Physics 139B **Solutions to Homework Set 2** **Fall 2009**

1. Liboff, problem 9.32 on page 395.

(a) The key equation is given in Table 9.4 on p. 379 of Liboff:

$$L_{\pm}\,|\ell\,,m\rangle = \hbar\left[(\ell \mp m)(\ell \pm m + 1)\right]^{1/2}|\ell\,,m \pm 1\rangle\ . \tag{1}$$

We will also need to use:

$$\vec{\boldsymbol{L}}^2\,|\ell\,,m\rangle = \hbar^2\ell(\ell+1)\,|\ell\,,m\rangle\ . \tag{2}$$

When coupling two p states, there are nine possible states in the product basis: $|1\,,m_{\ell 1}\rangle_1 \otimes |1\,,m_{\ell 2}\rangle_2$, where $m_{\ell 1}$, $m_{\ell 2} = +1\,,\,0\,,\,-1$. Using

$$\vec{\boldsymbol{L}}^2 = \vec{\boldsymbol{L}}_1^2 + \vec{\boldsymbol{L}}_2^2 + 2L_{1z}L_{2z} + L_{1+}L_{2-} + L_{1-}L_{2+}\,,$$

we can operate with $\vec{\boldsymbol{L}}^2$ on the nine states of the product basis:

$$\begin{aligned}
\vec{\boldsymbol{L}}^2\,|1\,,1\rangle_1 \otimes |1\,,1\rangle_2 &= 6\hbar^2\,|1\,,1\rangle_1 \otimes |1\,,1\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,1\rangle_1 \otimes |1\,,0\rangle_2 &= 4\hbar^2\,|1\,,1\rangle_1 \otimes |1\,,0\rangle_2 + 2\hbar^2\,|1\,,0\rangle_1 \otimes |1\,,1\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,1\rangle_1 \otimes |1\,,-1\rangle_2 &= 2\hbar^2\,|1\,,1\rangle_1 \otimes |1\,,-1\rangle_2 + 2\hbar^2\,|1\,,0\rangle_1 \otimes |1\,,0\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,0\rangle_1 \otimes |1\,,1\rangle_2 &= 4\hbar^2\,|1\,,0\rangle_1 \otimes |1\,,1\rangle_2 + 2\hbar^2\,|1\,,1\rangle_1 \otimes |1\,,0\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,0\rangle_1 \otimes |1\,,0\rangle_2 &= 4\hbar^2\,|1\,,0\rangle_1 \otimes |1\,,0\rangle_2 + 2\hbar^2\,|1\,,1\rangle_1 \otimes |1\,,-1\rangle_2 + 2\hbar^2\,|1\,,-1\rangle_1 \otimes |1\,,1\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,0\rangle_1 \otimes |1\,,-1\rangle_2 &= 4\hbar^2\,|1\,,0\rangle_1 \otimes |1\,,-1\rangle_2 + 2\hbar^2\,|1\,,-1\rangle_1 \otimes |1\,,0\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,-1\rangle_1 \otimes |1\,,1\rangle_2 &= 2\hbar^2\,|1\,,-1\rangle_1 \otimes |1\,,1\rangle_2 + 2\hbar^2\,|1\,,0\rangle_1 \otimes |1\,,0\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,-1\rangle_1 \otimes |1\,,0\rangle_2 &= 4\hbar^2\,|1\,,-1\rangle_1 \otimes |1\,,0\rangle_2 + 2\hbar^2\,|1\,,0\rangle_1 \otimes |1\,,-1\rangle_2\ ,\\
\vec{\boldsymbol{L}}^2\,|1\,,-1\rangle_1 \otimes |1\,,-1\rangle_2 &= 6\hbar^2\,|1\,,-1\rangle_1 \otimes |1\,,-1\rangle_2\ .
\end{aligned}$$

In evaluating the above, I replaced $\vec{\boldsymbol{L}}^2$ with $\vec{\boldsymbol{L}}_1^2 + \vec{\boldsymbol{L}}_2^2 + 2L_{1z}L_{2z} + L_{1+}L_{2-} + L_{1-}L_{2+}$, and then used eqs. (1) and (2) repeatedly.

Likewise, we can operate with $\vec{\boldsymbol{L}}^2$ on the nine states of the total angular momentum bases, $|\ell\,,m\,;\,1\,1\rangle$,

$$\begin{aligned}
\vec{\boldsymbol{L}}^2\,|2\,,m\,;\,1\,1\rangle &= 6\hbar^2\,|2\,,m\,;\,1\,1\rangle\ , &\qquad m &= 2\,,\,1\,,\,0\,,\,-1\,,\,-2\,,\\
\vec{\boldsymbol{L}}^2\,|1\,,m\,;\,1\,1\rangle &= 2\hbar^2\,|2\,,m\,;\,1\,1\rangle\ , &\qquad m &= 1\,,\,0\,,\,-1\,,\\
\vec{\boldsymbol{L}}^2\,|0\,,0\,;\,1\,1\rangle &= 0\,,
\end{aligned}$$

It is now straightforward to verify Table 9.5 on p. 394 of Liboff. First, by operating with $L_z = L_{z1} + L_{z2}$, it follows that in the expansion

$$|\ell\,m\,;\ell_1\,\ell_2\rangle = \sum_{m_1,m_2} C_{m_1 m_2}\,|\ell_1\,m_1\rangle_1 \otimes |\ell_2\,m_2\rangle_2\ ,$$

only terms with $m = m_1 + m_2$ appear on the right hand side above. Second, by operating with $\vec{\boldsymbol{L}}^2$ and using the results above, we can verify the entries of Table 9.5. We give two examples. To check that

$$|2\,,1\,;1\,,1\rangle = \sqrt{\tfrac{1}{2}}\Big[|1\,,1\rangle_1 \otimes |1\,,0\rangle_2 + |1\,,0\rangle_1 \otimes |1\,,1\rangle_2\Big]\,,$$

we note that $\vec{\boldsymbol{L}}^2\,|2\,,2\,;\,1\,1\rangle = 6\hbar^2\,|2\,,2\,;\,1\,1\rangle$, and

$$\vec{\boldsymbol{L}}^2\big[|1\,,1\rangle_1 \otimes |1\,,0\rangle_2 + |1\,,0\rangle_1 \otimes |1\,,1\rangle_2\big] = 6\hbar^2\big[|1\,,1\rangle_1 \otimes |1\,,0\rangle_2 + |1\,,0\rangle_1 \otimes |1\,,1\rangle_2\big]\,,$$

as expected. To check that

$$|0\,0\,;1\,1\rangle = \sqrt{\tfrac{1}{3}}\Big[|1\,,1\rangle_1 \otimes |1\,,-1\rangle_2 - |1\,,0\rangle_1 \otimes |1\,,0\rangle_2 + |1\,,-1\rangle_1 \otimes |1\,,1\rangle_2\Big]\,,$$

we note that $\vec{\boldsymbol{L}}^2\,|0\,0\,;1\,1\rangle = 0$, and

$$\begin{aligned}\vec{\boldsymbol{L}}^2\,&[|1\,,1\rangle_1 \otimes |1\,,-1\rangle_2 - |1\,,0\rangle_1 \otimes |1\,,0\rangle_2 + |1\,,-1\rangle_1 \otimes |1\,,1\rangle_2] \\ &= (2-2)\,|1\,,1\rangle_1 \otimes |1\,,-1\rangle_2 + (2+2-4)\,|1\,,0\rangle_1 \otimes |1\,,0\rangle_2 \\ &\quad +(2-2)\,|1\,,-1\rangle_1 \otimes |1\,,1\rangle_2 = 0\,.\end{aligned}$$

The other seven entries of Table 9.5 are easily checked in the same way using the results obtained above.

(b) The Clebsch-Gordon coefficients involved in the expansion of the state $|0\,0\,;1\,1\rangle$ are:

$$C_{1,-1} = -C_{0,0} = C_{-1,1} = \sqrt{\tfrac{1}{3}}\,.$$

(c) The inner product $\langle 2\,0\,;1\,1|\,0\,0\,;1\,1\rangle = 0$ as the total angular momentum basis states are orthonormal. This can be also be checked by expanding in terms of the direct product basis, and using the fact that the direct basis product states are orthonormal.

2. Liboff, problem 9.46 on page 400.

We are asked to evaluate $Y_\ell^m(\pi-\theta\,,\,\phi+\pi)$. Consider the formula,

$$Y_\ell^m(\theta\,,\,\phi) = \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)e^{im\phi}\,,$$

which follows from eqs. (9.70) and (9.77) of Liboff. First, we note that:

$$\cos(\pi-\theta) = -\cos\theta\,.$$

The last line of Table 9.2 indicates that $P_\ell(-\cos\theta) = (-1)^\ell P_\ell(\cos\theta)$. The definition of $P_\ell^m(\cos\theta)$ given in terms of $P_\ell(\cos\theta)$ in Table 9.3 implies that

$$P_\ell^m(-\cos\theta) = (-1)^{\ell+m} P_\ell^m(\cos\theta)\,.$$

Finally, using the definition of $Y_\ell^m(\theta\,,\,\phi)$ given above, it follows that:

$$\begin{aligned}
Y_\ell^m(\pi-\theta\,,\,\phi+\pi) &= (-1)^{\ell+m} e^{im\pi} Y_\ell^m(\theta\,,\,\phi) \\
&= (-1)^{\ell+m}(-1)^m Y_\ell^m(\theta\,,\,\phi) \\
&= (-1)^\ell Y_\ell^m(\theta\,,\,\phi)\,,
\end{aligned}$$

after noting that $e^{im\pi} = (e^{i\pi})^m = (-1)^m$ and $(-1)^{2m} = +1$ for integer m. Thus, we have proven that the parity of of the state $|\ell\, m\rangle$ (odd or even) is the same as that of ℓ.

3. Liboff, problem 11.61 on page 541.

The spin-one matrices are given by eq. (11.66) of Liboff:

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \qquad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\,, \qquad S_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,.$$

We can compute the expectation values of S_x, S_y and S_z with respect to the state

$$\xi = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\,,$$

as follows:

$$\langle S_x\rangle = \langle \xi\,|S_x|\,\xi\rangle = \xi^\dagger S_x\,\xi = \frac{\hbar}{\sqrt{2}}\,(a^* \;\; b^* \;\; c^*)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\, 2\mathrm{Re}[b^*(a+c)]\,,$$

$$\langle S_y\rangle = \langle \xi\,|S_y|\,\xi\rangle = \xi^\dagger S_y\,\xi = \frac{\hbar}{\sqrt{2}}\,(a^* \;\; b^* \;\; c^*)\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{-\hbar}{\sqrt{2}}\, 2\mathrm{Im}[b^*(a-c)]\,,$$

$$\langle S_z\rangle = \langle \xi\,|S_z|\,\xi\rangle = \xi^\dagger S_z\,\xi = \frac{\hbar}{\sqrt{2}}\,(a^* \;\; b^* \;\; c^*)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\,(|a|^2 - |c|^2)\,,$$

We shall also impose the normalization condition,

$$|a|^2 + |b|^2 + |c|^2 = 1\,.$$

To determine whether a state exists such that $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$, we must solve the simultaneous equations:

$$\begin{aligned} \mathrm{Re}[b^*(a+c)] &= 0\,, \\ \mathrm{Im}[b^*(a-c)] &= 0\,, \\ |a|^2 - |c|^2 &= 0\,, \\ |a|^2 + |b|^2 + |c|^2 &= 1\,. \end{aligned} \tag{3}$$

There are numerous solutions. For example, $b = 0$ and $|a| = |c| = \sqrt{\frac{1}{2}}$ is clearly a solution. Another possible solution is $a = c = 0$ and $|b| = 1$. To understand the physical meaning of these solutions, note that the vectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\,, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\,, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\,,$$

are eigenstates, respectively of S_x, S_y and S_z with eigenvalue zero in each case. Clearly, any eigenstate of S_x with eigenvalue zero will have $\langle S_x \rangle = 0$. But it is easy to see that eigenstates of S_y and S_z with eigenvalue zero will also satisfy $\langle S_x \rangle = 0$. For example, using eq. (11.68) of Liboff, we see that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \sqrt{\tfrac{1}{2}}\left[\xi_x^{(1)} + \xi_x^{(-1)}\right]\,,$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{\tfrac{1}{2}}\left[\xi_x^{(1)} - \xi_x^{(-1)}\right]\,,$$

where the $\xi_x^{(\pm 1)}$ are eigenstates of S_x with eigenvalues $\pm\hbar$. That is, these states are superpositions of states of opposite sign S_x values with amplitudes of equal magnitude. Hence the S_x value averages to zero, i.e., $\langle S_x \rangle = 0$. A similar argument can be made to show that $\langle S_y \rangle = \langle S_z \rangle = 0$.

NOTE ADDED: It is easy to prove the following general result. Let ξ be an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ with eigenvalue zero. Then, $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$, where the expectation values are defined by $\langle S_i \rangle \equiv \langle \xi | \, S_i \, | \xi \rangle$.

To prove this, we exploit the commutation relations satisfied by the S_i,

$$[S_i\,,\, S_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} S_k\,.$$

It follows that

$$[\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_{\mathbf{1}}\,,\, \vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_{\mathbf{2}}] = \sum_{i,j} \hat{n}_{1i}\hat{n}_{2j}[S_i\,,\, S_j] = i\hbar \sum_{ijk} \epsilon_{ijk}\hat{n}_{1i}\hat{n}_{2j}S_k = i\hbar\vec{\boldsymbol{S}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_{\mathbf{1}} \times \hat{\boldsymbol{n}}_{\mathbf{2}})\,.$$

Suppose that we choose $\boldsymbol{\hat{n}_1} = \boldsymbol{\hat{n}}$, where $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,|\xi\rangle = 0$. Then

$$\langle[\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,,\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}_2}]\rangle = i\hbar\langle\vec{\boldsymbol{S}}\boldsymbol{\cdot}(\boldsymbol{\hat{n}}\times\boldsymbol{\hat{n}_2})\rangle\,.$$

But,

$$\langle[\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,,\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}_2}]\rangle = \langle\xi|\,[\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,,\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}_2}]\,|\xi\rangle = \langle\xi|\,(\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}_2} - \vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}_2}\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,|\xi\rangle = 0\,.$$

after applying $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}\,|\xi\rangle = 0$ and $\langle\xi|\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}} = 0$. Hence, it follows that:

$$\langle\vec{\boldsymbol{S}}\boldsymbol{\cdot}(\boldsymbol{\hat{n}}\times\boldsymbol{\hat{n}_2})\rangle \equiv \langle\xi|\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}(\boldsymbol{\hat{n}}\times\boldsymbol{\hat{n}_2})\,|\xi\rangle = 0\,. \tag{4}$$

If $\boldsymbol{\hat{n}}$ points in some direction other than $\boldsymbol{\hat{x}}$, $\boldsymbol{\hat{y}}$, and $\boldsymbol{\hat{z}}$, then it is easy to show that one can always choose $\boldsymbol{\hat{n}_2}$ such that $\boldsymbol{\hat{n}}\times\boldsymbol{\hat{n}_2}$ points in either the $\boldsymbol{\hat{x}}$, $\boldsymbol{\hat{y}}$, or $\boldsymbol{\hat{z}}$ directions. Choosing the appropriate $\boldsymbol{\hat{n}_2}$, it then follows from eq. (4) that

$$\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = 0\,. \tag{5}$$

Next, suppose $\boldsymbol{\hat{n}} = \boldsymbol{\hat{z}}$. Then, $\langle S_z\rangle = \langle\xi|\,S_z\,|\xi\rangle = 0$, since $|\xi\rangle$ is an eigenstate of S_z with zero eigenvalue. In this case, if $\boldsymbol{\hat{n}_2} = \boldsymbol{\hat{x}}$, then $\boldsymbol{\hat{n}}\times\boldsymbol{\hat{n}_2} = \boldsymbol{\hat{y}}$. Similarly, if $\boldsymbol{\hat{n}_2} = -\boldsymbol{\hat{y}}$, then $\boldsymbol{\hat{n}}\times\boldsymbol{\hat{n}_2} = \boldsymbol{\hat{x}}$. Consequently, eq. (4) implies that $\langle S_x\rangle = \langle S_y\rangle = 0$. A similar argument can be made if $\boldsymbol{\hat{n}} = \boldsymbol{\hat{x}}$ or $\boldsymbol{\hat{n}} = \boldsymbol{\hat{y}}$. In all possible cases, one arrives at eq. (5). Hence, we conclude that if ξ is an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}$ with eigenvalue zero, then $\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = 0$.

Applying this general result to problem 3, we conclude that the most general ξ for which $\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = 0$ is an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}$ with eigenvalue zero, for any possible $\boldsymbol{\hat{n}}$. I will now demonstrate that

$$\xi = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\,,$$

is an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}}$ with eigenvalue zero (for some $\boldsymbol{\hat{n}}$) if and only if a, b and c satisfy the simultaneous equations given in eq. (3). Using the spin-one matrices $\vec{\boldsymbol{S}}$ given above,

$$\vec{\boldsymbol{S}}\boldsymbol{\cdot}\boldsymbol{\hat{n}} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} & 0 \\ \sin\theta e^{i\phi} & 0 & \sin\theta e^{-i\phi} \\ 0 & \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\,,$$

where $\boldsymbol{\hat{n}} = (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)$. Solving the equation,

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} & 0 \\ \sin\theta e^{i\phi} & 0 & \sin\theta e^{-i\phi} \\ 0 & \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\,,$$

we find:

$$a\cos\theta + b\sin\theta e^{-i\phi} = 0\,,$$
$$\sin\theta(e^{i\phi}a + e^{-i\phi}c) = 0\,,$$
$$b\sin\theta e^{i\phi} - c\cos\theta = 0\,,$$

where a, b and c satisfy the normalization condition $|a|^2 + |b|^2 + |c|^2 = 1$. I will now show that these equations are equivalent to eq. (3). For example, if $\sin\theta = 0$, then $a = c = 0$ and $|b| = 1$ [which is one of the solutions of eq. (3)]. If $\sin\theta \neq 0$, then $c = -e^{2i\phi}a$, which is equivalent to $|a| = |c|$. Finally, adding and subtracting the first and third equations above yield:

$$
\begin{aligned}
(a-c)\cos\theta + 2b\sin\theta\cos\phi &= 0\,, \\
(a+c)\cos\theta - 2ib\sin\theta\sin\phi &= 0\,.
\end{aligned}
$$

Multiply these equations by b^* to obtain:

$$
\begin{aligned}
b^*(a-c)\cos\theta &= -2|b|^2\sin\theta\cos\phi\,, \\
b^*(a+c)\cos\theta &= 2i|b|^2\sin\theta\sin\phi\,.
\end{aligned}
$$

These two equations imply that $\mathrm{Re}[b^*(a+c)] = \mathrm{Im}[b^*(a-c)] = 0$, which again reproduces the results of eq. (3). Thus, if ξ is an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ with eigenvalue zero, then the components of ξ satisfy eq. (3), which of course implies that $\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = 0$. The converse can also be proven. Namely, if the components of ξ satisfies eq. (3) then ξ is an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ with eigenvalue zero. Simply use the above equations to show that θ and ϕ (which determine $\hat{\boldsymbol{n}}$) can be determined from a, b, and c. One finds:

$$
e^{2i\phi} = -\frac{c}{a}\,, \qquad\qquad \tan\theta = -\frac{a}{b}e^{i\phi}\,.
$$

Note that the case of $a = 0$ implies that $c = 0$ and $\sin\theta = 0$, in which case ϕ is not well defined. There is one overall phase of a, b and c which is arbitrary (since it is not fixed by the normalization condition).

4. Start with the time-dependent Schrodinger equation for a charged particle in at external electromagnetic field in the Coulomb gauge. Denote the electromagnetic scalar and vector potentials by ϕ and $\vec{\boldsymbol{A}}$. Then,

$$
i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2\psi + \frac{ie\hbar}{mc}\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi + \frac{e^2}{2mc^2}\vec{\boldsymbol{A}}^2\psi + (V+e\phi)\psi\,. \tag{6}
$$

The complex conjugate of this equation is:

$$
-i\hbar\frac{\partial\psi^*}{\partial t} = \frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2\psi^* - \frac{ie\hbar}{mc}\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi^* + \frac{e^2}{2mc^2}\vec{\boldsymbol{A}}^2\psi^* + (V+e\phi)\psi^*\,. \tag{7}
$$

Multiply eq. (6) by ψ^* and eq. (7) by ψ and subtract the two resulting equations. The end result is:

$$
i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = \frac{-\hbar^2}{2m}\left[\psi^*\vec{\boldsymbol{\nabla}}^2\psi - \psi\vec{\boldsymbol{\nabla}}^2\psi^*\right] + \frac{ie\hbar}{mc}\left[\psi^*\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi + \psi\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi^*\right]. \tag{8}
$$

The above result can be simplified by applying the following three identities:

$$\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} = \frac{\partial}{\partial t}(\psi^* \psi)\,, \tag{9}$$

$$\psi^* \vec{\boldsymbol{\nabla}}^2 \psi - \psi \vec{\boldsymbol{\nabla}}^2 \psi^* = \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \left(\psi^* \vec{\boldsymbol{\nabla}} \psi - \psi \vec{\boldsymbol{\nabla}} \psi^* \right)\,, \tag{10}$$

$$\psi^* \vec{\boldsymbol{A}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} \psi + \psi \vec{\boldsymbol{A}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} \psi^* = \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\vec{\boldsymbol{A}} \psi^* \psi) - \psi^* \psi \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}}\,. \tag{11}$$

For example, in deriving eq. (11) above, we used the following properties of $\vec{\boldsymbol{\nabla}}$:

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\vec{\boldsymbol{A}} \psi^* \psi) = \vec{\boldsymbol{A}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} (\psi^* \psi) + \psi^* \psi \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}}\,,$$

$$\vec{\boldsymbol{\nabla}} (\psi^* \psi) = \psi^* \vec{\boldsymbol{\nabla}} \psi + \psi \vec{\boldsymbol{\nabla}} \psi^*\,.$$

However, in the Coulomb gauge, $\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} = 0$, in which case eq. (11) implies that:

$$\psi^* \vec{\boldsymbol{A}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} \psi + \psi \vec{\boldsymbol{A}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} \psi^* = \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\vec{\boldsymbol{A}} \psi^* \psi)\,. \tag{12}$$

Applying eqs. (9), (10) and (12) to eq. (8), one obtains:

$$\frac{\partial}{\partial t}(e \psi^* \psi) = -\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \left\{ \frac{e\hbar}{2im} \left[\psi^* \vec{\boldsymbol{\nabla}} \psi - \psi \vec{\boldsymbol{\nabla}} \psi^* \right] - \frac{e^2}{mc} \vec{\boldsymbol{A}} \psi^* \psi \right\}\,. \tag{13}$$

If we identify

$$\vec{\boldsymbol{J}} \equiv \frac{e\hbar}{2im} \left[\psi^* \vec{\boldsymbol{\nabla}} \psi - \psi \vec{\boldsymbol{\nabla}} \psi^* \right] - \frac{e^2}{mc} \vec{\boldsymbol{A}} \psi^* \psi\,, \tag{14}$$

$$\rho \equiv e \psi^* \psi\,, \tag{15}$$

then eq. (13) can be rewritten as:

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{J}} + \frac{\partial \rho}{\partial t} = 0\,,$$

which is the continuity equation. Note that if we employ the momentum operator $\vec{\boldsymbol{p}} = -i\hbar \vec{\boldsymbol{\nabla}}$, then we can rewrite $\vec{\boldsymbol{J}}$ as

$$\vec{\boldsymbol{J}} = \frac{e}{m} \,\mathrm{Re} \left[\psi^* \left(\vec{\boldsymbol{p}} - \frac{e\vec{\boldsymbol{A}}}{c} \right) \psi \right]\,. \tag{16}$$

In order to verify the above result, note that $\mathrm{Re}\, z \equiv \frac{1}{2}(z + z^*)$. Hence, eq. (16) yields:

$$\begin{aligned} \vec{\boldsymbol{J}} &= \frac{e}{2m} \left[\psi^* \left(\frac{\hbar}{i} \vec{\boldsymbol{\nabla}} - \frac{e\vec{\boldsymbol{A}}}{c} \right) \psi + \psi \left(\frac{\hbar}{i} \vec{\boldsymbol{\nabla}} - \frac{e\vec{\boldsymbol{A}}}{c} \right) \psi^* \right] \\ &= \frac{e\hbar}{2im} \left[\psi^* \vec{\boldsymbol{\nabla}} \psi - \psi \vec{\boldsymbol{\nabla}} \psi^* \right] - \frac{e^2}{mc} \vec{\boldsymbol{A}} \psi^* \psi\,, \end{aligned}$$

which is identical to the result of eq. (14).

Finally, we note that it we set $\vec{\boldsymbol{A}} = \phi = 0$, we recover the results of Liboff, eqs. (7.97) and (7.107) [apart from the overall factor of e, which is conventional]. In particular, the definition of ρ is unmodified by the presence of the external electromagnetic field.

ADDED NOTE: To verify that the expression obtained above for $\vec{\boldsymbol{J}}$ makes physical sense, recall that that the canonical momentum $\vec{\boldsymbol{p}}$ is related to the mechanical momentum $m\vec{\boldsymbol{v}}$ by

$$m\vec{\boldsymbol{v}} = \vec{\boldsymbol{p}} - \frac{e}{c}\vec{\boldsymbol{A}}\,.$$

Hence, we can write:

$$\vec{\boldsymbol{J}} = e\,\mathrm{Re}(\psi^*\vec{\boldsymbol{v}}\psi) = \mathrm{Re}(\rho\vec{\boldsymbol{v}})\,.$$

Classically, the current is related to the velocity field by $\vec{\boldsymbol{J}} = \rho\vec{\boldsymbol{v}}$, so the quantum mechanical result derived above is sensible.

5. The time-dependent Schrodinger equation for a charged particle in an external electromagnetic field (before fixing a choice of gauge) is given by:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2\psi + \frac{ie\hbar}{mc}\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi + \frac{ie\hbar}{mc}\psi(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}) + \frac{e^2}{2mc^2}\vec{\boldsymbol{A}}^2\psi + e\phi\psi\,, \tag{17}$$

where I have set the external potential $V = 0$, as it plays no roll in this problem. Note that I have not imposed a particular gauge such as the Coulomb gauge as we did in problem 4. This is important, since the goal of this problem is to show that the Schrodinger equation is form invariant under a generalized gauge transformation, in which the electromagnetic potentials, ϕ and $\vec{\boldsymbol{A}}$ are gauge-transformed and the wave function ψ is transformed by an appropriate phase factor.

(a) Consider a new wave function $\psi_1(\vec{\boldsymbol{r}}, t)$ defined by:

$$\psi_1(\vec{\boldsymbol{r}}, t) = \left[\exp\left(\frac{ieX(\vec{\boldsymbol{r}}, t)}{\hbar c}\right)\right]\psi(\vec{\boldsymbol{r}}, t)\,. \tag{18}$$

Substituting $\psi = \psi_1 \exp\left[-ieX(\vec{\boldsymbol{r}}, t)/(\hbar c)\right]$ into eq. (17), one needs to compute:

$$\frac{\partial\psi}{\partial t} = e^{-ieX(\vec{\boldsymbol{r}},t)/(\hbar c)}\left[\frac{\partial\psi_1}{\partial t} - \frac{ie}{\hbar c}\frac{\partial X}{\partial t}\psi_1\right],$$

$$\vec{\boldsymbol{\nabla}}\psi = e^{-ieX(\vec{\boldsymbol{r}},t)/(\hbar c)}\left[\vec{\boldsymbol{\nabla}}\psi_1 - \frac{ie}{\hbar c}\psi_1\vec{\boldsymbol{\nabla}}X\right],$$

$$\vec{\boldsymbol{\nabla}}^2\psi = e^{-ieX(\vec{\boldsymbol{r}},t)/(\hbar c)}\left[\vec{\boldsymbol{\nabla}}^2\psi_1 - \frac{2ie}{\hbar c}(\vec{\boldsymbol{\nabla}}\psi_1)\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}X) - \frac{ie}{\hbar c}\psi_1\vec{\boldsymbol{\nabla}}^2X - \frac{e^2}{\hbar^2c^2}\psi_1(\vec{\boldsymbol{\nabla}}X)\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}X)\right].$$

Inserting these results into eq. (17), the common factor of $\exp\left[-ieX(\vec{\boldsymbol{r}},t)/(\hbar c)\right]$ cancels, and we end up with:

$$
\begin{aligned}
i\hbar\frac{\partial\psi_1}{\partial t}+\frac{e}{c}\frac{\partial X}{\partial t}\psi_1 = {} & \frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2\psi_1+\frac{ie\hbar}{mc}\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi_1+\frac{ie\hbar}{2mc}\,\psi_1(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}})+\frac{e^2}{2mc^2}\vec{\boldsymbol{A}}^2\psi_1 \\
& +\frac{ie\hbar}{mc}(\vec{\boldsymbol{\nabla}}\psi_1)\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}X)+\frac{ie\hbar}{2mc}\psi_1\vec{\boldsymbol{\nabla}}^2X+\frac{e^2}{2mc^2}\,\psi_1(\vec{\boldsymbol{\nabla}}X)\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}X) \\
& +\frac{e^2}{mc^2}\,\psi_1(\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}X)+e\phi\psi_1\,.
\end{aligned}
$$

Collecting and rearranging terms, the above equation takes the following form:

$$
\begin{aligned}
i\hbar\frac{\partial\psi_1}{\partial t} = {} & \frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2\psi_1+\frac{ie\hbar}{mc}(\vec{\boldsymbol{A}}+\vec{\boldsymbol{\nabla}}X)\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi_1+\frac{ie\hbar}{2mc}\,\psi_1\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{A}}+\vec{\boldsymbol{\nabla}}X) \\
& +\frac{e^2}{2mc^2}\,\psi_1(\vec{\boldsymbol{A}}+\vec{\boldsymbol{\nabla}}X)^2+e\left(\phi-\frac{1}{c}\frac{\partial X}{\partial t}\right)\psi_1\,. \qquad (19)
\end{aligned}
$$

That is, eq. (19) is the time-dependent Schrodinger equation satisfied by $\psi_1(\vec{\boldsymbol{r}},t)$.

(b) If we perform a gauge transformation of the electromagnetic scalar and vector potentials,

$$
\vec{\boldsymbol{A}}'=\vec{\boldsymbol{A}}+\vec{\boldsymbol{\nabla}}X\,,\qquad\qquad \phi'=\phi-\frac{1}{c}\frac{\partial X}{\partial t}\,, \qquad (20)
$$

then, eq. (19) can be rewritten as:

$$
i\hbar\frac{\partial\psi_1}{\partial t}=\frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2\psi_1+\frac{ie\hbar}{mc}\,\vec{\boldsymbol{A}}'\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi_1+\frac{ie\hbar}{2mc}\,\psi_1(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}')+\frac{e^2}{2mc^2}\,\vec{\boldsymbol{A}}'^2\psi_1+e\phi'\psi_1\,. \quad (21)
$$

Note that eq. (21) has exactly the same form as eq. (17) if we replace

$$
\vec{\boldsymbol{A}}\to\vec{\boldsymbol{A}}'\,,\qquad \phi\to\phi'\qquad\text{and}\qquad \psi\to\psi_1\,.
$$

That is, the Schrodinger equation is invariant with respect to generalized gauge transformations in which $\vec{\boldsymbol{A}}$, ϕ and the wave function ψ are simultaneously transformed, as specified in eqs. (18) and (20).