1. Consider a spin-$\frac{1}{2}$ particle with magnetic moment $\vec{\boldsymbol{\mu}} = \gamma\vec{\boldsymbol{S}}$. At time $t = 0$, we measure S_y and find a value of $+\frac{1}{2}\hbar$ for its eigenvalue. Immediately after this measurement, we apply a uniform time-dependent magnetic field parallel to the z-axis. The B-field is chosen such that the Hamiltonian is:

$$H(t) = \omega_o(t) S_z \,,$$

where

$$\omega_o(t) = \begin{cases} 0\,, & \text{for } t < 0\,, \\ \dfrac{\omega_o t}{T}\,, & \text{for } 0 \le t \le T\,, \\ 0\,, & \text{for } t > T\,. \end{cases}$$

(a) Write down the time-dependent Schrodinger equation that governs the time evolution of the spin-$\frac{1}{2}$ particle of this problem.

The time-dependent Schrodinger equation is

$$H\,|\psi(t)\rangle = i\hbar\frac{d}{dt}\,|\psi(t)\rangle\ .$$

If we represent $|\psi(t)\rangle$ by a two-component spinor,

$$|\psi(t)\rangle = \begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix}\ ,$$

and take

$$H = \omega_o(t) S_z = \tfrac{1}{2}\hbar\omega_o(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,$$

after noting that $S_z = \frac{1}{2}\hbar\sigma_z$, then the time-dependent Schrodinger equation can be written as:

$$i\hbar\frac{d}{dt}\begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix} = \tfrac{1}{2}\hbar\omega_o(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix} = \tfrac{1}{2}\hbar\omega_o(t)\begin{pmatrix} a_+(t) \\ -a_-(t) \end{pmatrix}\ .$$

The above matrix differential equation can be written more explicitly as:

$$i\frac{d}{dt}a_+(t) = \tfrac{1}{2}\omega_o(t) a_+(t)\,,$$

$$i\frac{d}{dt}a_-(t) = -\tfrac{1}{2}\omega_o(t) a_-(t)\,.$$

(b) Show that at time t, the particle wave function is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{i\theta(t)}\alpha + ie^{-i\theta(t)}\beta\right] ,$$

where α and β are eigenfunctions of S_z with eigenvalues $\pm\frac{1}{2}\hbar$, respectively, and $\theta(t)$ is a real function of time that you should determine explicitly.

The two differential equations obtained in part (a) are uncoupled, and hence can be solved easily. Writing both equations simultaneously,

$$i\frac{d}{dt}a_\pm(t) = \pm\tfrac{1}{2}\omega_o(t)a_\pm(t) ,$$

where the upper [lower] signs yields the first [second] differential equation. Integrating, one obtains

$$\int \frac{da_\pm(t)}{a_\pm(t)} = \pm\frac{1}{2i}\int \omega_o(t')dt' ,$$

which yields

$$\ln\left(\frac{a_\pm(t)}{a_\pm(0)}\right) = \pm\frac{1}{2i}\int_0^t \omega_o(t')dt' .$$

Inserting the explicit form for $\omega_o(t)$ given at the beginning of this problem, one obtains:

$$\int_0^t \omega_o(t')dt' = \begin{cases} \dfrac{\omega_o}{T}\displaystyle\int_0^t t'dt' , & \text{for } 0 \leq t \leq T , \\ \dfrac{\omega_o}{T}\displaystyle\int_0^T t'dt' , & \text{for } t \geq T . \end{cases}$$

Evaluating the integrals yields,

$$\int_0^t \omega_o(t')dt' = \begin{cases} \dfrac{\omega_o t^2}{2T} , & \text{for } 0 \leq t \leq T , \\ \tfrac{1}{2}\omega_o T , & \text{for } t \geq T . \end{cases}$$

Hence, we conclude that:

$$a_\pm(t) = \begin{cases} a_\pm(0)\exp\left(\mp\dfrac{i\omega_o t^2}{4T}\right) , & \text{for } 0 \leq t \leq T , \\ a_\pm(0)\exp\left(\mp\tfrac{i}{4}\omega_o T\right) , & \text{for } t \geq T . \end{cases}$$

The problem states that at time $t = 0$, we measure S_y and find a value of $+\hbar$ for its eigenvalue. This means that $|\psi(0)\rangle$ is an eigenstate of S_y with eigenvalue $+\hbar$. Since

$$S_y = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ,$$

then the normalized initial state is represented by

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix},$$

or equivalently,

$$a_+(0) = \frac{1}{\sqrt{2}}, \qquad\qquad a_-(0) = \frac{i}{\sqrt{2}}.$$

Inserting these results into our expressions for $a_\pm(t)$ above then yields,

$$|\psi(t)\rangle = \begin{cases} \dfrac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t^2/(4T)} \\ \\ ie^{i\omega_o t^2/(4T)}\end{pmatrix}, & \text{for } 0 \leq t \leq T\,, \\ \\ \dfrac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0/4} \\ \\ ie^{i\omega_o T/4}\end{pmatrix}, & \text{for } t \geq T\,. \end{cases}$$

If we write:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{i\theta(t)}\alpha + ie^{-i\theta(t)}\beta\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta(t)} \\ \\ ie^{-i\theta(t)}\end{pmatrix},$$

then we can identify:

$$\theta(t) = \begin{cases} -\dfrac{\omega_o t^2}{4T}, & \text{for } 0 \leq t \leq T\,, \\ -\tfrac{1}{4}\omega_o T\,, & \text{for } t \geq T\,. \end{cases}$$

(c) At a time $t > T$, we measure S_y. What are the possible results of this measurement and with what probabilities?

The possible results of this measurement are $\pm\frac{1}{2}\hbar$. To determine the corresponding probabilities, we expand $|\psi(t)\rangle$ in terms of the eigenstates of S_y:

$$\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, \qquad\qquad \left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix},$$

where $S_y\left|\frac{1}{2}\,,\,\pm\frac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} = \pm\frac{1}{2}\hbar\left|\frac{1}{2}\,,\,\pm\frac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}}$. Here, we have used the notation $|s\,,\,m_s\rangle_{\hat{\boldsymbol{n}}}$ to represent a spin s state that is an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ with eigenvalue $\hbar m_s$. Thus, one can write:

$$|\psi(t)\rangle = c_+\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} + c_-\left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}}.$$

The probability of measuring S_y and obtaining $\pm\frac{1}{2}\hbar$ is $|c_\pm|^2$. To determine the $c_\pm$, we multiply on the left by $\left\langle\frac{1}{2}\,,\,\pm\frac{1}{2}\right|$. That is,

$$c_\pm = \left\langle\tfrac{1}{2}\,,\,\pm\tfrac{1}{2}\right|\psi(t)\rangle\,.$$

Since $t > T$, it follows that[1]

$$c_\pm = \frac{1}{\sqrt{2}}(1 \quad \mp i)\begin{pmatrix} e^{-i\omega_0/4} \\ ie^{i\omega_o T/4} \end{pmatrix} = \pm\tfrac{1}{2}\left(e^{i\omega_o T/4} \pm e^{-i\omega_o T/4}\right) ,$$

which can be rewritten as

$$c_+ = \cos(\tfrac{1}{4}\omega_o T) , \qquad\qquad c_- = -i\sin(\tfrac{1}{4}\omega_o T) .$$

Thus, the relevant probabilities corresponding to S_y measurements of $\pm\frac{1}{2}\hbar$ are:

$$\begin{aligned} \tfrac{1}{2}\hbar : &\qquad \cos^2\left(\tfrac{1}{4}\omega_o T\right) , \\ -\tfrac{1}{2}\hbar : &\qquad \sin^2\left(\tfrac{1}{4}\omega_o T\right) , \end{aligned}$$

(d) Find a relation between ω_0 and T such that the measurement of S_y yields a unique result. Interpret the physical significance of this result.

In order that the measurement of S_y yield a unique result, one must either have $\cos(\frac{1}{4}\omega_o T) = 0$ or $\sin(\frac{1}{4}\omega_o T) = 0$. Either the first or the second of these two conditions is satisfied if $\frac{1}{4}\omega_o T = \frac{1}{2}n\pi$. That is,

$$\boxed{\omega_o T = 2n\pi , \qquad\qquad n = 1\,,\,2\,,\,3\,,\,\ldots}$$

Note that only positive integers n are allowed, since by assumption $T > 0$.

The physical interpretation of this result is that in the presence of $H(t)$, the spin-$\frac{1}{2}$ particle of definite spin pointing along the y-axis begins to precess about the z-axis (corresponding to the direction of the uniform magnetic field). At time $t > T$, the total precession angle is given by

$$\int_o^T \omega_o(t')dt' = \tfrac{1}{2}\omega_o T .$$

If the total precession angle is some integer multiple of π, then the spin will end up still pointing along the (positive or negative) y-axis. This condition is

$$\tfrac{1}{2}\omega_o T = n\pi , \qquad\qquad n = 1\,,\,2\,,\,3\,,\,\ldots\,,$$

which reproduces our previous result.

[1]Note that the adjoint of the ket $\left|\frac{1}{2}\,,\,\pm\frac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ is the bra ${}_{\hat{\boldsymbol{y}}}\left\langle\frac{1}{2}\,,\,\pm\frac{1}{2}\right| = \frac{1}{\sqrt{2}}(1 \quad \mp i)$, respectively. That is, in obtaining the bra, one must remember to complex-conjugate the factor of i.

2. Consider a spin system made up of a spin-1/2 particle (with corresponding spin operator $\vec{\boldsymbol{S}}_1$) and a spin-1 particle (with corresponding spin operator $\vec{\boldsymbol{S}}_2$).

(a) Suppose that a bound state of the two spins exists in a state of zero relative orbital angular momentum. What are the possible values of the total angular momentum (i.e., the total spin) of the bound system?

In class, we learned that given a composite system made up of angular momentum j_1 and angular momentum j_2, then the possible values of the total angular momentum of the composite system are:

$$j = |j_1 - j_2|\,,\ |j_1 - j_2| + 1\,,\ \ldots\,,\ j_1 + j_2\,.$$

In this problem, $j_1 = 1$ and $j_2 = \frac{1}{2}$, and we conclude that the possible values of the total spin of the composite system are $j = \frac{1}{2},\,,\ \frac{3}{2}$.

(b) Consider the bound state of the two spins described in part (a). Suppose one determines that this state has total spin equal to $\frac{1}{2}$ and the spin points up with respect to the z-direction. Express this state as a linear combination of the product basis states of the two spins.

This is a simple exercise in reading the table of Clebsch-Gordon coefficients. A close examination of the entries in the $1 \times 1/2$ portion of the table reveals that:

$$\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\,;\,1\,,\,\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\;|1\,,\,1\rangle_1 \otimes \left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_2 - \sqrt{\tfrac{1}{3}}\;|1\,,\,0\rangle_1 \otimes \left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_2\,,$$

using the notation of Table 9.5 on p. 394 of Liboff. Note that Liboff suppresses the direct product symbol $\otimes$.

(c) The Hamiltonian of the spin system of part (a) is given by:

$$H = A + \frac{B\vec{\boldsymbol{S}}_{\mathbf{1}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}_{\mathbf{2}}}{\hbar^2} + \frac{C(S_{1z} + S_{2z})}{\hbar}\,.$$

Find the eigenvalues and eigenstates of the system.

First, we note that $S_z = S_{1z} + S_{2z}$, where $\vec{\boldsymbol{S}} = \vec{\boldsymbol{S}}_{\mathbf{1}} + \vec{\boldsymbol{S}}_{\mathbf{2}}$ is the total spin operator. The key step is to note that $\vec{\boldsymbol{S}}^2 = (\vec{\boldsymbol{S}}_{\mathbf{1}} + \vec{\boldsymbol{S}}_{\mathbf{2}})^2 = \vec{\boldsymbol{S}}_{\mathbf{1}}^2 + \vec{\boldsymbol{S}}_{\mathbf{1}}^2 + 2\vec{\boldsymbol{S}}_{\mathbf{1}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}_{\mathbf{2}}$. Thus,

$$\vec{\boldsymbol{S}}_{\mathbf{1}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}_{\mathbf{2}} = \tfrac{1}{2}\left(\vec{\boldsymbol{S}}^2 - \vec{\boldsymbol{S}}_{\mathbf{1}}^2 - \vec{\boldsymbol{S}}_{\mathbf{2}}^2\right)\,.$$

We can therefore rewrite H as:

$$H = A + \frac{B}{2\hbar^2}\left(\vec{\boldsymbol{S}}^2 - \vec{\boldsymbol{S}}_{\mathbf{1}}^2 - \vec{\boldsymbol{S}}_{\mathbf{2}}^2\right) + \frac{CS_z}{\hbar}\,.$$

Thus, the eigenstates of H are simultaneous eigenstates of $\vec{\boldsymbol{S}}^2$, S_z, $\vec{\boldsymbol{S}}_{\mathbf{1}}^2$ and $\vec{\boldsymbol{S}}_{\mathbf{2}}^2$, i.e., the total spin basis. Using

$$
\begin{aligned}
\vec{\boldsymbol{S}}^2 \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle &= \hbar^2 s(s+1) \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle\ , \\
S_z \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle &= \hbar m_s \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle\ , \\
\vec{\boldsymbol{S}}_{\mathbf{1}}^2 \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle &= \hbar^2 s_1(s_1+1) \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle\ , \\
\vec{\boldsymbol{S}}_{\mathbf{2}}^2 \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle &= \hbar^2 s_2(s_2+1) \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle\ ,
\end{aligned}
$$

with $s_1 = 1$ and $s_2 = \frac{1}{2}$, it follows that

$$
\begin{aligned}
H \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle &= \left(A + \tfrac{1}{2}B\left[s(s+1) - 1(1+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right] + Cm_s\right) \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle \\
&= \left(A + \tfrac{1}{2}B\left[s(s+1) - \tfrac{11}{4}\right] + Cm_s\right) \left|s\,,\, m_s\,;\, s_1\,,\, s_2\right\rangle\ .
\end{aligned}
$$

The possible values of s are $\frac{1}{2}$, $\frac{3}{2}$ and the corresponding $m_s = -s\,,\, -s+1\,,\, \ldots\,,\, s-1\,,\, s$. Thus the eigenstates and eigenvalues of the system are given in the following table:

s	m_s	eigenvalue of H
$\frac{3}{2}$	$\frac{3}{2}$	$A + \frac{1}{2}B + \frac{3}{2}C$
$\frac{3}{2}$	$\frac{1}{2}$	$A + \frac{1}{2}B + \frac{1}{2}C$
$\frac{3}{2}$	$-\frac{1}{2}$	$A + \frac{1}{2}B - \frac{1}{2}C$
$\frac{3}{2}$	$-\frac{3}{2}$	$A + \frac{1}{2}B - \frac{3}{2}C$
$\frac{1}{2}$	$\frac{1}{2}$	$A - B + \frac{1}{2}C$
$\frac{1}{2}$	$-\frac{1}{2}$	$A - B - \frac{1}{2}C$

3. Consider a charged particle (with charge q) whose motion is confined to a circle of radius R in the x–y plane, with its center at the origin. A thin magnetic flux tube of radius $r < R$ is located with its axis along the z-axis. The magnetic field is confined within the flux tube, and the total magnetic flux through the x–y plane is denoted by Φ. In particular, the charged particle moves in a region where there is no magnetic field. It is convenient to work in cylindrical coordinates (ρ, θ, z), where $x = \rho\cos\theta$ and $y = \rho\sin\theta$. In the region where there is no magnetic field, $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = 0$, which implies that

$$\vec{\boldsymbol{A}}(\rho, \theta, z) = \vec{\boldsymbol{\nabla}}\chi(\rho, \theta, z)\,. \qquad (1)$$

(a) Noting that Stokes' theorem relates Φ to the line integral of $\vec{\boldsymbol{A}}$ taken along the circle of radius R, show that the choice,

$$\chi(\rho, \theta, z) = \frac{\Phi\theta}{2\pi}\,,$$

satisfies Stokes' theorem and the Coulomb gauge condition.

As a first step, let us insert the value of χ into eq. (1) and evaluate $\vec{\boldsymbol{A}}$. Using:

$$\vec{\boldsymbol{\nabla}}\chi = \hat{\boldsymbol{\rho}}\frac{\partial\chi}{\partial\rho} + \hat{\boldsymbol{\theta}}\frac{1}{\rho}\frac{\partial\chi}{\partial\theta} + \hat{\boldsymbol{z}}\frac{\partial\chi}{\partial z}\,,$$

it follows that:

$$\vec{\boldsymbol{A}} = \frac{\Phi}{2\pi\rho}\hat{\boldsymbol{\theta}}\,.$$

We can now verify that Stokes' theorem is satisfied. Choose the curve C to be a circle of radius R that lies in the x–y plane centered at the origin. Then $d\vec{\boldsymbol{\ell}} = Rd\theta\hat{\boldsymbol{\theta}}$, and

$$\oint_C \vec{\boldsymbol{A}}\boldsymbol{\cdot} d\vec{\boldsymbol{\ell}} = \frac{\Phi}{2\pi R}R\int_0^{2\pi} d\theta = \Phi\,,$$

which indeed satisfies Stokes' theorem, since

$$\Phi \equiv \iint_S \vec{\boldsymbol{B}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,da = \iint_S (\boldsymbol{\nabla}\times\vec{\boldsymbol{A}})\boldsymbol{\cdot}\hat{\boldsymbol{n}}da = \oint_C \vec{\boldsymbol{A}}\boldsymbol{\cdot} d\vec{\boldsymbol{\ell}}.$$

In addition, the vector potential obtained above satisfies the Coulomb gauge condition, since

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} = 0\,,$$

after substituting $A_\theta = \Phi/(2\pi\rho)$ and $A_\rho = A_z = 0$ above.

(b) The wave function for the charged particle is only a function of θ (since $\rho = R$ and $z = 0$ are fixed due to the confined motion). Write down the time-independent Schrodinger equation for the charged particle wave function $\psi(\theta)$ in the cylindrical coordinate representation (simplifying your equation as much as possible).

The time-independent Schrodinger equation for a charged particle (with charge q) in an external electromagnetic field, in the Coulomb gauge, is given in the class handout:

$$\frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2\psi + \frac{iq\hbar}{mc}\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi + \frac{q^2}{2mc^2}\vec{\boldsymbol{A}}^2\psi + q\phi\psi = E\psi\,.$$

One can write this equation in cylindrical coordinates, using

$$\vec{\boldsymbol{\nabla}}^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}\,.$$

Noting that $\psi = \psi(\theta)$, since the other coordinates $\rho = R$ and $z = 0$ are fixed, the Schrodinger equation above can be written as:

$$\frac{-\hbar^2}{2mR^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{iq\hbar}{mcR}A_\theta\frac{\partial\psi}{\partial\theta} + \frac{q^2}{2mc^2}A_\theta^2\psi = E\psi\,, \qquad \text{where } A_\theta = \frac{\Phi}{2\pi R}\,. \tag{2}$$

This equation can be simplified. After inserting $A_\theta = \Phi/(2\pi R)$, we note that eq. (2) can be rewritten as:

$$\frac{1}{2mR^2}\left(i\hbar\frac{\partial}{\partial\theta} + \frac{q\Phi}{2\pi c}\right)^2\psi = E\psi\,. \tag{3}$$

(c) Solve the Schrodinger equation of part (b) for the energy eigenvalues and eigenfunctions. Show that the allowed energies depend on Φ even though the charged particle on the circle never encounters the magnetic field.

The solution to a second-order linear differential equation with constant coefficients is well known. Requiring that $\psi(\theta)$ is single-valued [that is, $\psi(\theta+2\pi) = \psi(\theta)$], we can take:

$$\psi(\theta) = \frac{1}{\sqrt{2\pi}}\,e^{in\theta}\,, \qquad n = 0\,, \pm1\,, \pm2\,, \ldots\,, \tag{4}$$

where the coefficient is chosen to normalize the wave function. Inserting the solution back into eq. (2) and solving for E, one obtains:

$$E = \frac{1}{2mR^2}\left[n^2\hbar^2 - \frac{q\hbar n\Phi}{\pi c} + \frac{q^2\Phi^2}{4\pi^2c^2}\right] = \frac{1}{2mR^2}\left(n\hbar - \frac{q\Phi}{2\pi c}\right)^2\,.$$

Alternatively, one can solve eq. (3) by noting that the energy eigenstates are also eigenstates of $\partial/\partial\theta$. This implies that the solution must have the form shown in eq. (4). Inserting this solution into eq. (3) immediately yields:

$$\boxed{E = \frac{1}{2mR^2}\left(n\hbar - \frac{q\Phi}{2\pi c}\right)^2\,, \qquad n = 0\,, \pm1\,, \pm2\,, \ldots}$$

as before. Indeed, the allowed energies depend on Φ even though the charged particle on the circle never encounters the magnetic field.