1. An electron is placed in a potential

$$V(\vec{\boldsymbol{r}}) = \frac{-e^2}{r} + \beta(r^2 - 3z^2)\,,$$

where β is a small parameter. Neglect the spin of the electron.

(a) Compute the shifts of the $n = 2$ energy levels (you may neglect fine-structure effects) using first order perturbation theory. Indicate the relative positions of the energy levels, assuming that $\beta > 0$.

The unperturbed $n = 2$ state of hydrogen are degenerate states with energy:

$$E^{(0)}_{2\,\ell\,m_\ell} = -\tfrac{1}{4}\ \text{Ry},$$

where 1 Ry $= me^4/(2\hbar^2) = 13.6$ eV. The corresponding unperturbed energy eigenfunctions are:

$$\psi_{200}(\vec{\boldsymbol{r}}) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2a_0}\right)^{3/2}\left(1 - \frac{r}{2a_0}\right)e^{-r/(2a_0)}\,,$$

$$\psi_{210}(\vec{\boldsymbol{r}}) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2a_0}\right)^{3/2}\frac{r}{a_0}\,e^{-r/(2a_0)}\cos\theta\,,$$

$$\psi_{21\pm1}(\vec{\boldsymbol{r}}) = \mp\frac{1}{\sqrt{8\pi}}\left(\frac{1}{a_0}\right)^{3/2}\frac{r}{a_0}\,e^{-r/(2a_0)}\sin\theta\,e^{\mp i\phi}\,,$$

where $a_0 = \hbar^2/(me^2)$.

The perturbing Hamiltonian is:

$$H' = \beta(r^2 - 3z^2) = \beta r^2(1 - 3\cos^2\theta)\,.$$

To compute the first-order energy shifts of degenerate states, we must construct the matrix elements of the matrix W, which is defined as:

$$W_{\ell' m'_\ell\,;\,\ell m_\ell} \equiv \langle 2\,\ell'\,m'_\ell|\,H'\,|2\,\ell\,m_\ell\rangle\ . \tag{1}$$

However, it is easy to show that the matrix W is diagonal. First, note that H' commutes with L_z, since $L_z = -i\hbar\frac{\partial}{\partial\phi}$ and H' is independent of ϕ. It follows that:

$$0 = \langle 2\,\ell'\,m'_\ell|\,[H'\,,\,L_z]\,|2\,\ell\,m_\ell\rangle = \langle 2\,\ell'\,m'_\ell|\,(H'L_z - L_zH')\,|2\,\ell\,m_\ell\rangle = (m_\ell - m'_\ell)\,\langle 2\,\ell'\,m'_\ell|\,H'\,|2\,\ell\,m_\ell\rangle\ ,$$

which implies that

$$\langle 2\,\ell'\,m'_\ell|\,H'\,|2\,\ell\,m_\ell\rangle = 0 \quad \text{if} \quad m_\ell \neq m'_\ell\,.$$

The only other non-diagonal elements to examine are $\langle 2\,0\,0|\, H'\, |2\,1\,0\rangle$ and $\langle 2\,1\,0|\, H'\, |2\,0\,0\rangle$. When we evaluate both matrix elements, we find that the angular integral vanishes, since

$$\int d\Omega\,(1-3\cos^2\theta) = 2\pi \int_{-1}^{1} d\cos\theta\,(1-3\cos^2\theta) = (\cos\theta - \cos^3\theta)\Big|_{-1}^{+1} = 0\,. \qquad (2)$$

It therefore follows that the first-order energy shifts of the $n = 2$ states are given by:

$$E^{(1)}_{2\ell m_\ell} = \langle 2\,\ell\, m_\ell|\, H'\, |2\,\ell\, m_\ell\rangle\ .$$

First we compute the first-order energy shift of the state $|2\,0\,0\rangle$:

$$E^{(1)}_{200} = \beta\,\langle 2\,0\,0|\, r^2(1-3\cos^2\theta)\,|2\,0\,0\rangle = \frac{\beta}{8\pi a_0^3}\int d^3r\,\left(1-\frac{r}{2a_0}\right)^2 r^2\, e^{-r/a_0}(1-3\cos^2\theta)\,.$$

Using eq. (2), it follows that:

$$\boxed{E^{(1)}_{200} = 0}$$

Turning to the $n = 2$, $\ell = 1$ states,

$$E^{(1)}_{210} = \frac{\beta}{32\pi a_0^5}\int d^3r\, r^4 e^{-r/a_0}\cos^2\theta(1-3\cos^2\theta)\,, \qquad (3)$$

$$E^{(1)}_{21\pm1} = \frac{\beta}{64\pi a_0^5}\int d^3r\, r^4 e^{-r/a_0}\sin^2\theta(1-3\cos^2\theta)\,, \qquad (4)$$

After writing $d^3r = r^2 dr d\Omega$, the radial integrals are all of the form:

$$\int_0^\infty r^n e^{-r/a_0}\, dr = a_0^{n+1} n!\,,$$

for non-negative integer n, and the angular integrals are:

$$\int d\Omega\,\cos^2\theta(1-3\cos^2\theta) = 2\pi\int_{-1}^{1} d\cos\theta\,\cos^2\theta(1-3\cos^2\theta) = 4\pi\left(\frac{1}{3}-\frac{3}{5}\right) = -\frac{16\pi}{15}\,,$$

$$\int d\Omega\,\sin^2\theta(1-3\cos^2\theta) = 2\pi\int_{-1}^{1} d\cos\theta\,(1-\cos^2\theta)(1-3\cos^2\theta) = 4\pi\left(1-\frac{4}{3}+\frac{3}{5}\right) = \frac{16\pi}{15}\,.$$

Applying the above results to eqs. (3) and (4) yields:

$$\boxed{E^{(1)}_{210} = -24\beta a_0^2\,, \qquad E^{(1)}_{21\pm1} = 12\beta a_0^2}$$

We conclude that if $\beta > 0$,

$$E_{210} < E_{200} < E_{21\pm1}\,,$$

where $|2\,1\,-1\rangle$ and $|2\,1,1\rangle$ remain degenerate at first order.

(b) Suppose that a weak uniform magnetic field B is applied in the z-direction. Determine its effect on the levels obtained in part (a) to first order in B.

In the presence of a weak uniform magnetic field B, new terms appear in the Hamiltonian of the electron (with charge $-e$),[1]

$$\frac{e}{2mc}\vec{\boldsymbol{B}}\boldsymbol{\cdot}\vec{\boldsymbol{L}} + \frac{e^2}{8mc^2}\left[r^2\vec{\boldsymbol{B}}^{\,2} - (\vec{\boldsymbol{r}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\right] .$$

To first order in B, we can neglect the second term above. Thus, the magnetic field adds another term to the perturbation,

$$H' = \beta r^2(1 - 3\cos^2\theta) + \frac{eB}{2mc}L_z\,. \tag{5}$$

However, since H' still commutes with L_z, the W matrix is still diagonal. Thus, the energy shift due to the last term in eq. (5) is given by:

$$\Delta E = \frac{eB}{2mc}\,\langle 2\,\ell\,m|\,L_z\,|2\,\ell\,m\rangle = \frac{e\hbar B m_\ell}{2mc}$$

Thus, the $n = 2$ energy levels at first order in perturbation theory are given by:

$$E_{2\ell m_\ell} = -\tfrac{1}{4}\ \mathrm{Ry} + E^{(1)}_{2\ell m_\ell} + \frac{e\hbar B m_\ell}{2mc}\,,$$

where $E^{(1)}_{2\ell m_\ell}$ are the first-order energy shifts in the absence of the magnetic field, which were obtained in part (a).

(c) Repeat part (b) assuming that the weak uniform magnetic field is applied in the x-direction.

If the uniform magnetic field points in the x direction, then the perturbed Hamiltonian is given by:

$$H' = \beta r^2(1 - 3\cos^2\theta) + \frac{eB}{2mc}L_x\,.$$

However, this time, H' does *not* commute with L_z. Hence the W matrix [see eq. (1)] is no longer diagonal. We shall solve this problem by redefining

$$H^{(0)} = \frac{\vec{\boldsymbol{p}}^{\,2}}{2m} - \frac{e^2}{r} + \beta(r^2 - 3z^2)\,, \qquad\qquad H' = \frac{eB}{2mc}L_x\,,$$

[1]See eq. (13) of the class handout on the quantum mechanics of a charged particle in an electromagnetic field.

with the "unperturbed" energies given by the energy levels obtained in part (a). In this case, only the states $|2\,1\,\pm 1\rangle$ are degenerate. Thus, we must evaluate a new W matrix:

$$W = \frac{eB}{2mc}\begin{pmatrix} \langle 2\,1\,1|\,L_x\,|2\,1\,1\rangle & \langle 2\,1\,1|\,L_x\,|2\,1\,-1\rangle \\ \langle 2\,1\,-1|\,L_x\,|2\,1\,1\rangle & \langle 2\,1\,-1|\,L_x\,|2\,1\,-1\rangle \end{pmatrix}.$$

Using eqs. (11.59) and (11.61) on p. 506 of Liboff, it follows that:

$$\langle \ell'\,m'_\ell|\,L_x\,|\ell\,m_\ell\rangle = \tfrac{1}{2}\hbar\delta_{\ell\ell'}\big\{[(\ell-m_\ell)(\ell+m_\ell+1)]^{1/2}\delta_{m'_\ell\,,\,m_\ell+1}+[(\ell+m_\ell)(\ell-m_\ell+1)]^{1/2}\delta_{m'_\ell\,,\,m_\ell-1}\big\}\,.$$

We immediately observe that all the matrix elements of W vanish. Thus, the energies of $|2\,1\,\pm 1\rangle$ are unshifted. Likewise, the non-degenerate states $|2\,0\,0\rangle$ and $|2\,1\,0\rangle$ are unshifted at first order, since

$$\langle 2\,0\,0|\,L_x\,|2\,0\,0\rangle = \langle 2\,1\,0|\,L_x\,|2\,1\,0\rangle = 0\,,$$

and we conclude that $H' \equiv (eB/2mc)L_x$ causes no energy shift to first order in B.

2. [25] Calculate the wavelength, in centimeters, of a photon emitted under a hyperfine transition in the ground state of deuterium. Deuterium is "heavy" hydrogen, with an extra neutron in the nucleus. The proton and neutron bind together to form a *deuteron*, with spin 1 and magnetic moment

$$\vec{\boldsymbol{\mu}}_d = \frac{g_d e}{2M_d}\,\vec{\boldsymbol{I}}\,,$$

where $\vec{\boldsymbol{I}}$ is the spin-vector of the deuteron, $g_d = 1.71$ is the deuteron g-factor and M_d is the mass of the deuteron.

In class, we computed the first-order energy shifts to the ground state energy of hydrogen due to the hyperfine interaction. The latter is given by:

$$H_{\rm HF} = \frac{e^2 g_N}{2mM_N}\left[\frac{\vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}}{r^3} + \frac{3(\vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{r}})(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{r}})}{r^5} - \frac{\vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}}{r^3} + \frac{8\pi}{3}\vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\,\delta^3(\vec{\boldsymbol{r}})\right],$$

where $\vec{\boldsymbol{I}}$ is the spin-vector of the nucleus, g_N is the g-factor of the nucleus, M_N is the mass of the nucleus and m is the mass of the electron. In the case of $\ell = 0$, we demonstrated in class that:

$$\left\langle\frac{\vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}}{r^3}\right\rangle = \left\langle\frac{3(\vec{\boldsymbol{I}}\boldsymbol{\cdot}\hat{\boldsymbol{r}})(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{r}}) - \vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}}{r^3}\right\rangle = 0\,.$$

Hence, in the ground state, which corresponds to $n = 1$ and $\ell = 0$, only the term in the hyperfine Hamiltonian proportional to $\vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\,\delta^3(\vec{\boldsymbol{r}})$ contributes to the first-order energy shift. In particular, after evaluating the expectation value of $H_{\rm HF}$ in the ground state,

$$E^{(1)} = \frac{4\pi e^2 g_N}{3mM_N}|\psi_{100}(0)|^2\langle\vec{\boldsymbol{I}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\rangle\,,$$

where

$$|\psi_{100}(0)|^2 = \frac{1}{\pi a_0^3}\,,$$

is the square of the ground-state unperturbed wave function evaluated at the origin. Recalling that:

$$a_0 = \frac{\hbar}{mc\alpha}\,, \qquad \alpha = \frac{e^2}{\hbar c}\,,$$

one can rewrite the shifted energy as:

$$E^{(1)} = \tfrac{4}{3} g_N \alpha^4 \frac{m^2c^2}{M_N \hbar} \langle \vec{\boldsymbol{I}} \boldsymbol{\cdot} \vec{\boldsymbol{S}} \rangle\,.$$

We now introduce the total angular momentum, $\vec{\boldsymbol{F}} \equiv \vec{\boldsymbol{L}} + \vec{\boldsymbol{S}} + \vec{\boldsymbol{I}}$. In the ground state (where $\ell = 0$), we have $\vec{\boldsymbol{F}} = \vec{\boldsymbol{S}} + \vec{\boldsymbol{I}}$. Hence, we can write:

$$\vec{\boldsymbol{I}} \boldsymbol{\cdot} \vec{\boldsymbol{S}} = \tfrac{1}{2}\left[(\vec{\boldsymbol{S}} + \vec{\boldsymbol{I}})^2 - \vec{\boldsymbol{S}}^{\,2} - \vec{\boldsymbol{I}}^{\,2}\right] = \tfrac{1}{2}\left[\vec{\boldsymbol{F}}^{\,2} - \vec{\boldsymbol{S}}^{\,2} - \vec{\boldsymbol{I}}^{\,2}\right]\,.$$

The electron has spin-$\frac{1}{2}$ and the deuteron has spin-1. Hence,

$$\left\langle \vec{\boldsymbol{S}}^{\,2} \right\rangle = \tfrac{3}{4}\hbar^2\,, \qquad \left\langle \vec{\boldsymbol{I}}^{\,2} \right\rangle = 2\hbar^2\,.$$

In general,

$$\left\langle \vec{\boldsymbol{F}}^{\,2} \right\rangle = f(f+1)\hbar^2\,,$$

with respect to the total angular momentum basis. Since $\ell = 0$, adding spin-$\frac{1}{2}$ to spin-1 yields a total angular momentum equal to either $\frac{1}{2}$ or $\frac{3}{2}$. That is, in the total angular momentum basis,

$$\left\langle \vec{\boldsymbol{I}} \boldsymbol{\cdot} \vec{\boldsymbol{S}} \right\rangle = \hbar^2\left[f(f+1) - \tfrac{11}{4}\right] = \begin{cases} 1\,, & \text{for } f = \frac{3}{2}\,, \\ -2\,, & \text{for } f = \frac{1}{2}\,. \end{cases}$$

Hence,

$$E^{(1)} = \tfrac{2}{3} g_d \alpha^4 \frac{m^2c^2}{M_d} \times \begin{cases} 1\,, & \text{for } f = \frac{3}{2}\,, \\ -2\,, & \text{for } f = \frac{1}{2}\,, \end{cases}$$

and so

$$\Delta E_d \equiv E^{(1)}(f = \tfrac{3}{2}) - E^{(1)}(f = \tfrac{1}{2}) = \frac{2g_d\alpha^4 m^2c^2}{M_d}\,.$$

The wavelength of the emitted photon in a transition from the $f = \frac{3}{2}$ to the $f = \frac{1}{2}$ state is $\lambda_d = 2\pi\hbar c/\Delta E_d$, or

$$\boxed{\lambda_d = \frac{\pi\hbar c}{\alpha^4 g_d mc^2}\left(\frac{M_d}{m}\right) = \frac{\pi(1.973\times 10^{-5})(137)^4}{(1.71)(5.11\times 10^5)}(2)(1836)\ \text{cm} \simeq 92\ \text{cm}}$$

<u>*REMARK:*</u> It is instructive to normalize the result obtained above for deuterium to the corresponding result for hydrogen, which was obtained in class. In class, we found that

$$\Delta E_p \equiv E^{(1)}(f = 1) - E^{(1)}(f = 0) = \frac{4g_p\alpha^4 m^2c^2}{3M_p}\,,$$

where the subscript p refers to the proton, which makes up the nucleus of the hydrogen atom. Thus,

$$\frac{\lambda_d}{\lambda_p} = \frac{\Delta E_p}{\Delta E_d} = \frac{2g_p M_d}{3g_d M_p}\,.$$

The deuteron mass is approximately given by twice the proton mass. Using $g_p = 5.59$, $g_d = 1.71$ and $\lambda_p = 21$ cm, we find:

$$\lambda_d = \frac{4}{3}\left(\frac{5.59}{1.71}\right)(21\text{ cm}) = 92\text{ cm}\,,$$

as before.

3. Consider a positively-charged spin-1/2 particle in an external magnetic field, governed by the Hamiltonian:

$$H = H_0\,\mathbb{I} - \gamma\vec{\boldsymbol{B}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\,,$$

where $\mathbb{I}$ is the identity operator in spin space, $\vec{\boldsymbol{S}}$ is the vector of spin-1/2 spin matrices, and γ is a constant (for a positively-charged particle, $\gamma > 0$). H_0 is spin-independent and is independent of the magnetic field $\vec{\boldsymbol{B}}$. For simplicity, assume that H_0 possesses exactly one eigenvalue, which is denoted by E.

(a) If the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ (where $B > 0$), determine the energy eigenstates and eigenvalues of H.

Since H_0 is spin-independent, it follows that the both H_0 and H commute with $\vec{\boldsymbol{S}}^2$ and S_z. Consequently, the eigenstates of H can be chosen to be simultaneous eigenstates of $\vec{\boldsymbol{S}}^2$ and S_z. Under the assumptions of this problem,

$$H_0\left|\tfrac{1}{2}\,m_s\right\rangle = E\left|\tfrac{1}{2}\,m_s\right\rangle\,, \qquad\qquad m_s = -\tfrac{1}{2}\,,\,+\tfrac{1}{2}\,.$$

Using $S_z\left|\tfrac{1}{2}\,m_s\right\rangle = \hbar m_s\left|\tfrac{1}{2}\,m_s\right\rangle$, it follows that:

$$H\left|\tfrac{1}{2}\,m_s\right\rangle = [H_0\mathbb{I} - \gamma B S_z]\left|\tfrac{1}{2}\,m_s\right\rangle = (E - \hbar m_s\gamma B)\left|\tfrac{1}{2}\,m_s\right\rangle$$

In what follows, we shall denote the two possible energy eigenvalues by:

$$E_\pm = E - \hbar m_s\gamma B\,, \qquad\qquad m_s = \pm\tfrac{1}{2}\,.$$

In particular, the energy difference of the two states is given by:

$$E_- - E_+ = \hbar\gamma B\,, \tag{6}$$

which implies that $E_- > E_+$ if $\gamma > 0$.

(b) Assume that the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ for time $t < 0$. The system is initially observed to be in a spin-up state. At $t = 0$, a time-dependent perturbation is added by modifying the magnetic field. The new magnetic field for $t > 0$ is given by:

$$\vec{\boldsymbol{B}} = b\,(\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t) + B\hat{\boldsymbol{z}}\,,$$

where $b > 0$. Using first-order time-dependent perturbation theory, derive an expression for the probability that the system will be found in a spin-down state at some later time $t = T$.

We shall denote $c_{m_s}(t)$ as the probability amplitude for the spin-$\frac{1}{2}$ to be in an eigenstate of S_z with eigenvalue $\hbar m_s$ at time t. Using first-order time-dependent perturbation theory [cf. eq. (13.51) on p. 711 of Liboff],

$$c_{-1/2}(t) = c_{-1/2}(0) - \frac{i}{\hbar}\int_0^t \big\langle \tfrac{1}{2}\ -\tfrac{1}{2}\big|\, H'(t)\, \big|\tfrac{1}{2}\ \ \tfrac{1}{2}\big\rangle\, e^{i(E_- - E_+)t'/\hbar}\, dt'\,, \tag{7}$$

The time-dependent perturbing Hamiltonian is given by:

$$H'(t) = -\gamma b(S_x\cos\omega t - S_y\sin\omega t) = -\tfrac{1}{2}\hbar\gamma b(\sigma_x\cos\omega y - \sigma_y\sin\omega t) = -\tfrac{1}{2}\hbar\gamma b\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0\end{pmatrix}.$$

Thus,

$$\big\langle \tfrac{1}{2}\ -\tfrac{1}{2}\big|\, H'(t)\, \big|\tfrac{1}{2}\ \ \tfrac{1}{2}\big\rangle = -\tfrac{1}{2}\gamma b\begin{pmatrix} 0 & 1\end{pmatrix}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix} = -\tfrac{1}{2}\gamma b e^{-i\omega t}\,.$$

By assumption, the system is observed in a spin-up state at $t = 0$, so that $c_{-1/2}(0) = 0$. Eqs. (6) and (7) then yield:

$$c_{-1/2}(t) = \tfrac{1}{2}i\gamma b\int_0^t e^{i(\gamma B-\omega)t'}\,dt' = \frac{\gamma b}{2(\gamma B-\omega)}\left[e^{i(\gamma B-\omega)t} - 1\right]\,.$$

The state is observed at time $t = T$. The probability that the state will be observed as spin-down is $P_{\frac{1}{2}\to-\frac{1}{2}} = |c_{-1/2}(T)|^2$. Explicitly,

$$\begin{aligned} P_{\frac{1}{2}\to-\frac{1}{2}} = |c_{-1/2}(T)|^2 &= \frac{\gamma^2 b^2}{4(\gamma B-\omega)^2}\left[e^{i(\gamma B-\omega)T} - 1\right]\left[e^{-i(\gamma B-\omega)T} - 1\right] \\ &= \frac{\gamma^2 b^2}{2(\gamma B-\omega)^2}\left(1 - \cos(\gamma B-\omega)T\right] \end{aligned}$$

Using the identity $\sin^2\left(\frac{1}{2}\theta\right) \equiv \frac{1}{2}(1-\cos\theta)$, it follows that:

$$\boxed{P_{\frac{1}{2}\to-\frac{1}{2}} = |c_{-1/2}(T)|^2 = \frac{\gamma^2 b^2}{(\gamma B-\omega)^2}\sin^2\left[\tfrac{1}{2}(\gamma B-\omega)T\right]} \tag{8}$$

(c) For what range of values of ω are the results of part (b) unreliable?

First-order perturbation theory is valid if $P_{\frac{1}{2}\to -\frac{1}{2}} \ll 1$. Since the sine function in eq. (8) can be as large as 1, it follows that the coefficient of the sine must be small. Hence,

$$\left|\frac{\gamma b}{\gamma B - \omega}\right| \ll 1\,. \tag{9}$$

That is, ω cannot be too close in value to γB.

In class, we studied the phenomena of magnetic resonance, in which we solved the Schrodinger equation exactly for this problem. In the notation of this problem, the result obtained in class was:

$$P_{\frac{1}{2}\to -\frac{1}{2}} = \frac{\gamma^2 b^2}{(\gamma B - \omega)^2 + \gamma^2 b^2}\,\sin^2\left[\tfrac{1}{2}T\sqrt{(\gamma B - \omega)^2 + \gamma^2 b^2}\right]\,.$$

Comparing this result with eq. (8), we confirm that the first-order perturbation theory result is accurate if eq. (9) is satisfied. However, if ω is close to the resonance condition of $\omega = \gamma B$, then the first-order perturbation theory computation is unreliable, and one must use the exact result given above.

REMARK: Normally, one might have guessed that the reliability of the first-order perturbation theory depends on the magnitude of b, since the perturbing Hamiltonian is proportional to b. This is true as long as one is far from the resonance condition. If $|\gamma B - \omega| \sim \mathcal{O}(\gamma B)$, then it is clear that the first-order perturbation theory result is reliable if $b \ll B$. However, if the resonance condition is exactly satisfied, then eq. (9) cannot be satisfied no matter how small b is, in which case the first-order perturbation theory result can never be reliable.

4. In class, we computed the phase shifts for the scattering by a hard sphere of radius a. In the low energy limit (*i.e.*, for $ka \ll 1$), the ℓth partial wave phase shift is given by:

$$\delta_\ell \simeq c_\ell (ka)^{2\ell+1}\,, \tag{10}$$

where $c_\ell = -1/\left\{(2\ell+1)[(2\ell-1)!!]^2\right\}$. We concluded that at low energies, only the lowest partial waves are important. One can prove a more general result: at low energies, δ_ℓ is of the form given by eq. (10) for any short-ranged spherically symmetric potential, where the parameters c_ℓ depend on the form of the potential.

(a) Using the results stated above, show that at low energies, the differential cross section for scattering off *any* short-ranged spherically symmetric potential is approximately isotropic (*i.e.*, independent of scattering angle). Compute the corresponding total cross-section and show that it is determined by one unknown parameter (which depends on the form of the potential).

Eq. (10) implies that at low energies where $ka \ll 1$, we have $\delta_\ell \ll 1$ and $\delta_{\ell+1} \ll \delta_\ell$. The partial wave expansion of the scattering amplitude is given by:

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta)\,, \tag{11}$$

and the differential and total cross-sections are given by:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2\,, \qquad\qquad \sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = 2\pi\int_{-1}^{1}|f(\theta)|^2\,d\cos\theta\,. \tag{12}$$

Since $\delta_\ell \ll 1$, one can approximate

$$e^{i\delta_\ell}\sin\delta_\ell \simeq \delta_\ell\,, \qquad \text{for} \quad ka \ll 1\,. \tag{13}$$

At low energies in first approximation, we can neglect all partial waves beyond $\ell = 0$. In this case, eqs. (11) and (13) yields

$$f(\theta) \simeq \frac{\delta_0}{k}\,, \qquad \text{for} \quad ka \ll 1\,.$$

Using eq. (10), it follows that $\delta_0 \simeq c_0 ka$ at low energies, in which case,

$$f(\theta) \simeq c_0 a\,,$$

which is independent of angle. Hence the scattering amplitude is approximately isotropic. Eq. (12) then yields:

$$\boxed{\frac{d\sigma}{d\Omega} = c_0^2 a^2\,, \qquad \sigma = 4\pi c_0^2 a^2\,, \qquad \text{for} \quad ka \ll 1}$$

(b) Now, raise the energy of scattering slightly so that *only the lowest-order corrections* that yield a non-trivial angular distribution are required for a good approximation. Assume that the energy is still sufficiently low such that the form for δ_ℓ given in eq. (10) is valid. In this case, find the most general form for the angular dependence of the differential cross-section, and show that it is determined in terms of two unknown parameters.

If the energy is raised slightly, then we must include both the $\ell = 0$ and the $\ell = 1$ partial waves. Higher partial waves can still be neglected. Then, eq. (11) yields:

$$f(\theta) \simeq \frac{1}{k}\left[e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1 P_1(\cos\theta)\right]\,.$$

Using $P_1(\cos\theta) = \cos\theta$ and again employing eq. (13) for both the $\ell = 0$ and $\ell = 1$ partial wave amplitudes, we obtain:

$$f(\theta) = \frac{1}{k}\left[\delta_0 + 3\delta_1 \cos\theta\right] .$$

Using eq. (10), we can write $\delta_0 \simeq c_0 ka$ and $\delta_1 \simeq c_1(ka)^3$. The scattering amplitude $f(\theta)$ can then be written as:

$$f(\theta) = c_0 a \left[1 + \frac{3c_1}{c_0}(ka)^2 \cos\theta\right] .$$

The cross-section is obtained by squaring the scattering amplitude. Since $ka \ll 1$ by assumption, and the problem instructs us to keep only the leading term that contributes to a non-trivial angular distribution, we shall neglect the term proportional to $(ka)^4 \cos^2\theta$ in the differential cross-section. Thus,

$$\boxed{\frac{d\sigma}{d\Omega} \simeq c_0^2 a^2 \left[1 + \frac{6c_1}{c_0}(ka)^2 \cos\theta\right]} \tag{14}$$

That is, the differential cross-section at low-energies is determined by two unknown parameters and has the general form

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta ,$$

where $B/A \sim \mathcal{O}(k^2a^2)$.

(c) [*EXTRA CREDIT*] Consider the case of a weakly attractive spherical potential well of depth V_0 and radius a. For this problem, "weakly attractive" means that the potential energy is given by $V(r) = -|V_0|$ for $r < a$ and $2m|V_0|a^2/\hbar^2 \ll 1$. Using the results of part (b), compute the leading contribution at low energies to the so-called forward-backward asymmetry, $A_{\rm FB}$, which is defined by

$$A_{FB} \equiv \frac{\sigma_{\rm F} - \sigma_{\rm B}}{\sigma_{\rm F} + \sigma_{\rm B}} ,$$

where (after integrating over the azimuthal angle ϕ),

$$\sigma_{\rm F} \equiv 2\pi \int_0^1 \frac{d\sigma}{d\Omega}\, d\cos\theta , \qquad \sigma_{\rm B} \equiv 2\pi \int_{-1}^0 \frac{d\sigma}{d\Omega}\, d\cos\theta .$$

Because the condition $2m|V_0|a^2/\hbar^2 \ll 1$ is satisfied, the Born approximation is a reliable approximation in the low energy limit. In class, the following result was given for the Born approximation to the phase shifts:

$$e^{i\delta_\ell} \sin\delta_\ell \simeq \delta_\ell = -\frac{2mk}{\hbar^2} \int_0^\infty V(r)[j_\ell(kr)]^2 r^2\, dr .$$

For the spherical attractive square well potential,

$$V(r) = \begin{cases} -|V_0|\,, & \text{for } r < a\,, \\ 0\,, & \text{for } r > a\,. \end{cases}$$

Moreover, in the low-energy limit ($ka \ll 1$), we can approximate

$$j_\ell(kr) \simeq \frac{(kr)^\ell}{(2\ell+1)!!}\,, \qquad \text{for } r < a\,.$$

Hence,

$$\delta_\ell \simeq \frac{2m|V_0|k}{\hbar^2}\int_0^a \left[\frac{(kr)^\ell}{(2\ell+1)!!}\right]^2 r^2\,dr = \frac{2m|V_0|k^{2\ell+1}}{[(2\ell+1)!!]^2\hbar^2}\int_0^a r^{2\ell+2}dr\,.$$

Evaluating the integral yields

$$\delta_\ell = \left(\frac{2m|V_0|a^2}{\hbar^2}\right)\frac{(ka)^{2\ell+1}}{(2\ell+3)[(2\ell+1)!!]^2}\,.$$

Comparing with eq. (10), we identify:

$$c_\ell = \frac{2m|V_0|a^2}{\hbar^2(2\ell+3)[(2\ell+1)!!]^2}\,. \tag{15}$$

If only the $\ell = 0$ and $\ell = 1$ partial waves are significant, then the results of part (b) indicate that the differential scattering cross-section is of the form:

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta\,,$$

where eq. (14) implies that

$$\frac{B}{A} = \frac{6c_1(ka)^2}{c_0}\,.$$

Using eq. (15), it follows that $c_1/c_0 = 1/15$. Thus, we conclude that

$$\frac{B}{A} = \frac{2}{5}(ka)^2\,. \tag{16}$$

To compute the forward-backward asymmetry, $A_{\rm FB}$, we compute:

$$\int_0^1 (A + B\cos\theta)\,d\cos\theta = A + \tfrac{1}{2}B\,, \qquad \int_{-1}^0 (A + B\cos\theta)\,d\cos\theta = A - \tfrac{1}{2}B\,.$$

Hence,

$$A_{\rm FB} = \frac{(A+\frac{1}{2}B) - (A-\frac{1}{2}B)}{(A+\frac{1}{2}B) + (A-\frac{1}{2}B)} = \frac{B}{2A}\,.$$

Using the result for B/A obtained in eq. (16), the leading contribution to $A_{\rm FB}$ at low-energies ($ka \ll 1$) is given by:

$$\boxed{A_{\rm FB} \simeq \frac{1}{5}(ka)^2}$$

REMARK: One can also solve this problem by explicitly computing the s-wave and p-wave phase shifts and then taking the low-energy limit. Liboff solves the relevant s-wave scattering problem on p. 770–771. On problem 2(d) of Homework set #5, we studied the low-energy behavior of the s-wave phase shift. One consequence of those results was that $c_0 = 2mV_0a^2/(3\hbar^2)$, in agreement with eq. (15). One can solve for the p-wave phase shift in a similar manner, although the algebra is somewhat tedious. Still, this is an exercise worth pursuing. You should check the corresponding low-energy limit and verify that the resulting value for c_1 also agrees with eq. (15).