INSTRUCTIONS: You have one hour and forty-five minutes to complete this exam. During the exam, you may refer to the textbook (Boas), the class handouts (including solution sets to the homeworks, the practice problems and the practice midterm) or your own personal notes. Collaboration with your neighbor is strictly forbidden.

In answering the questions, it is not sufficient to simply give a final result. You must provide the intermediate steps needed to arrive at the final solution in order to get full credit. *However, if you are employing any result already obtained in class or in the textbook, you do not need to re-derive it.*

The numerical value of each problem is indicated by a number inside square brackets. The total number of points for the midterm exam is 90. Use this information to manage your time appropriately during the exam. Part (d) of the first problem is extra credit worth 10 points. It should only be attempted after you have completed the rest of the exam.

1. [30] Consider the differential equation

$$2x(x+1)y'' + 3(x+1)y' - y = 0\,. \tag{1}$$

(a) Using the Frobenius method, obtain the two linearly independent solutions of eq. (1). If a solution takes the form of a generalized power series, determined the general term of the series.

(b) Compute the Wronskian of the two linearly independent solutions of eq. (1).

(c) If one solution of eq. (1) is known, then the second solution can be expressed as an indefinite integral in terms of the Wronskian. Note that of the two solutions obtained in part (a), one of them looks significantly simpler than the other. Using the simpler of the two solutions, express the second solution in terms of an indefinite integral.

(d) [*EXTRA CREDIT*] Evaluate the indefinite integral obtained in part (c). Then, expand it in a Taylor series and show that the result coincides with the second linearly independent solution obtained in part (a).

2. [20] Consider the definite integral involving the Bessel function,

$$\int_0^\infty \frac{J_p(x)}{x}\,dx\,. \tag{2}$$

(a) Assuming that p is a positive integer, evaluate the above integral in terms of p.

HINT: Use one of the recursion relations satisfied by the Bessel functions in your derivation. You may also use known integrals encountered in the homework or in the textbook.

(b) The result obtained in part (a) is actually valid for any positive real number p. Check this by explicitly evaluating eq. (2) for $p = \frac{1}{2}$. In your calculation, you may use the result of the following integral (which you do *not* need to prove here),

$$\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx = \sqrt{2\pi}\,,$$

which I have taken from a table of definite integrals.

3. [20] Consider Laplace's equation in two dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

In polar coordinates, we define $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \le \theta < 2\pi$. Laplace's equation takes the following form in polar coordinates,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,. \tag{3}$$

(a) Using the separation of variables technique, we can write $u(r,\theta) = R(r)\Theta(\theta)$, and obtain two ordinary differential equations for $R(r)$ and $\Theta(\theta)$, respectively. Solve for $\Theta(\theta)$ under the assumption that $\Theta(\theta + 2\pi) = \Theta(\theta)$. Then solve for $R(r)$ under the assumption that $R(r)$ is bounded (i.e. it is finite) as $r \to 0$.

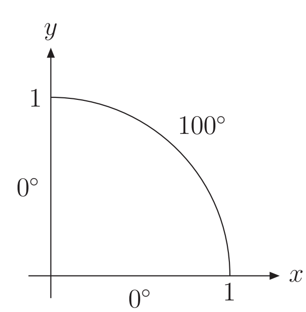
HINT: To solve the resulting equation for $R(r)$, assume a solution of the form $R(r) = r^s$, and determine the value of s. Keep only the solution that is bounded as $r \to 0$.

(b) Show that the most general solution to eq. (3) that is periodic in θ and bounded as $r \to 0$ is:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]\,, \tag{4}$$

where the a_n are arbitrary constants. Verify explicitly that eq. (4) solves eq. (3).

4. [20] A metal plate has the shape of a quarter circle of radius 1 as shown at the right. The boundaries of the plate that lie along the x and y axes are held at $T = 0°$ and the circular boundary is held at 100°. Find the interior steady-state temperature, $T(r,\theta)$.



HINT: You should use polar coordinates. Explain why eq. (4) is a good starting point for solving this problem. Keep in mind that the interior of the plate corresponds to $0 \le r \le 1$ and $0 \le \theta \le \frac{1}{2}\pi$. All you need to do is to determine the unknown constants a_n of eq. (4). You do not have to sum the series.