INSTRUCTIONS: This is a three-hour exam. During the exam, you may refer to the textbook, the class handouts, or your own personal notes. Collaboration with your neighbor is strictly forbidden. In answering the questions, you may quote results that have been derived in the textbook, homework solutions and class handouts (but if you do so, please cite explicitly the source of any such quotations).

The exam consists of six problems with a total of 16 parts. Each part is worth ten points, for a total of 160 points. In addition, part (e) of problem 6 is worth extra credit. Please do not attempt the extra credit part until you have completed the exam.

1. The associated Laguerre polynomials are denoted by $L_n^k(x)$. Evaluate $L_n^k(0)$ in terms of n and k.

<u>*HINT*</u>: Start with the definition of $L_n^k(x)$ given by Boas, and employ the explicit expression for the Laguerre polynomials given in summation form.

2. The Klein-Gordon equation is a generalization of the wave equation,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + \lambda^2 u\,,$$

where v and λ are constants.

(a) In two space dimensions, the Laplacian of $u(x,y)$ can be expressed in polar coordinates as:

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,,$$

where $x \equiv r\cos\theta$ and $y \equiv r\sin\theta$. Applying the separation of variables technique, find the most general solutions to the Klein-Gordon equation in two dimensions.

(b) Suppose that the vibrations of a circular membrane (e.g., a drumhead) are governed by the Klein-Gordon equation. Determine the characteristic frequencies of the membrane.

3. Consider the heat flow equation in one space dimension,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}\,. \tag{1}$$

We shall apply this to the problem of heat flow in a thin bar of length L with an initial temperature distribution of

$$u(x,0) = |x|\,, \qquad \text{for } -\tfrac{1}{2}L \leq x \leq \tfrac{1}{2}L\,. \tag{2}$$

Here we have chosen a convenient coordinate system where the origin is located at the midpoint of the bar. At $t = 0$ we instantaneously bend the bar into the shape of a circular ring, by tightly joining the ends of the bar. In the new configuration, we assume that eq. (1) still applies, where x labels the points along the circular ring. But now, the points $x = -\frac{1}{2}L$ and $x = \frac{1}{2}L$ represent the same point on the ring (where the two ends of the bar were joined). Thus, the relevant boundary conditions for $t > 0$ are:

$$u(-\tfrac{1}{2}L, t) = u(\tfrac{1}{2}L, t)\,, \tag{3}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=-\frac{1}{2}L} = \left.\frac{\partial u}{\partial x}\right|_{x=\frac{1}{2}L}\,, \tag{4}$$

for all $t > 0$.

(a) Write down the most general solution to eq. (1) subject to the boundary conditions specified in eqs. (3) and (4), *prior to* imposing the initial condition given in eq. (2). Your result should be expressed as an infinite sum with unknown coefficients.

HINT: You should be careful to allow for all possible solutions when applying the boundary conditions. Then, you should find that the application of the boundary conditions will imply that the separation constant for this problem can only take on discrete values.

(b) Impose the initial condition given in eq. (2) and determine all unknown constants. Write the final solution to the problem in summation form.

(c) Verify that your solution in part (b) is correct when $t = 0$ and $x = \pm\frac{1}{2}L$. Finally, show that your solution at $t = 0$ and $x = 0$ allows you to explicitly evaluate the infinite sum,

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

4. The following three questions are all related. In parts (b) and (c) you do not need to perform the separation of variables of the Laplace equation, as we have done this many times during this course. However, please cite the source for the solutions that you use.

(a) Write out an explicit expression for the Legendre polynomial $P_3(x)$.

(b) Find the electric potential *outside* of a sphere of radius R if the potential on the surface of the sphere is $\Phi(r = R, \theta, \phi) = V_0 \cos^3\theta$, where θ is the polar angle measured with respect to the z-axis. Assume that there are no electric charges outside of the sphere and that the potential approaches zero as $r \to \infty$.

(c) Find the steady-state temperature distribution *inside* of a *hemisphere* of radius R if the temperature of the curved surface is held at $T(r = R, \theta, \phi) = T_0 \cos^3\theta$ (for $0 \le \theta < \frac{1}{2}\pi$) and the equatorial plane is held at $0°$.

5. An experimental device is made up of either three or four individual components. The device will function correctly if and only if *at least* two of the components are functional. The probability that an individual component is functional is denoted by p. You may assume that the probabilities that each individual component is functional are independent. Both the three-component device (denoted by D_3) and the four-component device (denoted by D_4) are available for purchase.

(a) Assume that the probability that an individual component of D_4 is functional is $p = \frac{1}{2}$. What is the probability that device D_4 will function correctly?

(b) Denoting the probability that an individual component of D_3 is functional by p, write an expression for the probability that device D_3 will function correctly.

(c) Suppose that the probability that an individual component of D_3 is functional is $p = \frac{2}{3}$. Using the results of parts (a) and (b), which device is more reliable, D_3 or D_4? What is the minimum value of p needed for D_3 to out-perform D_4 [assuming that for the latter $p = \frac{1}{2}$ as in part (a)]?

<u>*HINT:*</u> For the last question of part (c), determine the minimum value of p to two significant figures. This is most easily accomplished by trial and error using your calculator.

6. The CIA monitors phone calls in Santa Cruz and finds that the length of time (in minutes) of a given phone call is a random phenomenon, with a probability function given by:

$$f(t) = \begin{cases} Ce^{-t/5}\,, & \text{for } t > 0\,, \\ 0\,, & \text{for } t \leq 0\,, \end{cases}$$

where t measures time in minutes.

(a) What is the value of C so that $f(t)$ is a properly normalized probability distribution.

(b) What is the average length of a phone call monitored by the CIA?

(c) What is the probability that a given phone call lasts more than 10 minutes?

(d) Suppose that a particular phone call is known to last more than 10 minutes. What is the probability that it lasted more than 20 minutes?

(e) [*EXTRA CREDIT*] Compare the results of parts (c) and (d) and explain how they are related. Generalize the result to phone calls of arbitrary duration and explain the significance of the relation that you found.