1. Liboff, problem 13.51 on pages 749–750.

A one-dimensional harmonic oscillator of charge-to-mass ratio e/m, and spring constant K oscillates parallel to the x-axis and is in its second excited state at $t < 0$ with energy $E_2 = \frac{5}{2}\hbar\omega_0$. An oscillating, uniform electric field $\vec{\boldsymbol{E}}(t) = 2\mathcal{E}_0 \cos\omega t\,\hat{\boldsymbol{x}}$ is turned on at $t = 0$.[1]

(a) The time-dependent Hamiltonian for $t > 0$ is most easily derived from the general formula for the Hamiltonian of a one-dimensional harmonic oscillator of charge e in an external electric field:

$$H = \frac{\vec{\boldsymbol{p}}^2}{2m} + \tfrac{1}{2}Kx^2 + e\phi\,,$$

where $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\phi$. It follows that $\phi = -2x\mathcal{E}_0\cos\omega t$, and

$$H = \frac{\vec{\boldsymbol{p}}^2}{2m} + \tfrac{1}{2}Kx^2 - 2ex\mathcal{E}_0\cos\omega t\,,$$

where $\omega_0 \equiv \sqrt{K/m}$ is the characteristic angular frequency of the harmonic oscillator.

(b) The harmonic perturbation is $H'(t) = 2\mathbb{H}'\cos\omega t$, where $\mathbb{H}' \equiv -ex\mathcal{E}_0$. Hence, the matrix element,[2]

$$\mathbb{H}'_{n2} \equiv \langle n|\,\mathbb{H}'\,|2\rangle = -e\mathcal{E}_0\,\langle n|\,x\,|2\rangle\ .$$

We now use eq. (11.45) on p. 500 of Liboff,

$$\langle n|\,x\,|k\rangle = \frac{1}{\sqrt{2}\beta}\left[\sqrt{k}\,\delta_{n,k-1} + \sqrt{k+1}\,\delta_{n,k+1}\right]\,, \qquad\qquad \beta \equiv \sqrt{m\omega_0/\hbar}\,.$$

Thus,

$$\mathbb{H}'_{n2} = -\frac{e\mathcal{E}_0}{\sqrt{2}\beta}\left[\sqrt{2}\,\delta_{n,1} + \sqrt{3}\,\delta_{n,3}\right]\ .$$

[1] Liboff writes $\vec{\boldsymbol{E}}(t) = 2\mathcal{E}_0\cos\omega_0 t\,\hat{\boldsymbol{x}}$. Although this is required in practice in order to drive the transitions (if t is large), it is mathematically cleaner to call the frequency of the oscillating field ω. The *resonance* condition that drives absorption or emission of energy is $\omega \simeq \omega_0$ for absorption and $\omega \simeq -\omega_0$ for emission. These conditions are a *consequence* of the analysis (see below).

[2] Liboff actually asks for $\mathbb{H}'_{2n}$, which is a little strange given that in part (c), he asks us to compute $P_{2\to n}$, which depends on $\mathbb{H}'_{n2}$. The computation is similar in both cases.

(c) The transition probability at time t is given by eq. (13.52) on p. 711 of Liboff,

$$P_{2\to n}(t) = \frac{|\mathbb{H}'_{n2}|^2}{\hbar^2}\left|\int_{-\infty}^{t} e^{i\omega_{n2}t'} f(t')dt'\right|^2 ,$$

where $\hbar\omega_{n2} \equiv E_n - E_2 = (n-2)\hbar\omega_0$ is the difference of the corresponding bound state energies of the unperturbed harmonic oscillator, and $H'(\vec{\boldsymbol{r}}, t) \equiv \mathbb{H}'(\vec{\boldsymbol{r}})f(t)$. In this notation, $\mathbb{H}'(\vec{\boldsymbol{r}}) \equiv -ex\mathcal{E}_0$, in which case we identify $f(t) \equiv 2\cos\omega t$. For this problem, we can replace the lower integration limit by $t' = 0$, since the perturbation is absent for $t' < 0$. Hence,

$$P_{2\to n}(t) = \frac{4|\mathbb{H}'_{n2}|^2}{\hbar^2}\left|\int_{0}^{t} e^{i\omega_{n2}t'} \cos\omega t' dt'\right|^2 .$$

The integral over t' is evaluated on p. 713 of Liboff,

$$\int_0^t e^{i\omega_{n2}t'}\cos\omega t' dt' = \frac{e^{i(\omega_{n2}-\omega)t/2}\sin\frac{1}{2}(\omega_{n2}-\omega)t}{\omega_{n2}-\omega} + \frac{e^{i(\omega_{n2}+\omega)t/2}\sin\frac{1}{2}(\omega_{n2}+\omega)t}{\omega_{n2}+\omega} .$$

Hence,

$$P_{2\to n}(t) = \frac{4|\mathbb{H}'_{n2}|^2}{\hbar^2}\left|\frac{e^{i(\omega_{n2}-\omega)t/2}\sin\frac{1}{2}(\omega_{n2}-\omega)t}{\omega_{n2}-\omega} + \frac{e^{i(\omega_{n2}+\omega)t/2}\sin\frac{1}{2}(\omega_{n2}+\omega)t}{\omega_{n2}+\omega}\right|^2 . \tag{1}$$

As described on pp. 713–714 of Liboff, if $\omega_{n2} = \hbar\omega_0(n-2) > 0$, corresponding to the case where the time-dependent electric field excites only *higher* energy oscillator eigenstates, then the first term inside the brackets of eq. (1) dominates assuming that $\omega \simeq \omega_{n2}$ is close to its resonant frequency. In this case,

$$P_{2\to n}(t) = \frac{4|\mathbb{H}'_{n2}|^2}{\hbar^2}\left|\frac{\sin\frac{1}{2}(\omega_{n2}-\omega)t}{\omega_{n2}-\omega}\right|^2 , \qquad \omega \simeq \omega_{n2} .$$

Likewise, if $\omega_{n2} = \hbar\omega_0(n-2) < 0$, corresponding to the case where the time-dependent electric field excites only *lower* energy oscillator eigenstates, then the second term inside the brackets of eq. (1) dominates assuming that $\omega \simeq -\omega_{n2}$ is close to its resonant frequency. In this case,

$$P_{2\to n}(t) = \frac{4|\mathbb{H}'_{n2}|^2}{\hbar^2}\left|\frac{\sin\frac{1}{2}(\omega_{n2}+\omega)t}{\omega_{n2}+\omega}\right|^2 , \qquad \omega \simeq -\omega_{n2} .$$

Using the results from part (b) to evaluate $\mathbb{H}'_{n2}$, there are only two possible transitions, due to the Kronecker delta factors in $\mathbb{H}'_{n2}$,

$$\boxed{P_{2\to 3}(t) = \left(\frac{6e^2\mathcal{E}_0^2}{\hbar m\omega_0}\right)\frac{\sin^2\frac{1}{2}(\omega_0-\omega)t}{(\omega_0-\omega)^2}}$$

corresponding to the absorption of a quantum $\hbar\omega_{32} = \hbar\omega_0$ of energy, and

$$\boxed{P_{2\to 1}(t) = \left(\frac{4e^2\mathcal{E}_0^2}{\hbar m\omega_0}\right)\frac{\sin^2\frac{1}{2}(\omega-\omega_0)t}{(\omega-\omega_0)^2}}$$

corresponding to the emission of a quantum of $-\hbar\omega_{12} = \hbar\omega_0$ of energy.

For short times $t \ll |\omega_0 - \omega|^{-1}$, one can use the small argument approximation for the sine functions above. This approximation yields:

$$P_{2\to 3}(t) = \frac{3e^2\mathcal{E}_0^2 t^2}{2\hbar m\omega_0}\,, \qquad\qquad P_{2\to 1}(t) = \frac{e^2\mathcal{E}_0^2 t^2}{\hbar m\omega_0}\,.$$

There are no first-order transitions to other oscillator energy eigenstates.

(d) If we wish to pump a harmonic oscillator to a higher energy state, we can simply apply the oscillating electric field of this problem. In part (c), we obtained $P_{2\to 3}(t) > P_{2\to 1}(t)$, so that applying the oscillating field will cause more transitions to the higher state as compared with the lower state. As this was only a first-order perturbation theory computation, one might guess that at higher order, one can induce transitions to even higher oscillator energy states. This is indeed correct, although the probabilities for such higher order transitions are suppressed if the applied oscillating field is weak.

2. Consider the scattering of particles by the square well potential in three dimensions:

$$V(r) = \begin{cases} -V_0\,, & \text{for } r < a\,, \\ \ \ 0\,, & \text{for } r > a\,, \end{cases}$$

where V_0 is positive.

(a) In the Born approximation for a spherical potential, the scattering amplitude is given by:

$$F(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r\sin(qr)V(r)dr\,,$$

where $q \equiv 2k\sin(\theta/2)$ and $p = \hbar k$ is the momentum of the incoming beam. Inserting the square well potential for $V(r)$,

$$f(\theta) = \frac{2mV_0}{\hbar^2 q}\int_0^a r\sin(qr)dr = \frac{2mV_0}{\hbar^2 q^3}\left[\sin(qa) - qa\cos(qa)\right]\,.$$

Hence,

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0}{\hbar^2 q^3}\right)^2\left[\sin(qa) - qa\cos(qa)\right]^2} \qquad (2)$$

(b) First consider the limit of low energy, $ka \ll 1$. Since $q \equiv 2k\sin(\theta/2)$, it follows that $qa \ll 1$. In this limit,

$$\begin{aligned}\sin(qa) - qa\cos(qa) &= \left(qa - \tfrac{1}{6}q^3a^3 + \tfrac{1}{120}q^5a^5 \ldots\right) - qa\left(1 - \tfrac{1}{2}q^2a^2 + \tfrac{1}{24}q^4a^4 \ldots\right) \\ &= \tfrac{1}{3}q^3a^3\left(1 - \tfrac{1}{10}q^2a^2 + \ldots\right) .\end{aligned}$$

Inserting this result into eq. (2),

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \left(1 - \tfrac{1}{10}q^2a^2 + \ldots\right)^2 .$$

We shall evaluate the square of the series above, keeping only the first two terms, which yields

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \left(1 - \tfrac{1}{5}q^2a^2\right) .$$

Integrating over angles, keeping in mind the θ dependence of $q \equiv 2k\sin(\theta/2)$, we need to compute

$$\begin{aligned}\int d\Omega\left(1 - \tfrac{1}{5}q^2a^2\right) &= \int d\Omega\left(1 - \tfrac{4}{5}k^2a^2\sin^2\frac{\theta}{2}\right) \\ &= 4\pi - \frac{4\pi k^2a^2}{5}\int_{-1}^{1} d\cos\theta(1-\cos\theta) \\ &= 4\pi\left(1 - \frac{2}{5}k^2a^2\right) .\end{aligned}$$

where we have used the identity $\sin^2(\theta/2) = \frac{1}{2}(1-\cos\theta)$. Hence, the total cross-section in the low-energy limit is given by:

$$\boxed{\sigma \simeq 4\pi\left(\frac{2mV_0a^3}{3\hbar^2}\right)^2\left[1 - \frac{2}{5}k^2a^2\right] , \qquad (ka \ll 1)}$$

Next, we consider the high energy limit, $ka \gg 1$. The expression that we must analyze is

$$\sigma = 2\pi\left(\frac{2mV_0}{\hbar^2q^3}\right)^2 \int_{-1}^{1} d\cos\theta\left[\sin(qa) - qa\cos(qa)\right]^2 ,$$

where we have integrated over the azimuthal angle ϕ. Let us change the integration variable from $\cos\theta$ to q. Since $q \equiv 2k\sin(\theta/2) = k[2(1-\cos\theta)]^{1/2}$, it follows that:

$$dq = -k[2(1-\cos\theta)]^{-1/2}d\cos\theta = -\frac{k^2}{q}d\cos\theta .$$

Hence,

$$\sigma = \frac{2\pi}{k^2}\left(\frac{2mV_0}{\hbar^2}\right)^2 \int_0^{2k} \frac{dq}{q^5}\left[\sin(qa) - qa\cos(qa)\right]^2 ,$$

where we have adjusted the integration limits to reflect the fact that $q = 0$ at $\cos\theta = 1$ and $q = 2k$ at $\cos\theta = -1$. It is more convenient to choose a dimensionless integration variable. Thus, we define $y \equiv qa$, in which case,

$$\sigma = \frac{2\pi}{k^2a^2}\left(\frac{2mV_0a^3}{\hbar^2}\right)^2 \int_0^{2ka} \frac{dy}{y^5}\left[\sin y - y\cos y\right]^2 . \tag{3}$$

In fact, this integral can be performed exactly, although the computation is long and tedious if performed by hand. But, since we are interested in the high energy limit, we may set the upper limit of the above integral to infinity. It follows that for $ka \gg 1$,

$$\sigma = \frac{2\pi}{k^2a^2}\left(\frac{2mV_0a^3}{\hbar^2q^3}\right)^2 \int_0^{\infty} \frac{dy}{y^5}\left[\sin y - y\cos y\right]^2 . \tag{4}$$

The integral above is still rather non-trivial. If one computes the square of the integrand, three terms are obtained. Each of the three resulting integrals is separately divergent due to the singularity of the integrand at $y = 0$, although the sum of the three terms is integrable.[3] To perform the integral, one can search a comprehensive table of integrals.[4] From such a table, you can obtain the following result:

$$\int_0^\infty \frac{[j_n(y)]^2}{y^p}\,dy = \frac{2^{p-2}\,\Gamma\left(\dfrac{2n+1-p}{2}\right)\Gamma^2\left(\dfrac{p+1}{2}\right)}{\Gamma(p+1)\,\Gamma\left(\dfrac{2n+p+3}{2}\right)}\,, \qquad (-1 < \mathrm{Re}\ p < 2n+1)\,. \tag{5}$$

where $j_n(y)$ is a spherical Bessel function and $\Gamma^2(z)$ is the square of the gamma function. Using eq. (10.55) on p. 416 of Liboff,

$$j_1(y) = \frac{\sin y - y\cos y}{y^2}\,, \tag{6}$$

the integral of eq. (4) can be written as:

$$\int \frac{dy}{y^5}\left[\sin y - y\cos y\right]^2 = \int_0^\infty \frac{[j_1(y)]^2}{y}dy = \frac{1}{4}\,, \tag{7}$$

after substituting $n = p = 1$ in eq. (5), and using the fact that $\Gamma(n) = (n-1)!$ for positive integers n. Inserting this result into eq. (4) then yields:

$$\boxed{\sigma \simeq \frac{2\pi}{k^2}\left(\frac{mV_0a^2}{\hbar^2}\right)^2\,, \qquad (ka \gg 1)}$$

[3] Although it appears that the full integrand, $[\sin y - y\cos y]^2/y^5$ is also singular at $y = 0$, you can check (by expanding out the numerator around $y = 0$) that the full integrand vanishes in the $y \to 0$ limit.

[4] My personal favorite is I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, 7th edition, edited by A. Jeffrey and D. Zwillinger (Academic Press, Burlington, MA, 2007).

BONUS MATERIAL

Although I did not ask you to compute the exact cross section, it is possible to evaluate it analytically. The integral given in eq. (3) can be evaluated by repeated integration by parts (after expanding out the numerator of the integrand). Alternatively, you can try one of the high powered mathematics software packages. Remarkably, Mathematica 7 is unable to perform the integral, whereas Maple 13 handles it with no problem!

Here is a brief treatment of the exact integral:

$$\begin{aligned}\int_0^{2ka}\frac{dy}{y^5}\left[\sin y - y\cos y\right]^2 &= \int_0^{2ka} dy\left(\frac{\sin^2 y}{y^5} - \frac{2\sin y\cos y}{y^4} + \frac{\cos^2 y}{y^3}\right)\\ &= \int_0^{2ka} dy\left(\frac{1}{2y^5} + \frac{1}{2y^3} - \frac{\cos 2y}{2y^5} - \frac{\sin 2y}{y^4} + \frac{\cos 2y}{2y^3}\right)\,,\end{aligned}$$

where we have used:

$$\sin^2 y = \tfrac{1}{2}(1-\cos 2y)\,,\qquad \cos^2 y = \tfrac{1}{2}(1+\cos 2y)\,,\qquad 2\sin x\cos x = \sin 2x\,.$$

Integrating the first two terms is trivial. As for the rest, integrating by parts repeatedly yields a very simple final result for the indefinite integral:

$$\mathcal{I}(y) \equiv \int\frac{dy}{y^5}\left[\sin y - y\cos y\right]^2 = -\frac{1}{8y^4} - \frac{1}{4y^2} + \frac{\cos 2y}{8y^4} + \frac{\sin 2y}{4y^3}\,. \tag{8}$$

You can verify this result by showing that its derivative yields the original integrand! By expanding $\sin 2y$ and $\cos 2y$ in eq. (8) around $y = 0$, one can check that:

$$\lim_{y\to 0}\mathcal{I}(y) = -\frac{1}{8y^4} - \frac{1}{4y^2} + \frac{1}{8y^4}\left[1 - 2y^2 + \tfrac{2}{3}y^4\right] + \frac{1}{4y^3}\left[2y - \tfrac{4}{3}y^3\right] = -\frac{1}{4}\,.$$

Moreover, $\lim\limits_{y\to\infty}\mathcal{I}(y) = 0$. Hence,

$$\equiv \int_0^\infty \frac{dy}{y^5}\left[\sin y - y\cos y\right]^2 = \mathcal{I}(y)\Big|_0^\infty = \frac{1}{4}\,,$$

which recovers the result of eq. (7).

Of course, having evaluated $\mathcal{I}(y)$ exactly, we can perform the integration in eq. (3) to obtain the total cross-section:

$$\sigma = \frac{2\pi}{k^2a^2}\left(\frac{2mV_0a^3}{\hbar^2}\right)^2\left[\mathcal{I}(2ka) - \mathcal{I}(0)\right]\,.$$

Using the above results, we end up with:

$$\boxed{\sigma = \frac{2\pi}{k^2}\left(\frac{mV_0a^2}{\hbar^2}\right)^2\left[1 - \frac{1}{4(ka)^2} - \frac{1}{32(ka)^4} + \frac{\sin(4ka)}{8(ka)^3} + \frac{\cos(4ka)}{32(ka)^4}\right]}$$

A nice check of this result is to verify the low-energy and high-energy limits previously obtained. In the limit of $ka \gg 1$, we immediately recover our previous result. In the limit of $ka \ll 1$, we must expand out the sine and cosine around zero. Setting $x = ka$,

$$\lim_{x\to 0} \frac{1}{x^2}\left[1 - \frac{1}{4x^2} - \frac{1}{32x^4} + \frac{1}{8x^3}\left(4x - \frac{32}{3}x^3 + \frac{128}{15}x^5 - \frac{1028}{315}x^7\right)\right.$$
$$\left. + \frac{1}{32x^4}\left(1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \frac{516}{315}x^8\right)\right] = \frac{8}{9}\left[1 - \frac{2x^2}{5}\right].$$

Hence,

$$\sigma \simeq 4\pi\left(\frac{2mV_0a^3}{3\hbar^2}\right)^2\left[1 - \frac{2}{5}(ka)^2\right], \qquad (ka \ll 1),$$

which confirms the result obtained previously.

(c) In class, we indicated that the Born approximation is valid for scattering in a central potential $V(r)$ if

$$\left|\frac{m}{\hbar^2 k}\int_0^\infty (e^{2ikr} - 1)V(r)dr\right|^2 \ll 1.$$

We then examined separately the low-energy limit ($k \to 0$) and the high-energy limit ($k \to \infty$).

The Born approximation is valid in the low-energy limit if

$$\left|\frac{2m}{\hbar^2}\int_0^\infty rV(r)dr\right|^2 \ll 1.$$

For the square well, this condition reads:

$$\left|-\frac{2mV_0}{\hbar^2}\int_0^a rdr\right|^2 \ll 1.$$

Evaluating the integral yields:

$$\boxed{\left(\frac{mV_0a^2}{\hbar^2}\right)^2 \ll 1}$$

Thus, if the square well potential is weak enough, then the Born approximation is valid for all energies.

The Born approximation is valid in the high energy limit if

$$\left|\frac{m}{\hbar^2 k}\int_0^\infty V(r)dr\right|^2 \ll 1.$$

Applying this to the square well,

$$\left|-\frac{mV_0}{\hbar^2 k}\int_0^a dr\right|^2 \ll 1.$$

Thus, we find that

$$\left(\frac{mV_0 a}{\hbar^2 k}\right)^2 \ll 1\,.$$

Using the fact that $E = \hbar^2 k^2/(2m)$, we can rewrite this inequality as:

$$\boxed{E \gg \frac{mV_0^2 a^2}{2\hbar^2}}$$

That is, no matter how strong the potential is, the Born approximation will be satisfied if the energy E is large enough, as indicated above.

(d) For s-wave scattering, the scattering amplitude is independent of angle. Likewise, $|f(\theta)|^2 = d\sigma/d\Omega$ is independent of the scattering angle, in which case the total cross section is simply $\sigma = 4\pi(d\sigma/d\Omega)$. Eq. (14.20) of Liboff on p. 771 provides the total cross-section. Hence, it follows that:

$$\frac{d\sigma}{d\Omega} = a^2\left(\frac{\tan k_1 a}{k_1 a} - 1\right)^2\,, \qquad\qquad \sigma = 4\pi a^2\left(\frac{\tan k_1 a}{k_1 a} - 1\right)^2\,, \tag{9}$$

where $E + V_0 = \hbar^2 k_1^2/(2m)$ and $E = \hbar^2 k^2/(2m)$ imply that:

$$k_1 = \sqrt{k^2 + \frac{2mV_0}{\hbar^2}}\,. \tag{10}$$

Since we have neglected the higher partial waves, eq. (9) should be valid only in the low-energy limit. Thus, we expect that eq. (9) should coincide with the lowest-order results obtained in parts (a) and (b) if the Born approximation is valid in the low-energy limit. In part (c) we saw that the latter is true if $mV_0 a^2/\hbar^2 \ll 1$. If this limit is satisfied, then in the low energy limit where $ka \ll 1$, it follows from eq. (10) that $k_1 a \ll 1$ as well.

Thus, we shall approximate the cross-section in eq. (9) in the limit of $k_1 a \ll 1$. In particular, since $\tan k_1 a \simeq k_1 a + \frac{1}{3}(k_1 a)^3$ for $k_1 a \ll 1$, it follows that

$$\frac{\tan k_1 a}{k_1 a} - 1 \simeq \tfrac{1}{3}(k_1 a)^2\,, \qquad \text{for} \quad k_1 a \ll 1\,.$$

Inserting this result into eq. (9) yields:

$$\sigma \simeq \frac{4\pi k_1^4 a^6}{9}\,.$$

If we now employ eq. (10) under the assumption that $ka \ll (2mV_0 a^2/\hbar^2)^{1/2}$, which defines more precisely what we mean by the low-energy limit, then we can approximate $k_1^2 \simeq 2mV_0/\hbar^2$. Inserting this value into the expression above yields:

$$\boxed{\sigma \simeq 4\pi\left(\frac{2mV_0 a^3}{3\hbar^2}\right)^2}$$

which reproduces the leading term of the Born-approximated total cross-section in the low-energy limit, obtained in part (b). Likewise, the leading term of the differential cross-section in the low-energy limit, obtained in part (a) is independent of angle [since the angular dependence resides in $q = 2k\sin(\theta/2)$].

Hence, in the low-energy, weak potential limit where $ka \ll (2mV_0a^2/\hbar^2)^{1/2}$ and $mV_0a^2/\hbar^2 \ll 1$, the leading term of the Born-approximated differential and total cross sections coincide with the corresponding results obtained by retaining only the s-wave contribution to the scattering amplitude.

3. Liboff, problem 14.8 on pages 782.

An important parameter in scattering theory is the scattering lenght a. This length is defined as the negative of the limiting value of the scattering amplitude as the energy of the incident particle goes to zero,

$$a = -\lim_{k\to 0} f(\theta)\,. \tag{11}$$

(a) The partial wave expansion of the scattering amplitude is given by:

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{e^{i\delta_\ell}\sin\delta_\ell}{k}\right)P_\ell(\cos\theta)\,.$$

In the limit of low-energies where $ka \ll 1$, only s-wave scattering is important. Thus, we can approximate

$$f(\theta) \simeq \frac{e^{i\delta_0}\sin\delta_0}{k}\,.$$

If $|\delta_0| \ll 1$, then at leading order $e^{i\delta_0} \simeq 1$ and $\sin\delta_0 \simeq \delta_0$, which yields

$$f(\theta) \simeq \delta_0/k\,.$$

Hence, according to the definition of the scattering length as given in eq. (11),

$$\boxed{a = -\lim_{k\to 0}\frac{\delta_0}{k}} \tag{12}$$

(b) We can easily compute the differential and total cross-sections.

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \simeq \left(\frac{\delta_0}{k}\right)^2 = a^2\,.$$

Since the differential cross-section is independent of angle, the total cross-section is obtained simply by multiplying the differential cross section by 4π, which yields

$$\boxed{\sigma \simeq 4\pi a^2}$$

(c) In class, we computed the phase shift for the scattering of particles from a rigid sphere (often called a hard sphere) of radius $\bar{a}$. The corresponding potential analyzed in class was given by:

$$V(r) = \begin{cases} \infty\,, & \text{for } \; r < \bar{a}\,, \\ 0\,, & \text{for } \; r > \bar{a}\,. \end{cases}$$

We found that the exact expression for the phase shifts was given by:

$$\tan\delta_\ell = \frac{j_\ell(k\bar{a})}{n_\ell(k\bar{a})}\,. \tag{13}$$

In particular, for $ka \ll 1$, one can use the small-argument approximation for the spherical Bessel functions, given in Table 10.1 on p. 418 of Liboff, to obtain:

$$\tan\delta_\ell \simeq \frac{-(k\bar{a})^{2\ell+1}}{(2\ell+1)[(2\ell-1)!!]^2}\,.$$

This result demonstrates that as $k \to 0$, the s-wave scattering dominates. Moreover, if we set $\ell = 0$ in eq. (13), we find that $\tan\delta_0 = -\tan k\bar{a}$, which implies that

$$\delta_0 = -k\bar{a}\,,$$

which again is an exact result. Inserting this result into eq. (12) yields

$$\boxed{a = \bar{a}}$$

Hence we conclude that in the zero energy limit, all obstacles scatter as though they were rigid spheres of radius equal to the scattering length.[5]

4. Consider the case of low-energy scattering from a spherical delta-function shell,

$$V(r) = V_0\delta(r-a)\,,$$

where V_0 and a are constants. First, we solve the Schrodinger equation. Following the standard steps (see e.g., Liboff p. 446), we write the solution as:

$$\psi(\vec{\boldsymbol{r}}) = \frac{u(r)}{r}Y_\ell^m(\theta,\phi)\,.$$

where $u(r) = rR(r)$ is related to the radial wave function $R(r)$, and satisfies the reduced radial equation,

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u(r) = Eu(r)\,,$$

[5]Here, I have quoted from a nice discussion of the low-energy behavior of the phase shifts given by Roger G. Newton, *Scattering Theory of Waves and Particles*, 2nd edition (Dover Publications, Inc., Mineola, NY, 2002), Section 11.2.

subject to the boundary condition that $u(r=0)=0$. In the low-energy limit, we may assume that $ka \ll 1$ so that only s-wave scattering is important. Thus, we can simply take $\ell = m = 0$. Hence, for the delta-function potential, we must solve:

$$-\frac{d^2u}{dr^2} + \frac{2mV_0}{\hbar^2}\delta(r-a)u(r) = k^2 u(r)\,. \tag{14}$$

where $E = \hbar^2 k^2/(2m)$ defines k as usual.

For $r \neq a$, the delta function vanishes, and we can solve the simple equation:

$$\frac{d^2u}{dr^2} = -k^2 u(r)\,. \tag{15}$$

The solution is thus given by:

$$u(r) = \begin{cases} A\sin kr\,, & \text{for } \; r < a\,, \\ B\sin kr + C\cos kr\,, & \text{for } \; r > a\,, \end{cases} \tag{16}$$

where we have imposed the boundary condition at the origin, $u(r=0)=0$. However, there is a more useful way to write the solution for $u(r)$ in the region $r > a$. In class, we showed that the asymptotic form for the wave function at large r is given by:[6]

$$\psi(\vec{\boldsymbol{r}}) \;\underset{r\to\infty}{\longrightarrow}\; \frac{1}{kr}\sum_{\ell=0}^{\infty} C_\ell \sin(kr - \tfrac{1}{2}\ell\pi + \delta_\ell)P_\ell(\cos\theta)\,.$$

Since we are focusing on s-wave scattering, we set $\ell = 0$, in which case the asymptotic form of the reduced radial wave function is simply:

$$u(r) \;\underset{r\to\infty}{\longrightarrow}\; \frac{1}{k}C_0\sin(kr+\delta_0)\,.$$

Our eventual goal is to determine δ_0. But the above result suggests that we should rewrite eq. (16) as:

$$u(r) = \begin{cases} A\sin kr\,, & \text{for } \; r < a\,, \\ B'\sin(kr+\delta_0)\,, & \text{for } \; r > a\,, \end{cases} \tag{17}$$

where there is a simple relation between B, C and B', δ_0. There is no need to write out this relation, as the above form for $u(r)$ clearly satisfies eq. (15) in the stated regions. The advantage of using the form of $u(r)$ given in eq. (17) is that we will be able to deduce directly an expression for the s-wave phase shift.

To make further progress, we must impose the correct boundary conditions at $r = a$. These are:

$$\text{(i)} \quad u(r) \text{ is continuous at } r = a\,,$$

$$\text{(ii)} \quad \left[\left.\frac{du}{dr}\right|_{a+\epsilon} - \left.\frac{du}{dr}\right|_{a-\epsilon}\right] = \frac{2mV_0}{\hbar^2}u(a)\,.$$

[6] We also showed in class that $C_\ell = i^\ell(2\ell+1)e^{i\delta_\ell}$. However in general, it is enough to isolate the $\sin(kr - \frac{1}{2}\ell\pi + \delta_\ell)$ term in order to determine the phase shift δ_ℓ.

Boundary condition (ii) arises after integrating eq. (14) from $r = a - \epsilon$ to $a + \epsilon$ (where $0 < \epsilon \ll 1$). In particular, in the limit of $\epsilon \to 0$,

$$-\int_{a-\epsilon}^{a+\epsilon} \frac{d^2u}{dr^2}\, dr + \frac{2mV_0}{\hbar^2} u(a) = 0\,.$$

The right hand side is zero as a consequence of boundary condition (i) above. I have also used the fact that for any $\epsilon \neq 0$ (no matter how small),

$$\int_{a-\epsilon}^{a+\epsilon} \delta(r-a) f(r) dr = f(a)\,,$$

for any well-behaved function $f(r)$. The remaining integral is:

$$\int_{a-\epsilon}^{a+\epsilon} \frac{d^2u}{dr^2}\, dr = \left.\frac{du}{dr}\right|_{a+\epsilon} - \left.\frac{du}{dr}\right|_{a-\epsilon}\,,$$

which establishes condition (ii) above.

From eq. (17), we obtain:

$$\frac{du}{dr} = \begin{cases} kA\cos kr\,, & \text{for } \; r < a\,, \\ kB'\cos(kr+\delta_0)\,, & \text{for } \; r > a\,. \end{cases}$$

Applying conditions (i) and (ii) then yield:

$$\begin{aligned} B'\sin(ka+\delta_0) &= A\sin ka\,, \\ B'\cos(ka+\delta_0) &= A\cos ka - \frac{2mV_0}{\hbar^2 k} A\sin ka\,. \end{aligned}$$

Diving these two equation, one obtains,

$$\cot(ka+\delta_0) = \cot ka + \frac{2mV_0}{\hbar^2 k}\,, \tag{18}$$

which is an implicit equation for the s-wave phase shift. To obtain δ_0, we make use of the trigonometric identity:

$$\cot(ka+\delta_0) = \frac{\cot ka \cot\delta_0 - 1}{\cot ka + \cot\delta_0}\,.$$

We can then rewrite eq. (18) as:

$$\frac{\cot ka \cot\delta_0 - 1}{\cot ka + \cot\delta_0} = \cot ka + \frac{2mV_0}{\hbar^2 k}\,.$$

Cross-multiplying and solving for $\cot\delta_0$, we find:

$$\cot\delta_0 = -\frac{\hbar^2 k}{2mV_0}\left[1 + \cot ka\left(\cot ka + \frac{2mV_0}{\hbar^2 k}\right)\right]\,.$$

It is slightly more convenient to rewrite the above in terms of $\tan\delta_0 = 1/\cot\delta_0$,

$$\tan\delta_0 = -\frac{2mV_0}{\hbar^2 k}\left[\frac{1}{\sin^2 ka} + \frac{2mV_0}{\hbar^2 k}\frac{\cos ka}{\sin ka}\right]^{-1},$$

where we have used the identity $1 + \cot^2 ka = 1/\sin^2 ka$ and the definition of the cotangent. A simple rearrangement yields the desired result,

$$\boxed{\tan\delta_0 = \frac{-\sin^2 ka}{\dfrac{\hbar^2 k}{2mV_0} + \sin ka\cos ka}} \tag{19}$$

The partial wave expansion of the scattering amplitude is given by:

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{e^{i\delta_\ell}\sin\delta_\ell}{k}\right)P_\ell(\cos\theta)\,.$$

In the limit of low-energies where $ka \ll 1$, only s-wave scattering is important. Thus, we can approximate

$$f(\theta) \simeq \frac{e^{i\delta_0}\sin\delta_0}{k}\,.$$

Applying the low-energy limit, $ka \ll 1$, to eq. (19) yields,

$$\tan\delta_0 \simeq -ka\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-1}.$$

Indeed, $|\tan\delta_0| \ll 1$ (as expected), in which case,

$$e^{i\delta_0}\sin\delta_0 \simeq \sin\delta_0 \simeq \tan\delta_0 \simeq \delta_0\,.$$

Hence,

$$\boxed{f(\theta) \simeq -a\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-1}}$$

Finally, the differential and total cross sections are given by:

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = a^2\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-2}}$$

and

$$\boxed{\sigma = 4\pi a^2\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-2}}$$

5. This problem provides a crude model for the photoelectric effect. Consider the hydrogen atom in its ground state (where we shall neglect the spins of the electron and proton). At time $t = 0$, the atom is placed in a high frequency uniform electric field that points in the z-direction,

$$\vec{\boldsymbol{\mathcal{E}}}(t) = \mathcal{E}_0 \hat{\boldsymbol{z}} \sin \omega t\,.$$

We wish to compute the transition probability per unit time that an electron is ejected into a solid angle lying between Ω and $\Omega + d\Omega$. From this result, we can obtain the differential and total ionization rates.

The time-dependent Hamiltonian that governs the electron (of charge $-e$) of the hydrogen atom. is given by $H'(t) = -e\phi$, where $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\phi$. That is,

$$H'(t) = ez\mathcal{E}_0 \sin \omega t = \frac{ez}{2i}\mathcal{E}_0 \left[e^{i\omega t} - e^{-i\omega t}\right]\,. \tag{20}$$

(a) The minimum frequency, ω_0, of the field necessary to ionize the atom is equal to the ionization energy divided by $\hbar$. The ionization energy of the ground state of hydrogen is equal to the negative of the bound state energy, and is given by 1 Ry = 13.6 eV. That is,

$$\omega_0 = \frac{me^4}{2\hbar^3}\,. \tag{21}$$

(b) Fermi's golden rule for the transition rate for the absorption of energy from an harmonic perturbation is given by [cf. eq. (13.64) on p. 717 of Liboff]:

$$w_{a\to b}(t) = \frac{2\pi}{\hbar}\left|\left\langle b^{(0)}\right| \mathbb{H}' \left|a^{(0)}\right\rangle\right|^2 \rho(E_b^{(0)})\,, \tag{22}$$

where $\rho(E_b^{(0)})$ is the density of states of the ionized electron. In eq. (22), we have written $H'(t) = 2\mathbb{H}' \sin \omega t$. Using eq. (20), it follows that[7]

$$\mathbb{H}' = \tfrac{1}{2} ez\mathcal{E}_0\,.$$

The state $\left|a^{(0)}\right\rangle$ is the unperturbed wave function for the ground state of hydrogen,

$$\left|a^{(0)}\right\rangle = \Psi_{100}(r) = \frac{1}{(\pi a_0^3)^{1/2}}\, e^{-r/a_0}\,, \qquad\qquad a_0 \equiv \frac{\hbar^2}{me^2}\,.$$

The state $\left|b^{(0)}\right\rangle$ is the unperturbed wave function for the ionized wave function. This wave function is actually quite complicated, since one cannot really neglect the effects of the long-range Coulomb potential. Nevertheless, we shall simplify the computation by assuming the wave function of the ejected electron is a free-particle plane wave, with wave number vector $\vec{\boldsymbol{k}}$, where the direction of $\vec{\boldsymbol{k}}$ corresponds to

[7] Although the derivation in class was performed for an harmonic perturbation proportional to $\cos\omega t$, it is easy to check that the same result for Fermi's golden rule is obtained in the case of $\sin\omega t$.

that of the ejected electron. That is, $\left|b^{(0)}\right\rangle = e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}}/\sqrt{V}$. Taking the hermitian conjugate yields,

$$\left\langle b^{(0)}\right| = \frac{1}{\sqrt{V}}\, e^{-i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}}\,.$$

Note that we have normalized the free-particle plane wave by placing the system in a very large box of volume V. Imposing periodic boundary conditions, the possible values of $\vec{\boldsymbol{k}}$ are quantized as discussed in class. This will be convenient since we can later use the expression derived in class for the free-particle density of states.

We are now ready to compute the matrix element, $\left\langle b^{(0)}\right|\mathbb{H}'\left|a^{(0)}\right\rangle$. Employing spherical coordinates, $z = r\cos\theta$ and

$$\left\langle b^{(0)}\right|\mathbb{H}'\left|a^{(0)}\right\rangle = \frac{e\mathcal{E}_0}{2(V\pi a_0^3)^{1/2}}\int_0^\infty dr'\, r'^3 e^{-r'/a_0}\int d\Omega' e^{-i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}'}\cos\theta'\,. \tag{23}$$

In order to perform this integral, we make use of the following trick. There is a very useful identity given by Liboff in eq. (10.67) on p.421 (see also, problem 10.11 on p. 427 of Liboff):[8]

$$e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}'} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kr')[Y_\ell^m(\theta',\phi')]^* Y_\ell^m(\theta,\phi)\,, \tag{24}$$

where the vector $\vec{\boldsymbol{r}}'$ points in a direction with polar and azimuthal angles θ', ϕ' with respect to a fixed z-axis, and the vector $\vec{\boldsymbol{k}}$ points in a direction with polar and azimuthal angles θ, ϕ with respect to a fixed z-axis. Taking the complex conjugate of eq. (24) and inserting the result into eq. (23) yields:

$$\left\langle b^{(0)}\right|\mathbb{H}'\left|a^{(0)}\right\rangle = \frac{4\pi e\mathcal{E}_0}{2(V\pi a_0^3)^{1/2}}\sum_{\ell=0}^{\infty} i^\ell\int_0^\infty dr'\, r'^3 e^{-r'/a_0} j_\ell(kr')\sum_{m=-\ell}^{\ell}[Y_\ell^m(\theta,\phi)]^*\int d\Omega'\cos\theta'\, Y_\ell^m(\theta',\phi')\,. \tag{25}$$

It may look like we have made things more complicated, but the reverse is true! Noting that we can write (with the help of Table 9.1 on p. 373 of Liboff):

$$\cos\theta' = \left(\frac{4\pi}{3}\right)^{1/2}[Y_1^0(\theta',\phi')]^*\,,$$

---

[8] This identity should look familiar. In particular, if $\vec{\boldsymbol{r}}'$ points along the z-direction, then $\theta' = \phi' = 0$, in which case $Y_\ell^m(\theta',\phi') = Y_\ell^m(0,0) = \left(\frac{2\ell+1}{4\pi}\right)^{1/2}\delta_{m0}$. Thus, only the $m = 0$ term in the sum over m survives. Using $Y_\ell^0(\theta,\phi) = \left(\frac{2\ell+1}{4\pi}\right)^{1/2}P_\ell(\cos\theta)$, eq. (24) reduces to:

$$e^{ikz} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta)\,,$$

which is a result obtained in class. In fact using the addition theorem for spherical harmonics (see p. 390 of Liboff, identity (b) in the caption to Figure 9.16), one can use the above equation to derive eq. (24) or vice versa (cf. problem 10.12 of Liboff on p. 427).

the integration over solid angles in eq. (25) can be immediately performed:

$$\int d\Omega'\,\cos\theta'\,Y_\ell^m(\theta',\phi') = \left(\frac{4\pi}{3}\right)^{1/2}\int d\Omega'\,\cos\theta'\,Y_\ell^m(\theta',\phi')[Y_1^0(\theta',\phi')]^* = \left(\frac{4\pi}{3}\right)^{1/2}\delta_{\ell 1}\delta_{m0}\,, \tag{26}$$

where we have used the orthogonality relations of the spherical harmonics,

$$\int d\Omega\,Y_\ell^m(\Omega)\,[Y_{\ell'}^{m'}(\Omega)]^* = \delta_{\ell\ell'}\delta_{mm'}\,.$$

Inserting eq. (26) back into eq. (25) collapses both the sums over m and ℓ, respectively. Only the $\ell = 1$, $m = 0$ term of the sums survives. Thus, using $Y_1^0(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$, eq. (26) reduces to:

$$\left\langle b^{(0)}\right|\mathbb{H}'\left|a^{(0)}\right\rangle = \frac{2\pi i e\mathcal{E}_0\cos\theta}{(V\pi a_0^3)^{1/2}}\int_0^\infty dr\,r^3 e^{-r/a_0}\left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right)\,,$$

where we have substituted for $j_1(kr)$ using eq. (6). (For notational convenience, I have now dropped the primes on the integration variable r.) My integral tables provide the following results:[9]

$$\int_0^\infty r\,e^{-r/a_0}\sin kr\,dr = \frac{2ka_0^3}{(1+k^2a_0^2)^2}\,,$$

$$\int_0^\infty r^2\,e^{-r/a_0}\cos kr\,dr = \frac{2a_0^3(1-3k^2a_0^2)}{(1+k^2a_0^2)^3}\,.$$

Thus,

$$\int_0^\infty dr\,r^3 e^{-r/a_0}\left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right) = \frac{8ka_0^5}{(1+k^2a_0^2)^3}\,.$$

Hence, it follows that:

$$\left\langle b^{(0)}\right|\mathbb{H}'\left|a^{(0)}\right\rangle = 16ie\mathcal{E}_0\cos\theta\left(\frac{\pi a_0^5}{V}\right)^{1/2}\frac{ka_0}{(1+k^2a_0^2)^3}\,.$$

We are now ready to compute the transition rate. Using the density of states derived in class,

$$\rho(E) = \frac{Vm\hbar k}{(2\pi\hbar)^3}\,d\Omega\,,$$

the transition rate [see eq. (22)] is given by:

$$w_{a\to b} = \frac{2\pi}{\hbar}\,\frac{Vm\hbar k}{(2\pi\hbar)^3}\,d\Omega\left(\frac{256\pi e^2\mathcal{E}_0^2 a_0^5\cos^2\theta}{V}\right)\frac{(ka_0)^2}{(1+k^2a_0^2)^6}\,.$$

[9] For simple integrals, my reference table of choice is H.B. Dwight, *Table of Integrals and other Mathematical Data* (Macmillan Publishing Co., Inc., New York, 1961).

Simplifying the above result, and noting that $me^2/\hbar^2 = 1/a_0$, we end up with:

$$\boxed{\frac{dw_{a\to b}}{d\Omega} = \frac{64\mathcal{E}_0^2 a_0^3 \cos^2\theta}{\pi\hbar} \frac{(ka_0)^3}{(1+k^2a_0^2)^6}}$$

The factors of the volume V have canceled out, which indicates that the transition rate for ionization is a physical quantity.

Fermi's golden rule also imposes energy conservation. The initial energy is the ground state energy of hydrogen, which is given by $E_a^{(0)} = -\hbar\omega_0$, as noted in part (a). The final state energy is $E_b^{(0)} = \hbar^2k^2/(2m)$. Since this is an absorption process, a quantum of energy $\hbar\omega$ from the harmonic perturbation must account for the energy difference between the final and initial state energies. Therefore,

$$\hbar\omega = \frac{\hbar k^2}{2m} + \hbar\omega_0\,.$$

Solving for k^2, we can write:

$$k^2a_0^2 = \frac{2ma_0^2}{\hbar}(\omega-\omega_0) = \frac{2\hbar^3}{me^4}(\omega-\omega_0) = \frac{\omega-\omega_0}{\omega_0}\,,$$

where we have used the definition of the Bohr radius, $a_0 \equiv \hbar^2/(me^2)$, and the results of part (a). Thus, we can rewrite the differential transition rate for ionization as:

$$\boxed{\frac{dw_{a\to b}}{d\Omega} = \frac{64\mathcal{E}_0^2 a_0^3}{\pi\hbar}\left(\frac{\omega_0}{\omega}\right)^6\left(\frac{\omega}{\omega_0}-1\right)^{3/2}\cos^2\theta}$$

Note that as ω_0 is the minimum frequency of the field necessary to ionize the hydrogen atom, it follows that $\omega \geq \omega_0$.

(c) Integrating over solid angles [using $\int d\Omega\cos^2\theta = 4\pi/3$], we find that the total ionization rate is given by:

$$\boxed{w_{a\to b} = \frac{256\mathcal{E}_0^2 a_0^3}{3\hbar}\left(\frac{\omega_0}{\omega}\right)^6\left(\frac{\omega}{\omega_0}-1\right)^{3/2}}$$

Note that the ionization rate approaches zero both in the limit of $\omega \to \omega_0$ and in the limit of $\omega \to \infty$. Moreover, the ionization rate (which is a physical observable) must be non-negative for $\omega_0 \leq \omega < \infty$. Thus, there must be some value of ω in the range $\omega_0 \leq \omega < \infty$ for which the ionization rate is maximal. To find this value of ω, take the derivative of the expression above with respect to ω and set it to zero. Thus, we solve:

$$-\frac{6}{\omega^7}\left(\frac{\omega}{\omega_0}-1\right)^{3/2} + \frac{3}{2\omega^6\omega_0}\left(\frac{\omega}{\omega_0}-1\right)^{1/2} = 0\,.$$

This can be easily simplified, and one finds that the the above equation is satisfied for only one value, $\omega = \frac{4}{3}\omega_0$. We conclude that at this frequency, the ionization rate must be maximal. (Of course, one can also verify this by computing the sign of the second derivative.)