*FINAL EXAM INSTRUCTIONS*: This is an open book exam. You are permitted to consult the textbook by Liboff, your handwritten notes, and class handouts. No other consultations or collaborations are permitted during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.* However, you are *not* required to re-derive any formulae that you cite from the textbook or the class handouts. The point value of each problem is indicated in the square brackets below.

1. An electron is placed in a potential

$$V(\vec{\boldsymbol{r}}) = \frac{-e^2}{r} + \beta(r^2 - 3z^2)\,,$$

where $\beta$ is a small parameter. Neglect the spin of the electron.

(a) [10] Compute the shifts of the $n = 2$ energy levels (you may neglect fine-structure effects) using first order perturbation theory. Indicate the relative positions of the energy levels.

(b) [10] Suppose that a weak uniform magnetic field $B$ is applied in the $z$-direction. Determine its effect on the levels obtained in part (a) to first order in $B$.

(c) [10] Repeat part (b) assuming that the weak uniform magnetic field is applied in the $x$-direction.

*HINTS:* When evaluating matrix elements, think before you calculate. Often, you can show that a particular matrix element is zero without fully computing it. In part (c), use first order degenerate perturbation theory by taking the "unperturbed" energy eigenstates and eigenvalues to be the ones you obtained in part (a).

2. [20] Calculate the wavelength, in centimeters, of a photon emitted under a hyperfine transition in the ground state of deuterium. Deuterium is "heavy" hydrogen, with an extra neutron in the nucleus. The proton and neutron bind together to form a *deuteron*, with spin 1 and magnetic moment

$$\vec{\boldsymbol{\mu}}_d = \frac{g_d e}{2M_d}\,\vec{\boldsymbol{I}}\,,$$

where $\vec{\boldsymbol{I}}$ is the spin-vector of the deuteron, $g_d = 1.71$ is the deuteron $g$-factor and $M_d$ is the mass of the deuteron.

<u>*DATA:*</u> Take the deuteron mass to be roughly twice the proton mass. You can also use the following results. The ratio of the proton to electron mass is $M_p/m_e \simeq 1836$. The rest-mass of the electron is $m_e c^2 \simeq 5.11 \times 10^5$ eV. Other useful numbers are:

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,, \qquad \hbar c = 1.973 \times 10^{-5}\ \text{eV-cm}\,, \qquad a_0 = \frac{\hbar^2}{me^2} = \frac{\hbar}{mc\alpha} \simeq 0.5 \times 10^{-8}\ \text{cm}\,.$$

3. Consider a positively-charged spin-1/2 particle in an external magnetic field, governed by the Hamiltonian:

$$H = H_0\,\mathbb{I} - \gamma\vec{\boldsymbol{B}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\,,$$

where $\mathbb{I}$ is the identity operator in spin space, $\vec{\boldsymbol{S}}$ is the vector of spin-1/2 spin matrices, and $\gamma$ is a constant (for a positively-charged particle, $\gamma > 0$). $H_0$ is spin-independent and is independent of the magnetic field $\vec{\boldsymbol{B}}$. For simplicity, assume that $H_0$ possesses exactly one eigenvalue, which is denoted by $E$.

(a) [5] If the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ (where $B > 0$), determine the energy eigenstates and eigenvalues of $H$.

(b) [10] Assume that the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ for time $t < 0$. The system is initially observed to be in a spin-up state. At $t = 0$, a time-dependent perturbation is added by modifying the magnetic field. The new magnetic field for $t > 0$ is given by:

$$\vec{\boldsymbol{B}} = b\,(\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t) + B\hat{\boldsymbol{z}}\,,$$

where $b > 0$. Using first-order time-dependent perturbation theory, derive an expression for the probability that the system will be found in a spin-down state at some later time $t = T$.

(c) [10] For what range of values of $\omega$ are the results of part (b) unreliable?

HINT: This is a problem that we solved exactly earlier in the quarter. See if you can compare the results obtained in part (b) with the exact solution derived in class.

4. In class, we computed the phase shifts for the scattering by a hard sphere of radius $a$. In the low energy limit (*i.e.*, for $ka \ll 1$), the $\ell th$ partial wave phase shift is given by:

$$\delta_\ell \simeq c_\ell (ka)^{2\ell+1}\,, \tag{1}$$

where $c_\ell = -1/\left\{(2\ell+1)[(2\ell-1)!!]^2\right\}$. We concluded that at low energies, only the lowest partial waves are important. One can prove a more general result: at low energies, $\delta_\ell$ is of the form given by eq. (1) for any short-ranged spherically symmetric potential, where the parameters $c_\ell$ depend on the form of the potential.

(a) [10] Using the results stated above, show that at low energies, the differential cross section for scattering off *any* short-ranged spherically symmetric potential is approximately isotropic (*i.e.*, independent of scattering angle). Compute the corresponding total cross-section and show that it is determined by one unknown parameter (which depends on the form of the potential).

(b) [15] Now, raise the energy of scattering slightly so that *only the lowest-order corrections* that yield a non-trivial angular distribution are required for a good approximation. Assume that the energy is still sufficiently low such that the form for $\delta_\ell$ given in eq. (1) is valid. In this case, find the most general form for the angular dependence of the differential cross-section, and show that it is determined in terms of two unknown parameters.

(c) [*EXTRA CREDIT*] Consider the case of a weakly attractive spherical potential well of depth $V_0$ and the radius $a$. For this problem, "weakly attractive" means that the potential energy is given by $V(r) = -|V_0|$ for $r < a$ and $2m|V_0|a^2/\hbar^2 \ll 1$. Using the results of part (b), compute the leading contribution at low energies to the so-called forward-backward asymmetry, $A_{\rm FB}$, which is defined by

$$A_{FB} \equiv \frac{\sigma_{\rm F} - \sigma_{\rm B}}{\sigma_{\rm F} + \sigma_{\rm B}}\,,$$

where (after integrating over the azimuthal angle $\phi$),

$$\sigma_{\rm F} \equiv 2\pi \int_0^1 \frac{d\sigma}{d\Omega}\, d\cos\theta\,, \qquad\qquad \sigma_{\rm B} \equiv 2\pi \int_{-1}^0 \frac{d\sigma}{d\Omega}\, d\cos\theta\,.$$

(*HINT:* The simplest way to solve this problem is to use the Born approximation.)

## <u>**USEFUL INFORMATION**</u>

The $n = 1$ and $n = 2$ wave functions of the hydrogen atom are:

$$\psi_{100}(\vec{\boldsymbol{r}}) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}\,,$$

$$\psi_{200}(\vec{\boldsymbol{r}}) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2a_0}\right)^{3/2}\left(1 - \frac{r}{2a_0}\right) e^{-r/(2a_0)}\,,$$

$$\psi_{210}(\vec{\boldsymbol{r}}) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0}\, e^{-r/(2a_0)} \cos\theta\,,$$

$$\psi_{21\pm1}(\vec{\boldsymbol{r}}) = \mp\frac{1}{\sqrt{8\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0}\, e^{-r/(2a_0)} \sin\theta\, e^{\mp i\phi}\,,$$

where $a_0 \equiv \hbar^2/(me^2)$ is the Bohr radius.

A useful integral that arises in evaluating matrix elements involving hydrogen atom wave functions is:

$$\int_0^\infty r^n e^{-r/a}\, dr = a^{n+1} n!\,, \qquad \text{for integer } n = 0\,,\,1\,,\,2\,,\,\ldots$$

The matrix elements of the angular momentum operators are given in Section 11.5 of Liboff. For example, defining $L_\pm \equiv L_x \pm iL_y$, it follows that:

$$\langle \ell\, m_\ell |\, L_\pm\, | \ell'\, m'_\ell \rangle = \hbar\left[(\ell' \mp m'_\ell)((\ell' \pm m'_\ell + 1)\right]^{1/2} \delta_{\ell\ell'}\, \delta_{m_\ell\,,\,m'_\ell \pm 1}\,.$$

In case you need it, the small argument approximations for the spherical Bessel functions $j_\ell(x)$ and $n_\ell(x)$ are given by:

$$j_\ell(x) \simeq \frac{x^\ell}{(2\ell+1)!!}\,, \qquad n_\ell(x) \simeq \frac{(2\ell-1)!!}{x^{\ell+1}}\,, \qquad \text{for } |x| \ll 1\,,$$

where $(2\ell+1)!! \equiv 1\cdot 3\cdot 5\cdots(2\ell+1)$, and $(-1)!! \equiv 1$.