MIDTERM INSTRUCTIONS: You have one hour and forty five minutes to complete the midterm exam. This is an open book exam. You are permitted to consult the textbook (Liboff), your handwritten notes, and class handouts. No other consultations or collaborations are permitted during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.* The point value of each problem is indicated in the square brackets below.

1. [40] Consider a spin-$\frac{1}{2}$ particle with magnetic moment $\vec{\boldsymbol{\mu}} = \gamma\vec{\boldsymbol{S}}$. At time $t = 0$, we measure S_y and find a value of $+\frac{1}{2}\hbar$ for its eigenvalue. Immediately after this measurement, we apply a uniform time-dependent magnetic field parallel to the z-axis. The B-field is chosen such that the Hamiltonian is:

$$H(t) = \omega_o(t) S_z\,,$$

where

$$\omega_o(t) = \begin{cases} 0\,, & \text{for } t < 0\,, \\ \dfrac{\omega_o t}{T}\,, & \text{for } 0 \leq t \leq T\,, \\ 0\,, & \text{for } t > T\,. \end{cases}$$

(a) Write down the time-dependent Schrodinger equation that governs the time evolution of the spin-$\frac{1}{2}$ particle of this problem.

(b) Show that at time t, the particle wave function is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{i\theta(t)}\alpha + ie^{-i\theta(t)}\beta\right]\,,$$

where α and β are eigenfunctions of S_z with eigenvalues $\pm\frac{1}{2}\hbar$, respectively, and $\theta(t)$ is a real function of time that you should determine explicitly.

(c) At a time $t > T$, we measure S_y. What are the possible results of this measurement and with what probabilities?

(d) Find a relation between ω_0 and T such that the measurement of S_y yields a unique result. Interpret the physical significance of this result.

2. [30] Consider a spin system made up of a spin-1/2 particle (with corresponding spin operator $\vec{\boldsymbol{S}}_1$) and a spin-1 particle (with corresponding spin operator $\vec{\boldsymbol{S}}_2$).

(a) Suppose that a bound state of the two spins exists in a state of zero relative orbital angular momentum. What are the possible values of the total angular momentum (i.e., the total spin) of the bound system?

(b) Consider the bound state of the two spins described in part (a). Suppose one determines that this state has total spin equal to $\frac{1}{2}$ and the spin points up with respect to the z-direction. Express this state as a linear combination of the product basis states of the two spins.

(c) The Hamiltonian of the spin system of part (a) is given by:

$$H = A + \frac{B\vec{\boldsymbol{S}}_{\mathbf{1}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}_{\mathbf{2}}}{\hbar^2} + \frac{C(S_{1z} + S_{2z})}{\hbar}\,.$$

Find the eigenvalues and eigenstates of the system.

HINT: The eigenstates correspond to simultaneous eigenstates of the maximal commuting set of operators. Determine which operators belong in this set. Then identify an eigenstate by its eigenvalues with respect to this set of commuting operators.

3. [30] Consider a charged particle (with charge q) whose motion is confined to a circle of radius R in the x–y plane, with its center at the origin. A thin magnetic flux tube of radius $r < R$ is located with its axis along the z-axis. The magnetic field is confined within the flux tube, and the total magnetic flux through the x–y plane is denoted by Φ. In particular, the charged particle moves in a region where there is no magnetic field. It is convenient to work in cylindrical coordinates (ρ, θ, z), where $x = \rho\cos\theta$ and $y = \rho\sin\theta$. In the region where there is no magnetic field, $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = 0$, which implies that

$$\vec{\boldsymbol{A}}(\rho, \theta, z) = \vec{\boldsymbol{\nabla}}\chi(\rho, \theta, z)\,. \tag{1}$$

(a) Noting that Stokes' theorem relates Φ to the line integral of $\vec{\boldsymbol{A}}$ taken along the circle of radius R, show that the choice,

$$\chi(\rho, \theta, z) = \frac{\Phi\theta}{2\pi}\,,$$

satisfies Stokes' theorem and the Coulomb gauge condition.

FIRST HINT: Insert the value of χ into eq. (1) and evaluate $\vec{\boldsymbol{A}}$. Show that the vector potential points in the $\hat{\boldsymbol{\theta}}$ direction.

(b) The wave function for the charged particle is only a function of θ (since $\rho = R$ and $z = 0$ are fixed due to the confined motion). Write down the time-independent Schrodinger equation for the charged particle wave function $\psi(\theta)$ in the cylindrical coordinate representation (simplifying your equation as much as possible).

(c) Solve the Schrodinger equation of part (b) for the energy eigenvalues and eigenfunctions. Show that the allowed energies depend on Φ even though the charged particle on the circle never encounters the magnetic field.

SECOND HINT: Show that the energy eigenstates are also eigenstates of $\partial/\partial\theta$.

In problem 3, it is simplest to employ cylindrical coordinates. Some of the following results might be useful.

$$\vec{\boldsymbol{\nabla}}\chi = \hat{\boldsymbol{\rho}}\frac{\partial\chi}{\partial\rho} + \hat{\boldsymbol{\theta}}\frac{1}{\rho}\frac{\partial\chi}{\partial\theta} + \hat{\boldsymbol{z}}\frac{\partial\chi}{\partial z}\,,$$

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}\,,$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\boldsymbol{z}}\,\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\theta) - \frac{\partial A_\rho}{\partial\theta}\right)\,,$$

$$\vec{\boldsymbol{\nabla}}^2\chi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\chi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\chi}{\partial\theta^2} + \frac{\partial^2\chi}{\partial z^2}\,,$$

where χ denotes a scalar field and $\vec{\boldsymbol{A}} \equiv A_\rho\hat{\boldsymbol{\rho}} + A_\theta\hat{\boldsymbol{\theta}} + A_z\hat{\boldsymbol{z}}$ denotes a vector field.

For your convenience, I remind you that Stokes' theorem states that for a suitably behaved vector field,

$$\iint_S (\boldsymbol{\nabla}\times\vec{\boldsymbol{A}})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,da = \oint_C \vec{\boldsymbol{A}}\boldsymbol{\cdot}d\vec{\boldsymbol{\ell}},$$

where S is an open surface that caps the closed curve C, $d\vec{\boldsymbol{\ell}}$ is an infinitesimal tangent vector to the curve C, and $\hat{\boldsymbol{n}}$ is a unit vector that is normal to the surface S.