Uniqueness of solutions to the Laplace and Poisson equations

1. Introduction

In these notes, I shall address the uniqueness of the solution to the Poisson equation,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} u(\vec{\boldsymbol{x}}) = f(\vec{\boldsymbol{x}}) , \tag{1}$$

subject to certain boundary conditions. That is, suppose that there is a region of space of volume V and the boundary of that surface is denoted by S. Then, assuming that $f(\vec{\boldsymbol{x}})$ is a known function of $\vec{\boldsymbol{x}}$, is there a unique solution to eq. (1) for all points $\vec{\boldsymbol{x}}$ inside the volume V given certain "data" on S (for example, the data could consist of the values of $u(\vec{\boldsymbol{x}})$ for all points $\vec{\boldsymbol{x}} \in S$)?[1] The question of uniqueness also applies to solutions of the Laplace equation, $\vec{\boldsymbol{\nabla}}^{\mathbf{2}} u(\vec{\boldsymbol{x}}) = 0$, which corresponds to choosing $f(\vec{\boldsymbol{x}}) = 0$ in eq. (1).

Another question one could ask is whether a nontrivial solution[2] to eq. (1) subject to certain boundary conditions is guaranteed to exist. The question of existence is usually more difficult to address as compared with the question of uniqueness. But, in these notes, we imagine that we have discovered a solution to eq. (1) subject to certain boundary conditions on S by following some procedure (such as the separation of variables technique or by the time honored method of the educated guess). Having found one solution, we would like to know whether this solution is unique. If yes, then the problem is completely solved.

In some cases, it is sufficient to prove that a given solution is unique up to an arbitrary additive constant. In certain physical applications, this is equivalent to finding a unique solution to the problem of interest. For example, in electrostatics, the electric potential $\Phi(\vec{\boldsymbol{x}})$, in the absence of charge, is a solution to Laplace's equation, $\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\Phi = 0$. The actual physical quantity of interest is the electric field, $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi$. Clearly, it is sufficient to determine $\Phi(\vec{\boldsymbol{x}})$ up to an arbitrary additive constant, which has no impact on the value of the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ at the point $\vec{\boldsymbol{x}}$.

In the theory of linear partial differential equations, a *well-posed problem* consists of a linear partial differential equation subject to certain boundary conditions such that the solution is unique.[3] Before imposing the boundary condition, the general solution to the linear partial differential equation typically is expressed in terms of some unknown function of a certain form. By imposing the boundary conditions, these unknown functions are

[1] The considerations of these notes also apply to solving for $u(\vec{\boldsymbol{x}})$ for all points $\vec{\boldsymbol{x}}$ that lie *outside* the volume V given the values of $u(\vec{\boldsymbol{x}})$ for all points $\vec{\boldsymbol{x}} \in S$.

[2] The trivial solution, $u(\vec{\boldsymbol{x}}) = 0$, is always a solution to Laplace's equation, but such a solution is of no interest when solving practical problems.

[3] For convenience, we shall also regard the problem as well-posed if the solution is unique up to an overall additive constant.

then determined precisely with no attendant free parameters (except perhaps an overall additive constant). The corresponding problem is considered *ill-posed* if the boundary conditions imposed are either (i) not sufficient to yield a unique solution to the problem, or (ii) over-constrained so that no nontrivial solution to the problem exists.

2. A taxonomy of boundary conditions

We shall consider four classes of possible boundary conditions that could be applied to the solution of Poisson's equation. As before, we assume that a closed surface S is the boundary of some volume V. The volume V may be infinite, in which case the closed surface would include the point of infinity.

1. **Dirichlet boundary conditions.** The values of $u(\vec{\boldsymbol{x}})$ are specified for all points $\vec{\boldsymbol{x}} \in S$.

2. **Neumann boundary conditions.** The values of the normal derivative of $u(\vec{\boldsymbol{x}})$,

$$\frac{\partial u(\vec{\boldsymbol{x}})}{\partial n} \equiv \hat{\boldsymbol{n}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} u(\vec{\boldsymbol{x}})$$

are specified for all points $\vec{\boldsymbol{x}} \in S$. The normal vector $\hat{\boldsymbol{n}}$ (which depends on $\vec{\boldsymbol{x}}$) is the unit vector that is perpendicular to the surface S and points outwards from S at the point $\vec{\boldsymbol{x}}$.

3. **Mixed boundary conditions.** Suppose that the closed surface S is comprised of two sub-surfaces S_1 and S_2. Mixed boundary conditions consist of Dirichlet boundary conditions on S_1 and Neumann boundary conditions on S_2 (or vice versa).

4. **Cauchy boundary conditions.** The values of $u(\vec{\boldsymbol{x}})$ and $\partial u(\vec{\boldsymbol{x}})/\partial n$ are simultaneously specified for all points $\vec{\boldsymbol{x}} \in S$.

Then, one can prove that the Poisson equation subject to certain boundary conditions is ill-posed if Cauchy boundary conditions are imposed. In the Cauchy case, the boundary conditions are too constraining and in general there is no solution (or in the case of Laplace's equation only the trivial solution exists). For the case of Dirichlet boundary conditions or mixed boundary conditions, the solution to Poisson's equation always exists and is unique. Finally, for the case of the Neumann boundary condition, a solution may or may not exist (depending on whether a certain condition [cf. eq. (4)] is satisfied). If a solution exists, then it is unique up to an overall additive constant.

3. The uniqueness of solutions to the Poisson equation with Dirichlet boundary conditions

As remarked at the beginning of these notes, we are primarily interested in determining the uniqueness of the solution assuming that a solution has been exhibited. The proof of uniqueness is straightforward. Consider the closed surface S and the enclosed volume V where $\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}} u(\vec{\boldsymbol{x}}) = f(\vec{\boldsymbol{x}})$. If $u_1(\vec{\boldsymbol{x}})$ and $u_2(\vec{\boldsymbol{x}})$ are two solutions of the Poisson equation, both of which satisfy the same Dirichlet boundary conditions, then $u_1(\vec{\boldsymbol{x}}) = u_2(\vec{\boldsymbol{x}})$ for all points $\vec{\boldsymbol{x}} \in S$. It follows that the quantity $\Phi(\vec{\boldsymbol{x}}) \equiv u_1(\vec{\boldsymbol{x}}) - u_2(\vec{\boldsymbol{x}})$ satisfies the Laplace equation, $\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\Phi(\vec{\boldsymbol{x}}) = 0$, subject to the condition that $\Phi(\vec{\boldsymbol{x}}) = 0$ for all $\vec{\boldsymbol{x}} \in S$.

Consider the volume integral,

$$J \equiv \int_V (\vec{\boldsymbol{\nabla}}\Phi)^2\, dV > 0\,, \qquad \text{assuming that } \Phi(\vec{\boldsymbol{x}}) \neq c \text{ inside } V\text{, for some constant } c\,. \quad (2)$$

Note that if $\Phi(\vec{\boldsymbol{x}}) = c$ for some constant c, then $\vec{\boldsymbol{\nabla}}\Phi = 0$ for all points $\vec{\boldsymbol{x}} \in V$, which yields $J = 0$. Otherwise J is positive, since it picks up positive contributions from volume elements where $\vec{\boldsymbol{\nabla}}\Phi \neq 0$. Using the vector identity (which is a consequence of the derivative of the product rule),

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\Phi\vec{\boldsymbol{\nabla}}\Phi) = (\vec{\boldsymbol{\nabla}}\Phi)^2 + \Phi(\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\Phi)\,,$$

it follows that when Φ satisfies Laplace's equation, $\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}}\Phi = 0$, then

$$(\vec{\boldsymbol{\nabla}}\Phi)^2 = \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\Phi\vec{\boldsymbol{\nabla}}\Phi)\,.$$

Inserting this result into eq. (2) yields,

$$J = \int_V \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\Phi\vec{\boldsymbol{\nabla}}\Phi) = \oint_S \Phi\, \hat{\boldsymbol{n}}\boldsymbol{\cdot}\, \vec{\boldsymbol{\nabla}}\Phi\, dS\,,$$

where we have used the divergence theorem [cf. eq. (10.17) on p. 318 of Boas] to convert the volume integral over V into a surface integral over the closed surface S. That is, we can write

$$J = \oint_S \Phi \frac{\partial\Phi}{\partial n}\, dS\,. \qquad (3)$$

Applying this result to $\Phi(\vec{\boldsymbol{x}}) \equiv u_1(\vec{\boldsymbol{x}}) - u_2(\vec{\boldsymbol{x}})$, we note that $\Phi(\vec{\boldsymbol{x}}) = 0$ for all $\vec{\boldsymbol{x}} \in S$. Hence, it immediately follows that $J = 0$. But $J = 0$ is possible only if $\Phi(\vec{\boldsymbol{x}})$ is a constant inside V [cf. eq. (2) above]. This means that in the limit where $\vec{\boldsymbol{x}}$ approaches the surface S, $\Phi(\vec{\boldsymbol{x}})$ is a constant. Consequently, this constant must be zero since $\Phi(\vec{\boldsymbol{x}}) = 0$ for all $\vec{\boldsymbol{x}} \in S$. Thus, we have proven that $\Phi(\vec{\boldsymbol{x}}) = 0$ inside V, which is equivalent to the statement that $u_1(\vec{\boldsymbol{x}}) = u_2(\vec{\boldsymbol{x}})$ for all points $\vec{\boldsymbol{x}} \in V$. Therefore, we have demonstrated that if $u_1(\vec{\boldsymbol{x}})$ and $u_2(\vec{\boldsymbol{x}})$ are two solutions, both of which satisfy the same Dirichlet boundary conditions, then $u_1(\vec{\boldsymbol{x}}) = u_2(\vec{\boldsymbol{x}})$ for all points $\vec{\boldsymbol{x}} \in V$. In other words, the solution to eq. (1) subject to Dirichlet boundary conditions is unique, assuming that the solution exists in the first place. The proof just presented does not address the question of the existence of a solution. To prove existence requires techniques beyond the scope of these notes (see e.g., Ref. 1).

4. The uniqueness of solutions to the Poisson equation (up to an additive constant) subject to Neumann boundary conditions

Alternatively, consider the case where $u_1(\vec{\boldsymbol{x}})$ and $u_2(\vec{\boldsymbol{x}})$ are two solutions, both of which satisfy the same Neumann boundary conditions. In this case,

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}\,, \qquad \text{for all points } \vec{\boldsymbol{x}} \in S\,.$$

Then, the quantity $\Phi(\vec{\boldsymbol{x}}) \equiv u_1(\vec{\boldsymbol{x}}) - u_2(\vec{\boldsymbol{x}})$ satisfies Laplace's equation, $\vec{\boldsymbol{\nabla}}^2\Phi(\vec{\boldsymbol{x}}) = 0$, subject to the condition that $\partial\Phi/\partial n = 0$ for all $\vec{\boldsymbol{x}} \in S$. We can again define the volume integral J as in eq. (2) and show that eq. (3) is satisfied. Applying eq. (3) to the quantity $\Phi(\vec{\boldsymbol{x}}) \equiv u_1(\vec{\boldsymbol{x}}) - u_2(\vec{\boldsymbol{x}})$, it immediately follows that $J = 0$, since $\partial\Phi/\partial n = 0$ for all $\vec{\boldsymbol{x}} \in S$. By the same argument as before, we conclude that $\Phi(\vec{\boldsymbol{x}})$ is a constant inside V, which is equivalent to the statement that $u_1(\vec{\boldsymbol{x}}) - u_2(\vec{\boldsymbol{x}})$ is a constant for all points $\vec{\boldsymbol{x}} \in V$. But the condition that $\partial\Phi/\partial n = 0$ for all $\vec{\boldsymbol{x}} \in S$ does not determine the value of $\Phi(\vec{\boldsymbol{x}})$ for $\vec{\boldsymbol{x}} \in S$, so we cannot conclude in this case that the constant is zero. Therefore, we have demonstrated that if $u_1(\vec{\boldsymbol{x}})$ and $u_2(\vec{\boldsymbol{x}})$ are two solutions, both of which satisfy the same Neumann boundary conditions, then $u_1(\vec{\boldsymbol{x}}) - u_2(\vec{\boldsymbol{x}}) = c$ for all points $\vec{\boldsymbol{x}} \in V$, where c is an undetermined constant. In other words, the solution to eq. (1) subject to Neumann boundary conditions is unique up to an overall undetermined additive constant, assuming that the solution exists in the first place. The existence of the undetermined overall constant is not a surprise, since one can always add a constant to the solution to the Neumann problem and still have a solution that satisfies the original Poisson equation subject to the Neumann boundary conditions.

Once again, the proof just presented does not address the question of existence of a solution. However, there is an important consistency condition that must be satisfied in order that a solution to the Neumann problem exist. Staring with eq. (1), we shall integrate both sides of the equation over the volume V. Since,

$$\int_V \vec{\boldsymbol{\nabla}}^2 u = \int_V \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} u = \oint_S \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\nabla}} u \,,$$

due to the divergence theorem, eq. (1) yields the consistency condition,

$$\oint \frac{\partial u}{\partial n}\, dS = \int_V f(\vec{\boldsymbol{x}})\, dV \,. \tag{4}$$

The right hand side of eq. (4) is known a priori, since the function $f(\vec{\boldsymbol{x}})$ is known. The left hand side of eq. (4) is determined solely by the Neumann boundary conditions. Thus, these boundary conditions must respect eq. (4) in order for there to be a nontrivial solution to eq. (1) subject to Neumann boundary conditions. If eq. (4) is satisfied, then one can show that a solution must exist (and using the arguments above, this solution is unique up to an overall additive constant). However to prove existence requires techniques beyond the scope of these notes (see e.g., Ref. 1).

As an example, given the Laplace equation subject to homogeneous Neumann boundary conditions, i.e. $\partial u/\partial n = 0$ for all $\vec{\boldsymbol{x}} \in S$, it follows that a solution always exists and is unique up to an overall additive constant. Such a problem arises in electrostatics when one is asked to compute the electric potential in a charge free volume, given the normal component of the electric field at all points on the surface.

5. The uniqueness of solutions to the Poisson equation with mixed boundary conditions

In the case of mixed boundary conditions (Dirichlet on part of S and Neumann on the rest of S), we can again use eq. (3) to conclude that if $u_1(x)$ and $u_2(x)$ are solutions to eq. (1) subject to the same mixed boundary conditions, then $u_1(\vec{\boldsymbol{x}}) - u_2(\vec{\boldsymbol{x}})$ is a constant for all points $\vec{\boldsymbol{x}} \in V$. Since the constant is zero on that part of S where Dirichlet boundary conditions apply, we conclude that $u_1(x) = u_2(x)$ for all points $\vec{\boldsymbol{x}} \in V$. Thus, if a solution exists it must be unique. Once again, uniqueness can be proved using techniques beyond the scope of these notes.

6. Why is the Poisson equation with Cauchy boundary conditions ill-posed?

In the case of Cauchy boundary conditions (where both u and $\partial u/\partial n$ are simultaneously specified on S), the same arguments employed above can again be used to prove that if a solution exists it must be unique. Unfortunately, in almost all cases no solution exists. This is why the Poisson equation subject to Cauchy boundary conditions is ill-posed. The proof of this assertion is simple. Consider the problem of solving the Poisson equation subject to Cauchy boundary conditions. Let us first neglect the data corresponding to $\partial u/\partial n$ on S. In this case, we solve the corresponding Poisson equation subject to Dirichlet boundary conditions. The resulting solution exists and is unique according to the results quoted in Section 3 above. From this unique solution, one can evaluate $\partial u/\partial n$ everywhere in V. Now take the limit as $\vec{\boldsymbol{x}}$ approaches the surface S. This yields $\partial u/\partial n$ for all $\vec{\boldsymbol{x}} \in S$, which in principle could coincide with the data of the Poisson equation with the original Cauchy boundary conditions. In such an exceptional case, the solution to the Dirichlet problem is also the solution to the Cauchy problem, and the solution is unique. However, in general there is no reason to expect the resulting $\partial u/\partial n$ for all $\vec{\boldsymbol{x}} \in S$ to be consistent with the original Cauchy boundary conditions. In this latter case, one must conclude that no solution to the original Cauchy problem exists. This is the reason why no nontrivial solutions exist for the Poisson equation with generic Cauchy boundary conditions. That is, the Poisson equation with Cauchy boundary conditions ill-posed.

7. Generalizations to the diffusion equations and the wave equation

Finally, we note that the uniqueness theorems of the Laplace and Poisson equations can be extended to the time-dependent diffusion and wave equations. The diffusion equation,

$$\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}} u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}\,,$$

is first order in time. Thus, the diffusion equation is well-posed if one specifies an initial condition, $u(\vec{\boldsymbol{x}}, t_0)$ at some initial time t_0 (usually chosen to be $t_0 = 0$) for all $\vec{\boldsymbol{x}} \in V$, and either Dirichlet, Neumann or mixed boundary conditions on the closed surface S for all times $t \geq t_0$. Likewise, the wave equation,

$$\vec{\boldsymbol{\nabla}}^{\boldsymbol{2}} u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}\,,$$

is second order in time. Thus, the wave equation is well-posed if one specifies two initial conditions, $u(\vec{\boldsymbol{x}}, t_0)$ and $(\partial u(\vec{\boldsymbol{x}}, t)/\partial t)_{t=t_0}$ at some initial time t_0 for all $\vec{\boldsymbol{x}} \in V$, and either Dirichlet, Neumann or mixed boundary conditions on the closed surface S for all times $t \geq t_0$. For further details, see Ref. 1.

References:

There are numerous books that treat the topics discussed in these notes. I find that the discussion in Ref. 1 to be especially enlightening, and I have made liberal use of the material on pp. 92–94 of this text in preparing these notes. Other references that you may find useful are also listed below.

1. G. Barton, *Elements of Green's Functions and Propagation: Potentials, Diffusion and Waves* (Oxford Science Publications, Oxford, UK, 1989).

2. Erich Zauderer, *Partial Differential Equations of Applied Mathematics*, 2nd Edition (John Wiley & Sons, New York, NY, 1989).

3. Alan Jeffrey, *Applied Partial Differential Equations* (Academic Press, San Diego, CA, 2003).

4. Yehuda Pinchover and Jacob Rubenstein, *An Introduction to Partial Differential Equations* (Cambridge University Press, Cambridge, UK, 2005).

5. James Brown and Ruel Churchill, *Fourier Series and Boundary Value Problems*, 8th Edition (McGraw-Hill Companies, Inc., New York, NY, 2011).