Regular points and singular points of second-order linear differential equations

Consider the most general homogeneous second-order linear differential equation,[1]

$$y'' + a(x)y' + b(x)y = 0\,, \tag{1}$$

where $y' \equiv dy/dx$ and $y'' \equiv d^2y/dx^2$. The point $x = x_0$ is called an *ordinary point* of the differential equation if $a(x)$ and $b(x)$ possess Taylor series when expanded about x_0 with a nonzero radius of convergence. In this case, $a(x)$ and $b(x)$ are analytic functions of x in a neighborhood of $x = x_0$. If these conditions are not fulfilled, then $a(x)$ or $b(x)$ [or both] are singular $x = x_0$, in which case the point $x = x_0$ is called a *singular point* of the differential equation.

The concepts of ordinary and singular points are easily extended to the case of homogeneous nth order linear differential equations.[2] However, the theory of regular and singular points discussed in these notes is *not* applicable to nonlinear differential equations.

1. A power series solution about an ordinary point

It is especially useful to consider the nature of the origin (corresponding to the point $x_0 = 0$). Suppose that the origin is an ordinary point of eq. (1). Then for values of x very close to the origin, we can approximate $a(x) \simeq a$ and $b(x) \simeq b$ by the leading terms of their Taylor series about the origin. In this case, a good approximation to eq. (1) in a small neighborhood about the origin is given by

$$y'' + ay' + b = 0\,. \tag{2}$$

The general solution of eq. (2) is well known—it is a linear combination of exponentials (or in a special case the product of a linear function and an exponential). These solutions are also analytic in the neighborhood of the origin and can be approximated by the first term of their Taylor series.

Consequently, it should not be surprising that if the origin is an ordinary point then the solution to eq. (1) can be expressed in general by a power series expanded about the origin,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n\,.$$

[1]Note that one can in principle have a more general form in which an arbitrary function $c(x)$ multiplies the y'' term in eq. (1). However, since eq. (1) is homogeneous [i.e. the right-hand side of eq. (1) is zero], one can simply divide the more general equation by $c(x)$ to obtain the form given in eq. (1). Here there is a subtlety, since the function $c(x)$ may be zero for specific values of x. Thus, the general solution $y(x)$ may only be strictly valid when x does not coincide with one of the zeros of $c(x)$.

[2]See, e.g., Chapter 3 of Ref. 1.

The technique for computing the coefficients c_n is developed on pp. 562–564 of Boas. The relevant theorem is given below.[3]

Theorem 1: For the differential equation, $y'' + a(x)y' + b(x)y = 0$, if $a(x)$ and $b(x)$ have power series expansions for $|x| < r$ where $r > 0$, then there is a pair of solutions, $y_1(x)$ and $y_2(x)$, such that *both* $y_1(x)$ and $y_2(x)$ have power series expansions around the origin, each with a radius of convergence that is at least as large as r.

This theorem is the basis of the series solution technique for solving a homogeneous second-order linear differential equation in the case where the origin is an ordinary point of eq. (1).

2. A Frobenius series solution about a regular singular point

Consider the homogeneous second-order linear differential equation,

$$x^2y'' + xA(x)y' + B(x) = 0\,. \tag{3}$$

We can convert this into the form of eq. (1) by dividing by x^2 and identifying

$$a(x) \equiv A(x)/x\,, \qquad\qquad b(x) \equiv B(x)/x^2\,. \tag{4}$$

Suppose that $A(x)$ and $B(x)$ possess Taylor series when expanded about the origin $x = 0$ with a nonzero radius of convergence. In this case, $A(x)$ and $B(x)$ are analytic functions of x in a neighborhood of the origin. However, according to eq. (4), the functions $a(x)$ and $b(x)$ are in general *singular* at $x = 0$. Nevertheless, this singular behavior is not particularly disturbing. After all, in the neighborhood of the origin we can approximate $A(x) \simeq A$ and $B(x) \simeq B$, where $A \equiv A(0)$ and $B \equiv B(0)$. Consequently, a good approximation to eq. (3) in a small neighborhood about the origin is given by

$$x^2y'' + Axy' + B = 0\,. \tag{5}$$

We recognize eq. (5) as an Euler equation [cf. eq. (7.17) on p. 434 of Boas]. The general solution of eq. (5) is well known—it is a linear combination of powers (or in a special case the product of a power and a logarithm). Indeed, if one inserts $y = x^s$ into eq. (5), it follows that

$$s(s-1) + As + B = 0\,, \tag{6}$$

which is a quadratic equation for s that is called the *indicial equation.*

As a result, it should not be surprising that the general solution to eq. (3) can be expressed as a Frobenius series,

$$y(x) = x^s \sum_{n=0}^{\infty} c_n x^n\,, \qquad\qquad \text{where } c_0 \neq 0\,, \tag{7}$$

[3] This theorem is proven on pp. 282–284 of Ref. 3.

and s is a root of the indicial equation given in eq. (6). This is the basis for the method of Frobenius discussed on pp. 585–586 of Boas, which applies to eq. (1) under the assumption that $A(x) \equiv xa(x)$ and $B(x) \equiv x^2b(x)$ possess Taylor series when expanded about the origin with a nonzero radius of convergence. For this reason, we say that if the origin is a singular point of eq. (1) such that $xa(x)$ and $x^2b(x)$ are analytic functions in the neighborhood of the origin, then the origin is called a *regular singular point.*[4] If these latter conditions are not fulfilled, then the origin is called an *irregular singular point.*[5] Indeed, in the case of an irregular singular point at the origin, eq. (3) cannot be approximated by an Euler equation in the neighborhood of the origin since $A(x)$ or $B(x)$ [or both] are singular at this point.

Thus, the method of Frobenius applies in the case that the origin is a regular singular point. In this case, the analog of Theorem 1 is more complicated. Here we quote one of several relevant theorems.[6]

Theorem 2: For the differential equation, $x^2y'' + xA(x)y' + B(x)y = 0$, if $A(x)$ and $B(x)$ have power series expansions for $|x| < r$ where $r > 0$, and if s_1 and s_2 are roots of the indicial equation, so labeled such that $s_1 \geq s_2$ when both roots are real, then one of the two linearly independent solutions $y_1(x)$ can be expressed as a Frobenius series [cf. eq. (7)] with $s = s_1$. If $s_1 - s_2$ is not an integer, then there is a second solution $y_2(x)$ that can be expressed as a Frobenius series with $s = s_2$. In each case, the series converges with a radius of convergence that is at least as large as r, and the solutions $y_1(x)$ and $y_2(x)$ satisfy the differential equation at least in the range $0 < x < r$.[7]

If $s_1 - s_2$ is a nonzero integer, then in some cases Theorem 2 applies to the second solution $y_2(x)$, while in other cases the second solution involves a Frobenius series multiplied by $\ln x$. The second solution always involves a logarithm if $s_1 = s_2$. We shall not provide the details of the extension of Theorem 2 to the logarithmic cases here. Suffice it to say that each of the relevant summations that appear in the logarithmic cases also possesses a radius of convergence that is at least as large as r.

In the case of a regular singular point at the origin, one can work out a systematic method for obtaining the corresponding Frobenius series solutions for $y_1(x)$ and if applicable for $y_2(x)$. Moreover, the corresponding solution for $y_2(x)$ in the logarithmic case can also be determined. Details of this analysis for all the possible cases are presented in the class handout entitled: *Series solutions to a second order linear differential equation with regular singular points.*

The case of an irregular singular point is much more difficult to address, although some techniques do exist for obtaining the solution to a homogeneous second-order linear

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[4] On pp. 605–606, Boas indicates that the method of Frobenius is applicable to a series solution at a regular singular point as a consequence of a general theorem known as Fuchs's theorem. Indeed, it is common to define a *Fuchsian differential equation* as a linear differential equation for which *every* singular point (possibly including the point of infinity) is a regular singular point.

[5] More generally, if $x = x_0$ is a singular point of eq. (1), then $a(x)$ and/or $b(x)$ [or both] are singular at $x = x_0$. Then, the point $x = x_0$ is a regular singular point if $(x - x_0)a(x)$ and $(x - x_0)^2b(x)$ are analytic at $x = x_0$. If these latter conditions are not fulfilled, then the point $x = x_0$ is an irregular singular point.

[6] This theorem is stated on pp. 291–292 of Ref. 3.

[7] These solutions can be extended to the region of $-r < x < 0$ by replacing x^s by $|x|^s$ in eq. (7).

differential equation in the vicinity of an irregular singular point. This topic lies beyond the scope of Physics 116C. However, if you would like to explore some of these techniques, a good starting point would be Chapter 3 of Ref. 1.

3. Ordinary and singular points of Legendre's differential equation

One can consider any point x_0 and ask whether it is an ordinary point or a singular point of eq. (1). For example, consider Legendre's differential equation,

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0\,. \tag{8}$$

It is convenient to divide this equation by $1-x^2$ to obtain the form of eq. (1). It follows immediately that $x=1$ and $x=-1$ are regular singular points and all other finite values of x (including the origin) are ordinary points of Legendre's differential equation. The point of infinity is also a regular singular point of eq. (8). For further details on the analysis of the point of infinity see, e.g., Section 2.5 of Ref. 6.

Applications of these concepts to other second order linear differential equations encountered in Chapter 12 of Boas are left as exercises for the reader.

References:

There are numerous books that treat the topics discussed in these notes. Here are some of my favorites.

1. Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill Book Company, New York, 1978).

2. Mayer Humi and William Miller, *Second Course in Ordinary Differential Equations for Scientists and Engineers* (Springer-Verlag, New York, 1988).

3. Ray Redheffer, *Differential Equations: Theory and Applications* (Jones and Bartlett Publishers, Boston, MA, 1991).

4. Earl D. Rainville, Phillip E. Bedient and Richard E. Bedient, *Elementary Differential Equations*, 8th Edition (Pearson Education, Inc., Upper Saddle River, NJ, 1997).

5. Ravi P. Agarwal and Donal O'Regan, *Ordinary and Partial Differential Equations* (Springer Science, New York, 2009)

6. Gerhard Kristensson, *Second Order Differential Equations: Special Functions and Their Classification* (Springer Science, New York, 2010).