INSTRUCTIONS: This is a three-hour exam. During the exam, you may refer to the textbook, the class handouts, or your own personal notes. Collaboration with your neighbor is strictly forbidden. In answering the questions, it is not sufficient to simply write the final result. You must provide the intermediate steps needed to arrive at the solution in order to get full credit. You may quote results that have been derived in the textbook, homework solutions and class handouts (but if you do so, please cite explicitly the source of any such quotations).

The exam consists of six problems with a total of 20 parts. Each part is worth five points, for a total of 100 points.

1. Consider the differential equation,

$$y'' + \frac{1}{x^2}y' - \frac{c}{x^2}y = 0\,, \tag{1}$$

where c is a real number. Assume that a solution exists of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+p}\,, \qquad \text{where } a_0 \neq 0\,, \tag{2}$$

where p is a number to be determined.

(a) Find the recurrence relation satisfied by the a_n.

HINT: After plugging eq. (2) into eq. (1), isolate the term in the resulting equation that is proportional to x^{p-3}. Show that there is only one consistent solution for p. Using this value for p, deduce the recurrence relation that is satisfied by the a_n.

(b) For what values of c is $y(x)$ a polynomial of finite degree?

(c) Write down the three polynomial solutions of lowest degree that satisfy eq. (1). In each case, indicate the corresponding value of c.

(d) Suppose c is not equal to any of the values obtained in part (b). In this case, the series given in eq. (2) does not terminate. Does the series for $y(x)$ converge for any nonzero x? If yes, what is the radius of convergence? If no, explain why the Frobenius method has failed.

2. (a) Evaluate the following integral

$$\int_0^\infty e^{-pt} I_0(2\sqrt{at})dt\,, \qquad \text{for } p > 0\,, \tag{3}$$

by inserting the series representation for the modified Bessel function $I_0(t)$ into eq. (3) and integrating term by term. Write your final result in summation notation, where the general form of the nth term of the series is given.

HINT: It is convenient to replace any gamma functions that appear in this problem with the corresponding factorials.

(b) Sum the series and show that eq. (3) can be expressed as an elementary function.

3. Assume that the temperature distribution on the surface of the earth is given by:

$$T = T_2 + (T_1 - T_2)\cos^2\theta\,, \tag{4}$$

where θ is the polar angle (e.g., $\theta = \pi/2$ at the equator), T_1 is the temperature at the north and south poles, and T_2 is the temperature at the equator. Let r be the distance as measured from the center of the earth. The units of distance are chosen such that the earth is a sphere of radius 1.

(a) Show that eq. (4) can be expressed as a linear combination of two Legendre polynomials $P_\ell(x)$ (which ones?) where $x = \cos\theta$.

(b) We shall model the steady state temperature distribution of the earth by assuming that $T(r,\theta)$ is a solution to Laplace's equation in the region $0 \le r \le 1$, with the boundary condition $T(1,\theta)$ specified by eq. (4) [The symmetry of the problem implies that T does not depend on the azimuthal angle ϕ.] Compute $T(r,\theta)$ for all angles θ and $0 \le r \le 1$.

HINT: You need not rederive the general solutions to Laplace's equation in spherical coordinates. But, if you make use of solutions found in Boas, make sure you choose the appropriate solutions relevant for this problem and justify your selections.

(c) Evaluate $T(r,\theta)$ at the center of the earth ($r = 0$). What feature of the earth have we neglected which may be responsible for such an unrealistic final result?

4. In quantum mechanics, the time-independent Schrödinger equation for a free particle of energy $E = k^2$ in three dimensions is[*]

$$(\vec{\boldsymbol{\nabla}}^{\mathbf{2}} + k^2)\psi(\vec{\boldsymbol{r}}) = 0\,. \tag{5}$$

By convention, we take k to be positive. Suppose we confine the particle to a spherical region $0 \leq r \leq a$, where $r \equiv |\vec{\boldsymbol{r}}|$. It is convenient to employ spherical coordinates. The Laplacian can then be written as

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{6}$$

Then, the relevant boundary conditions for this problem are:

(i) $\psi(\vec{\boldsymbol{r}})$ is non-singular at the origin, and
(ii) $\psi(a,\theta,\phi) = 0\,.$

(a) Using the technique of separation of variables, write $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)\,,$ and find the ordinary differential equation that is satisfied by $R(r)$. This equation is called the *radial equation*.

HINT: You do not have to solve this problem from scratch, since a similar computation has already been given a number of times in this class (and in Boas). If you quote any "known" results, please cite your source. If the computation that you cite is similar but not exactly the same as the one required here, indicate how the cited computation should be modified to arrive at the desired result.

WARNING: do not confuse k^2 in eq. (5) with the separation constant. Indeed, k^2 is an eigenvalue of the Sturm-Liouville problem that arises when you solve the radial equation subject to the boundary conditions specified above.

(b) What are the most general solutions to the radial equation before imposing the boundary conditions?

(c) Now, impose the relevant boundary conditions and write an implicit equation for the allowed values of k.

(d) Using the asymptotic form for the solutions to the radial equation, write down an explicit equation for the allowed values of k (which provide a good approximation if k is large enough).

[*]To make the algebra easier, I have taken the mass of the particle equal to $\frac{1}{2}$ and have chosen units where Planck's constant is equal to 2π.

5. In poker, each player is dealt five cards from a deck of 52 cards. The five cards dealt is called a *poker hand.* A *full house* consists of three cards of one value and two cards of another value, for example three kings and two queens. If the poker hand has four cards of one value, for example four aces (one from each of the four possible suits), then the poker hand is called a *four-of-a-kind.* Poker players sometimes wonder why four-of-a-kind beats a full house. Let's see why this is true.

(a) How many possible poker hands are there?

(b) How many possible four-of-a-kind poker hands are there? What is the probability of being dealt a four-of-a-kind hand?

(c) How many possible full-house poker hands are there? What is the probability of being dealt a full house?

(d) A *royal flush* is a poker hand consisting of an ace, king, queen, jack and ten of the same suit. This is the rarest of all poker hands as there are only four possible royal flush poker hands (one for each of the four suits). Suppose that two players are each dealt a poker hand. In the absence of privileged information, each player initially has the same probability of being dealt a royal flush. But suppose your opponent announces that she has a royal flush before you look at your hand. Is the probability that you have a royal flush now larger, smaller or unchanged as compared to the initial probability prior to receiving this information? Justify your response.

6. A continuous random variable x is said to be *uniformly distributed* over the interval $a \leq x \leq b$ (where a and b are real numbers and $a < b$) if it probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}\,, & \text{if } a \leq x \leq b\,, \\ 0\,, & \text{if } x > b \text{ or if } x < a. \end{cases}$$

(a) Compute the mean of the uniformly distributed random variable x.

(b) Compute the variance of the uniformly distributed random variable x.

(c) What is the probability that the random variable x lies within one standard deviation of the mean?