, \qquad \text{for } n = 1, 2, 3, \ldots} \tag{8}$$

To prove eq. (8), we employ the following trick. We pretend that $f(x)$ is an odd function of x, i.e. $f(-x) = -f(x)$, that is defined on the larger interval $-\ell \leq x \leq \ell$. One can now make use of the results of section 1. In particular, the extended function $f(x)$ can now be represented by the Fourier series given in eq. (1). Then it immediately follows that

$$a_n = \frac{1}{\ell(1+\delta_{n0})} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right)\, dx = 0\,,$$

since the integrand above, $f(x)\cos(n\pi x/\ell)$, is an odd function of x that vanishes when integrated symmetrically about the origin. Hence, $f(x)$ can be represented by a Fourier sine series. Moreover, using eq. (3),

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx\,, \qquad \text{for } n = 1, 2, 3, \ldots,$$

since $f(x)\sin(n\pi x/\ell)$ is an even function of x. Thus, we have confirmed eq. (8). Since these results have been established on the interval $-\ell \leq x \leq \ell$, they clearly apply for $0 \leq x \leq \ell$.

Indeed, the set of functions $\{\sin(n\pi x/\ell)\}$, for $n = 1, 2, 3, \ldots$ is orthogonal and complete on the interval $0 \le x \le \ell$. This is proven using the methods of section 1. In particular, for $m \neq n$,

$$\int_0^\ell \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_0^\ell \left\{\cos\left(\frac{(m-n)\pi x}{\ell}\right) - \cos\left(\frac{(m+n)\pi x}{\ell}\right)\right\}\,dx$$

$$= \frac{\ell}{2(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{\ell}\right)\bigg|_0^\ell - \frac{\ell}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{\ell}\right)\bigg|_0^\ell = 0\,,$$

and for $m = n$,

$$\int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_0^\ell \left\{1 - \cos\left(\frac{2n\pi x}{\ell}\right)\right\}\,dx = \tfrac{1}{2}\ell\,, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Thus, the orthogonality relations are given by

$$\boxed{\int_0^\ell \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)\,dx = \tfrac{1}{2}\ell\,\delta_{mn}\,, \qquad \text{for } m, n = 1, 2, 3, \ldots} \tag{9}$$

We can use eq. (9) to establish eq. (8) by multiplying both sides of eq. (7) by $\sin(m\pi x/\ell)$ and integrating the resulting equation from 0 to ℓ.

3. Fourier cosine series on the interval $0 \le x \le \ell$

In solving second-order linear partial differential equations subject to boundary conditions, one often encounters a Fourier series, defined on the interval $0 \le x \le \ell$, that is composed only of cosine functions,

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)\,, \qquad \text{for } 0 \le x \le \ell} \tag{10}$$

This arises in Neumann problems, where the derivative of the solutions to the partial differential equation are required to vanish at $x = 0$. Then, it follows that

$$\boxed{a_n = \frac{2}{\ell(1+\delta_{n0})}\int_0^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots} \tag{11}$$

To prove eq. (11), we can employ the following trick. We pretend that $f(x)$ is an even function of x, i.e. $f(-x) = f(x)$, that is defined on the interval $-\ell \le x \le x$. One can now make use of the results of section 1. In particular, the extended function $f(x)$ can now be represented by the Fourier series given in eq. (1). Then it immediately follows that

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx = 0\,,$$

since the integrand above, $f(x)\sin(n\pi x/\ell)$, is an odd function of x that vanishes when integrated symmetrically about the origin. Hence, $f(x)$ can be represented by a Fourier cosine series. Moreover, using eq. (2),

$$a_n = \frac{1}{\ell(1+\delta_{n0})}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{2}{\ell(1+\delta_{n0})}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx\,, \qquad \text{for } n=0,1,2,3,\ldots,$$

since $f(x)\cos(n\pi x/\ell)$ is an even function of x. Thus, we have confirmed eq. (11). Since these results have been established on the interval $-\ell \leq x \leq \ell$, they clearly apply for $0 \leq x \leq \ell$.

Indeed, the set of functions $\{\cos(n\pi x/\ell)\}$, for $n = 0,1,2,3,\ldots$ is orthogonal and complete on the interval $0 \leq x \leq \ell$. This is proven using the methods of section 1. In particular, for $m \neq n$,

$$\begin{aligned}\int_0^{\ell}\cos\left(\frac{m\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\,dx &= \frac{1}{2}\int_0^{\ell}\left\{\cos\left(\frac{(m-n)\pi x}{\ell}\right)+\cos\left(\frac{(m+n)\pi x}{\ell}\right)\right\}\,dx \\ &= \frac{\ell}{2(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{\ell}\right)\bigg|_0^{\ell} + \frac{\ell}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{\ell}\right)\bigg|_0^{\ell} = 0\,,\end{aligned}$$

and for $m = n$,

$$\int_0^{\ell}\cos^2\left(\frac{n\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_0^{\ell}\left\{1+\cos\left(\frac{2n\pi x}{\ell}\right)\right\}\,dx = \tfrac{1}{2}\ell(1+\delta_{n0})\,, \qquad \text{for } n=0,1,2,3,\ldots.$$

Thus, the orthogonality relations are given by

$$\boxed{\int_0^{\ell}\cos\left(\frac{m\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\,dx = \tfrac{1}{2}\ell(1+\delta_{n0})\,\delta_{mn}\,, \qquad \text{for } m,n=0,1,2,3,\ldots} \tag{12}$$

We can use eq. (12) to establish eq. (11) by multiplying both sides of eq. (10) by $\cos(m\pi x/\ell)$ and integrating both sides of the resulting equation from 0 to ℓ.

4. Fourier sine series summed over odd integers on the interval $0 \leq x \leq \frac{1}{2}\ell$

In solving second-order linear partial differential equations subject to boundary conditions, one sometimes encounters a Fourier series, defined on the interval $0 \leq x \leq \frac{1}{2}\ell$, that is composed only of sine functions summed over odd integers,[2]

$$\boxed{f(x) = \sum_{\substack{n=1\\ \text{odd } n}}^{\infty} b_n\sin\left(\frac{n\pi x}{\ell}\right) = \sum_{n=0}^{\infty} b_{2n+1}\sin\left(\frac{(2n+1)\pi x}{\ell}\right)\,, \qquad \text{for } 0 \leq x \leq \tfrac{1}{2}\ell} \tag{13}$$

[2] A Fourier series, defined on the interval $0 \leq x \leq \frac{1}{2}\ell$ and composed only of sine functions summed over even integers, reduces to eq. (7) with ℓ replaced by $\frac{1}{2}\ell$.

This occurs in problems with mixed boundary conditions, e.g. where the solutions to the partial differential equation vanishes at $x = 0$ and its derivative vanishes at $x = \frac{1}{2}\ell$. It follows that

$$\boxed{b_{2n+1} = \frac{4}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\sin\left(\frac{(2n+1)\pi x}{\ell}\right)dx\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots} \tag{14}$$

Indeed, the set of functions $\{\sin[(2n+1)\pi x/\ell]\}$, for $n = 0, 1, 2, 3, \ldots$ is orthogonal and complete on the interval $0 \leq x \leq \frac{1}{2}\ell$. This is proven using the methods of section 1. In particular, for $m \neq n$,

$$\begin{aligned}
&\int_0^{\frac{1}{2}\ell} \sin\left(\frac{(2m+1)\pi x}{\ell}\right)\sin\left(\frac{(2n+1)n\pi x}{\ell}\right)dx \\
&\qquad = \frac{1}{2}\int_0^{\frac{1}{2}\ell}\left\{\cos\left(\frac{2(m-n)\pi x}{\ell}\right) - \cos\left(\frac{2(m+n+1)\pi x}{\ell}\right)\right\}dx \\
&\qquad = \frac{\ell}{4(m-n)\pi}\sin\left(\frac{2(m-n)\pi x}{\ell}\right)\Bigg|_0^{\frac{1}{2}\ell} - \frac{\ell}{4(m+n+1)\pi}\sin\left(\frac{2(m+n+1)\pi x}{\ell}\right)\Bigg|_0^{\frac{1}{2}\ell} = 0\,,
\end{aligned}$$

and for $m = n$,

$$\int_0^{\frac{1}{2}\ell}\sin^2\left(\frac{(2n+1)\pi x}{\ell}\right)dx = \frac{1}{2}\int_0^{\frac{1}{2}\ell}\left\{1 - \cos\left(\frac{2(2n+1)\pi x}{\ell}\right)\right\}dx = \tfrac{1}{4}\ell\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots.$$

Thus, the orthogonality relations are given by:

$$\boxed{\int_0^{\frac{1}{2}\ell}\sin\left(\frac{(2m+1)\pi x}{\ell}\right)\sin\left(\frac{(2n+1)\pi x}{\ell}\right)dx = \tfrac{1}{4}\ell\,\delta_{mn}\,, \qquad \text{for } m, n = 0, 1, 2, 3, \ldots}$$

To understand the origin of eq. (13), we note that $\sin[(2n+1)\pi x/\ell]$ is even with respect to $x \to \ell - x$, since

$$\begin{aligned}
\sin\left(\frac{(2n+1)\pi(\ell-x)}{\ell}\right) &= \sin\left((2n+1)\pi - \frac{(2n+1)\pi x}{\ell}\right) \\
&= \sin\bigl((2n+1)\pi\bigr)\cos\left(\frac{(2n+1)\pi x}{\ell}\right) - \cos\bigl((2n+1)\pi\bigr)\sin\left(\frac{(2n+1)\pi x}{\ell}\right) \\
&= \sin\left(\frac{(2n+1)\pi x}{\ell}\right).
\end{aligned} \tag{15}$$

One can pretend that $f(x)$ is an even function with respect to $x \to \ell - x$ that is defined on the interval $0 \leq x \leq \ell$. Then, $f(x)$ possesses the Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right), \qquad \text{for } 0 \leq x \leq \ell\,, \tag{16}$$

where $f(\ell - x) = f(x)$. Hence, we may use eq. (8) to obtain the coefficients in eq. (16).

First, we compute

$$b_{2n} = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{2n\pi x}{\ell}\right)dx = 0\,. \tag{17}$$

To verify eq. (17), we note that $\sin(2n\pi x/\ell)$ is odd with respect to $x \to \ell - x$ as shown below,

$$\sin\left(\frac{2n\pi(\ell-x)}{\ell}\right) = \sin\left(2n\pi - \frac{n\pi x}{\ell}\right)$$
$$= \sin(2n\pi)\cos\left(\frac{2n\pi x}{\ell}\right) - \cos(2n\pi)\sin\left(\frac{2n\pi x}{\ell}\right)$$
$$= -\sin\left(\frac{2n\pi x}{\ell}\right). \tag{18}$$

Thus, if we define $y \equiv \ell - x$, then

$$b_{2n} = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{2n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\sin\left(\frac{2n\pi x}{\ell}\right)dx + \frac{2}{\ell}\int_{\frac{1}{2}\ell}^{\ell} f(x)\sin\left(\frac{2n\pi x}{\ell}\right)dx$$
$$= \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\sin\left(\frac{2n\pi x}{\ell}\right)dx - \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(y)\sin\left(\frac{2n\pi y}{\ell}\right)dy = 0\,,$$

where we have used $f(\ell - y) = f(y)$ and eq. (18).

Second, by employing eq. (15), we find

$$b_{2n+1} = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{(2n+1)\pi x}{\ell}\right)dx \tag{19}$$
$$= \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\sin\left(\frac{(2n+1)\pi x}{\ell}\right)dx + \frac{2}{\ell}\int_{\frac{1}{2}\ell}^{\ell} f(x)\sin\left(\frac{(2n+1)\pi x}{\ell}\right)dx$$
$$= \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\sin\left(\frac{(2n+1)\pi x}{\ell}\right)dx + \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(y)\sin\left(\frac{(2n+1)\pi y}{\ell}\right)dy$$
$$= \frac{4}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\sin\left(\frac{(2n+1)\pi x}{\ell}\right)dx\,,$$

which confirms that the Fourier sine series corresponds to a sum over odd integers, with coefficients given by eq. (14).

5. Fourier cosine series summed over odd integers on the interval $0 \leq x \leq \frac{1}{2}\ell$

In solving second-order linear partial differential equations subject to boundary conditions, one sometimes encounters a Fourier series, defined on the interval $0 \leq x \leq \frac{1}{2}\ell$, that is composed only of cosine functions summed over odd integers,[3]

$$\boxed{f(x) = \sum_{\substack{n=1\\ \text{odd } n}}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) = \sum_{n=0}^{\infty} a_{2n+1}\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,, \qquad \text{for } 0 \leq x \leq \tfrac{1}{2}\ell} \tag{20}$$

[3]A Fourier series, defined on the interval $0 \leq x \leq \frac{1}{2}\ell$ and composed only of cosine functions summed over even integers, reduces to eq. (10) with ℓ replaced by $\frac{1}{2}\ell$.

This occurs in problems with mixed boundary conditions, e.g. where the solutions to the partial differential equation vanishes at $x = \frac{1}{2}\ell$ and its derivative vanishes at $x = 0$. It follows that

$$\boxed{a_{2n+1} = \frac{4}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots} \tag{21}$$

Indeed, the set of functions $\{\cos[(2n+1)\pi x/\ell]\}$, for $n = 0, 1, 2, 3, \ldots$ is orthogonal and complete on the interval $0 \leq x \leq \frac{1}{2}\ell$. This is proven using the methods of section 1. In particular, for $m \neq n$,

$$\begin{aligned}
&\int_0^{\frac{1}{2}\ell} \cos\left(\frac{(2m+1)\pi x}{\ell}\right)\cos\left(\frac{(2n+1)n\pi x}{\ell}\right)\,dx \\
&\qquad = \frac{1}{2}\int_0^{\frac{1}{2}\ell}\left\{\cos\left(\frac{2(m-n)\pi x}{\ell}\right) + \cos\left(\frac{2(m+n+1)\pi x}{\ell}\right)\right\}\,dx \\
&\qquad = \frac{\ell}{4(m-n)\pi}\sin\left(\frac{2(m-n)\pi x}{\ell}\right)\bigg|_0^{\frac{1}{2}\ell} + \frac{\ell}{4(m+n+1)\pi}\sin\left(\frac{2(m+n+1)\pi x}{\ell}\right)\bigg|_0^{\frac{1}{2}\ell} = 0\,,
\end{aligned}$$

and for $m = n$,

$$\int_0^{\frac{1}{2}\ell}\cos^2\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx = \frac{1}{2}\int_0^{\frac{1}{2}\ell}\left\{1 + \cos\left(\frac{2(2n+1)\pi x}{\ell}\right)\right\}\,dx = \tfrac{1}{4}\ell\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots.$$

Thus, the orthogonality relations are given by:

$$\boxed{\int_0^{\frac{1}{2}\ell}\cos\left(\frac{(2m+1)\pi x}{\ell}\right)\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx = \tfrac{1}{4}\ell\,\delta_{mn}\,, \qquad \text{for } m, n = 0, 1, 2, 3, \ldots}$$

To understand the origin of eq. (20), we note that $\cos[(2n+1)\pi x/\ell]$ is odd with respect to $x \to \ell - x$, since

$$\begin{aligned}
\cos\left(\frac{(2n+1)\pi(\ell - x)}{\ell}\right) &= \cos\left((2n+1)\pi - \frac{(2n+1)\pi x}{\ell}\right) \\
&= \cos\bigl((2n+1)\pi\bigr)\cos\left(\frac{(2n+1)\pi x}{\ell}\right) + \sin\bigl((2n+1)\pi\bigr)\sin\left(\frac{(2n+1)\pi x}{\ell}\right) \\
&= -\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,.
\end{aligned} \tag{22}$$

One can pretend that $f(x)$ is an odd function with respect to $x \to \ell - x$ that is defined on the interval $0 \leq x \leq \ell$. Then, $f(x)$ possesses the Fourier cosine series,

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)\,, \qquad \text{for } 0 \leq x \leq \ell\,, \tag{23}$$

where $f(\ell - x) = -f(x)$. Hence, we may use eq. (11) to obtain the coefficients in eq. (23).

First, we compute

$$a_{2n} = \frac{2}{\ell(1+\delta_{n0})}\int_0^\ell f(x)\cos\left(\frac{2n\pi x}{\ell}\right)\,dx = 0\,. \tag{24}$$

To verify eq. (24), we note that $\cos(2n\pi x/\ell)$ is even with respect to $x \to \ell - x$ as shown below,

$$\begin{aligned}
\cos\left(\frac{2n\pi(\ell-x)}{\ell}\right) &= \cos\left(2n\pi - \frac{2n\pi x}{\ell}\right) \\
&= \cos(2n\pi)\cos\left(\frac{2n\pi x}{\ell}\right) + \cos(2n\pi)\sin\left(\frac{2n\pi x}{\ell}\right) \\
&= \cos\left(\frac{2n\pi x}{\ell}\right)\,.
\end{aligned} \tag{25}$$

Thus, if we define $y \equiv \ell - x$, then

$$\begin{aligned}
a_{2n} &= \frac{2}{\ell(1+\delta_{n0})}\int_0^\ell f(x)\cos\left(\frac{2n\pi x}{\ell}\right)\,dx \\
&= \frac{2}{\ell(1+\delta_{n0})}\int_0^{\frac{1}{2}\ell} f(x)\cos\left(\frac{2n\pi x}{\ell}\right)\,dx + \frac{2}{\ell(1+\delta_{n0})}\int_{\frac{1}{2}\ell}^{\ell} f(x)\cos\left(\frac{2n\pi x}{\ell}\right)\,dx \\
&= \frac{2}{\ell(1+\delta_{n0})}\int_0^{\frac{1}{2}\ell} f(x)\cos\left(\frac{2n\pi x}{\ell}\right)\,dx - \frac{2}{\ell(1+\delta_{n0})}\int_0^{\frac{1}{2}\ell} f(y)\cos\left(\frac{2n\pi y}{\ell}\right)\,dy = 0\,,
\end{aligned}$$

where we have used $f(\ell - y) = -f(y)$ and eq. (25).

Second, by employing eq. (22), we find

$$\begin{aligned}
a_{2n+1} &= \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx \qquad (26)\\
&= \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx + \frac{2}{\ell}\int_{\frac{1}{2}\ell}^{\ell} f(x)\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx \\
&= \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx + \frac{2}{\ell}\int_0^{\frac{1}{2}\ell} f(y)\cos\left(\frac{(2n+1)\pi y}{\ell}\right)\,dy \\
&= \frac{4}{\ell}\int_0^{\frac{1}{2}\ell} f(x)\cos\left(\frac{(2n+1)\pi x}{\ell}\right)\,dx\,,
\end{aligned}$$

which confirms that the Fourier cosine series corresponds to a sum over odd integers, with coefficients given by eq. (21).

Reference

Georgi P. Tolstov, *Fourier Series* (Dover Publications, Inc., Mineola, NY, 1976).