Physics 195a
Course Notes
Path Integrals: An Example
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1 Introduction

This note illustrates the use of path integral arguments in quantum mechanics via a famous example, the "Aharonov-Bohm" effect. Along the way, we get a glimpse of the more fundamental role which electromagnetic potentials take on in quantum mechanics, compared with classical electrodynamics.

2 Electromagnetism in Quantum Mechanics

Consider the motion of a charge q in an electromagnetic field $(\phi, \mathbf{A})$, where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. Classically, the force on the charge is the "Lorentz force":

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1}$$

where $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$ is the electric field, $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field, and $\mathbf{v}$ is the velocity of the charge. The classical Hamiltonian is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi, \tag{2}$$

where m is the mass of the charge and $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}}$ is the generalized momentum conjugate to $\mathbf{x}$ (L is the Lagrangian). Since $H = \frac{1}{2}mv^2 + V$, $\mathbf{v} = (\mathbf{p} - q\mathbf{A})/m$. Thus, $\mathbf{p} - q\mathbf{A}$ is the ordinary kinematic momentum. In quantum mechanics we presume a form:

$$\begin{aligned} H &= \frac{1}{2m}\left[\mathbf{p} - q\mathbf{A}(\mathbf{x}, t)\right]^2 + q\phi(\mathbf{x}, t) & (3) \\ &= \frac{1}{2m}\left[-i\nabla - q\mathbf{A}(\mathbf{x}, t)\right]^2 + q\phi(\mathbf{x}, t). & (4) \end{aligned}$$

3 Is the Vector Potential Real?

In classical electrodynamics, we invented the electromagnetic potentials as a mathematical aid. The real "physics" (*i.e.*, forces affecting motion) is in the fields. Interestingly, when we pursue the correspondence in quantum physics, we find a new phenomenon. In this section, we look at a specific example, and use path integral arguments as a tool to investigate this.

Let us consider the following problem: We are given a very long, thin "solenoid" (Fig. 1). Assume that the current is static, and that the net charge density is everywhere zero. Thus, we can take $\phi(\mathbf{x}, t) = 0$, and hence $\mathbf{E} = 0$ everywhere.

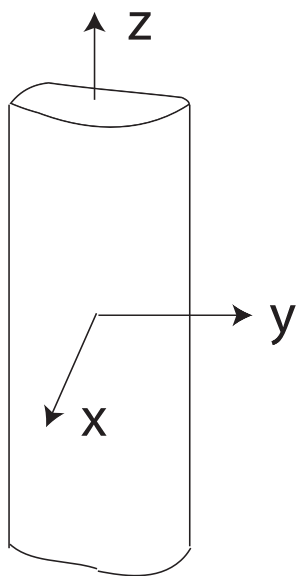

Figure 1: A section of the long solenoid, with coordinate system indicated.

Let us further assume that we have made the pitch of the winding very fine, so that we may assume that the current is perpendicular to the z direction, to whatever approximation we wish (perhaps we could do this by setting up a persistent current in a cylindrical superconductor). Then, to whatever approximation we desire, we have $\mathbf{B}_{\text{outside}} = 0$, where "outside" refers to the region outside the solenoid windings. Inside the solenoid, we have a non-zero magnetic flux:

$$\Phi = \int_{\text{solenoid cross section}} \mathbf{B} \cdot d\mathbf{S}. \tag{5}$$

That is, we have a magnetic field given by:

$$\mathbf{B} = \begin{cases} 0 & \text{outside solenoid} \\ B_0\mathbf{e}_z & \text{inside solenoid,} \end{cases} \tag{6}$$

where B_0 is a constant. This may be demonstrated with Maxwell's equations and symmetry arguments.

Thus, we have a situation where the electromagnetic field is zero everywhere outside of a thin cylindrical region, to whatever approximation we need. What is the vector potential outside this solenoid? Starting with $\mathbf{B} = \nabla \times \mathbf{A}$, and using Stoke's theorem we find:

$$\mathbf{A}_{\text{outside}}(\mathbf{x}, t) = A\mathbf{e}_\phi = \frac{\Phi}{2\pi r}\mathbf{e}_\phi. \tag{7}$$

We have made a particular choice of gauge here, in which $A_r = A_z = 0$. Since $J_r = J_z = 0$, these two components of the vector potential must be constants.

Notice now that the line integral of $\mathbf{A}$ along a closed curve outside and around the solenoid (see Fig. 2) is equal to the enclosed flux, which is non-zero. Thus, it is impossible to find a gauge transformation such that $\mathbf{A} = 0$ everywhere outside the solenoid. Let's check this more explicitly, by seeing

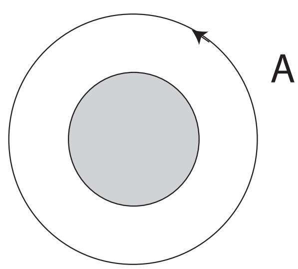

Figure 2: Cross section view of the solenoid, with a path around it in a region with vector potential $\mathbf{A}$.

how we fail: $\mathbf{A}$ is ambiguous up to a gauge transformation:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi, \tag{8}$$

where χ is an arbitrary differentiable function of position. We already have a solution in the form of Eqn. 7. If we attempt to find a gauge transformation so that $\mathbf{A}' = 0$, we must have

$$\nabla\chi = -\frac{\Phi}{2\pi r}\mathbf{e}_\phi. \tag{9}$$

In cylindrical coordinates,

$$\nabla\chi = \mathbf{e}_r\frac{\partial\chi}{\partial r} + \mathbf{e}_\phi\frac{1}{r}\frac{\partial\chi}{\partial\phi} + \mathbf{e}_z\frac{\partial\chi}{\partial z}. \tag{10}$$

Only the $\mathbf{e}_\phi$ component contributes, hence

$$\chi = -\Phi\frac{\phi}{2\pi} \quad \text{(plus any constant)}. \tag{11}$$

Following a brief moment of elation that we have succeeded, we realize that there is a problem. This solution is not a single-valued function of spatial position. Any attempt to patch this, *e.g.*, by introducing a point of discontinuity in $\chi(\phi)$, results in a vector potential which is non-zero outside the soelnoid for some value(s) of ϕ.

We have constructed a physical situation, to a good approximation, in which the magnetic field vanishes in a region, but the vector potential does not. Since there is something which we cannot get rid of by a gauge transformation, we might wonder whether there really is "something" outside the solenoid that "knows" about the field inside, even if the field outside is zero. Is there an experimentally observable consequence, or is this merely a mathematical oddity?

Classically, it appears that we in fact see nothing outside the solenoid: If we probe the outside region with charged particles, their trajectories are unaltered, because

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = 0. \tag{12}$$

What about quantum mechanics? We expect that the only hope is that something observable happens in the phase of the amplitude, since there are no classical effects. Let us imagine an experiment in which phase effects could appear.

We consider a wave packet (particle) suitably localized, and split, away from the solenoid (see Fig. 3). To begin the analysis of what happens to this wave packet, suppose first that the magnet is off, $\Phi = 0$. Let

$$\psi_0(\mathbf{x},t) = \psi_\ell(\mathbf{x},t) + \psi_r(\mathbf{x},t), \tag{13}$$

where ψ_ℓ is the wave passing to the left, and ψ_r is the wave passing to the right. That is, ψ_ℓ is a solution to the ($\mathbf{A} = 0$) Schrödinger equation with the property that it vanishes, for all times, on the half plane to the right of the solenoid. Switching right and left, ψ_r has the corresponding interpretation. We assume that at some early time, t_i, the wave packet is in front of the solenoid, and that at a later time, t_f, the wave packet is predominantly behind the solenoid. We look for interference patterns behind the solenoid at t_f.

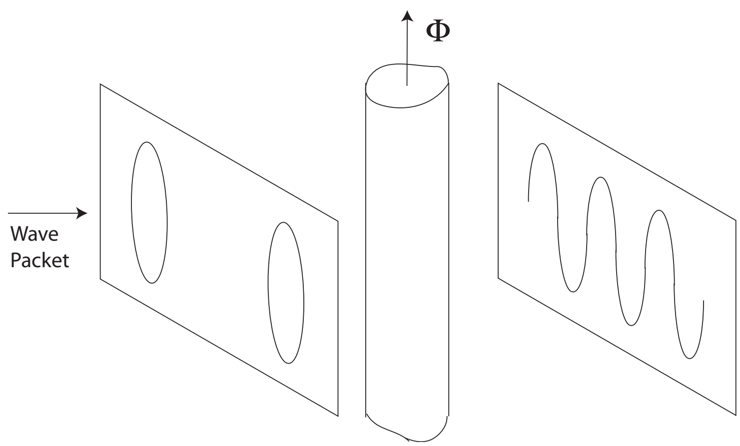

Figure 3: Schematic of the experimental arrangement.

With this general setup, and magnet-off solutions, let us now consider what happens when there is a flux Φ in the solenoid. We'll assume that this flux is time-independent. Let

$$s_\ell(\mathbf{x}) \equiv \int_{\substack{\mathbf{x}_i \\ \text{left path}}}^{\mathbf{x}} \mathbf{A}(\mathbf{x}') \cdot d\mathbf{x}', \tag{14}$$

$$s_r(\mathbf{x}) \equiv \int_{\substack{\mathbf{x}_i \\ \text{right path}}}^{\mathbf{x}} \mathbf{A}(\mathbf{x}') \cdot d\mathbf{x}', \tag{15}$$

where "left path" ("right path") is a path which does not intersect the right(left) half-plane at the solenoid (or perhaps intersects it an even number of times).

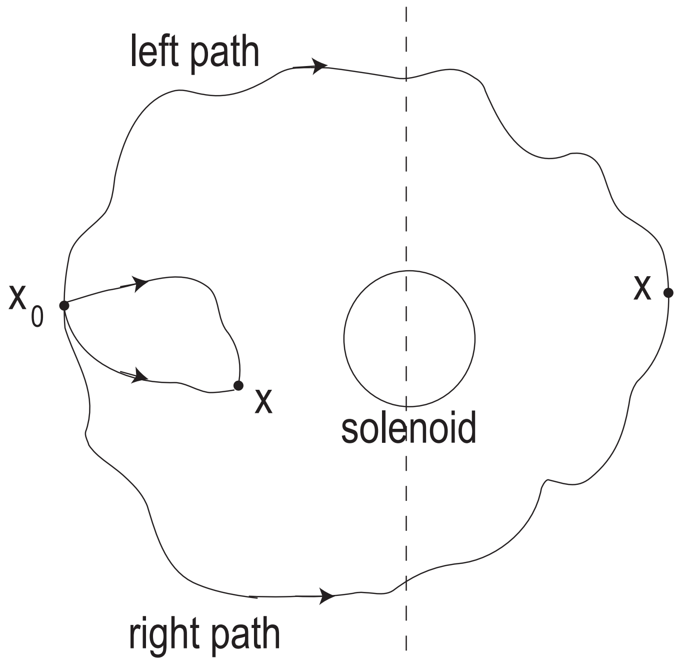

Figure 4: A section of the long solenoid, with sample left and right paths from x_0 to x.

We know that, independent of the details of the paths taken (Fig. 4),

$$\oint_{\substack{-\text{left path}\\+\text{right path}}} \mathbf{A}(\mathbf{x}')\cdot d\mathbf{x}' = \begin{cases} 0 & \mathbf{x} \text{ in front of solenoid,} \\ \Phi & \mathbf{x} \text{ behind solenoid.} \end{cases} \tag{16}$$

Actually, we could worry about less-probable scenarios, such as paths which circulate the solenoid multiple times, which would require modification to the above statement. We thus have that $s_\ell(\mathbf{x})$ and $s_r(\mathbf{x})$ are independent of their particular paths, as long as we maintain the left, right constraints.

So, what is the solution to the Scrödinger equation in the presence of Φ? It will be left to the reader to verify that the solution is:

$$\psi_\Phi(\mathbf{x},t) = \psi_\ell(\mathbf{x},t)e^{iqs_\ell(\mathbf{x})} + \psi_r(\mathbf{x},t)e^{iqs_r(\mathbf{x})}. \tag{17}$$

Verification may be accomplished by direct substitution into the Schrödinger equation; the left and right pieces independently satisfy the equation.

If $\mathbf{x}$ is in front of the solenoid, then $s_\ell(\mathbf{x}) = s_r(\mathbf{x})$, since $\Phi_{\text{enclosed}} = 0$. Then

$$\begin{aligned} \psi_\Phi(\mathbf{x},t) &= \left[\psi_\ell(\mathbf{x},t) + \psi_r(\mathbf{x},t)\right] e^{iqs_\ell(\mathbf{x})} \\ &= \psi_0(\mathbf{x},t) e^{iqs_\ell(\mathbf{x})} \quad \text{(early times, } t_i). \end{aligned} \tag{18}$$

Thus, at early times, ψ_Φ and ψ_0 represent the same state, because they differ only in phase.

At later times we consider the region "behind" the solenoid. In this case,

$$s_r(\mathbf{x}) = \Phi + s_\ell(\mathbf{x}), \tag{19}$$

and thus,

$$\psi_\Phi(\mathbf{x},t) = \left[\psi_\ell(\mathbf{x},t) + e^{iq\Phi}\psi_r(\mathbf{x},t)\right] e^{iqs_\ell(\mathbf{x})} \quad \text{(late times, } t_f). \tag{20}$$

Thus, the flux manifests itself in a phase shift by $q\Phi$ of the wave passing to the right relative to the wave passing to the left. The interference pattern behind the solenoid will then be affected, unless $q\Phi = 2\pi n$, where n is an integer. The flux inside the solenoid may be "observed" with a probe outside the solenoid in a purely quantum mechanical way. This has been experimentally verified. An early theoretical paper, for which this effect is named, is Aharonov and Bohm, *Physical Review* **115** (1959) 485. An early experimental paper is Chambers, *Physical Review Letters* **5** (1960) 3.

4 Exercises

1. Show that the $\mathbf{B}$ field is as given in Eqn. 6, and that the vector potential is as given in Eqn. 7, up to gauge transformations.

2. Verify that the wave function in Eqn. 17 satisfies the Schrödinger equation.

3. A number of assumptions have been made, possibly implicitly in the discussion of this effect.

   (a) Critique the discussion, pointing out areas where the argument may break down.

   (b) Resolve the problematic areas in your critique, or else demonstrate that the argument really does break down.

4. We have discussed the interesting Aharonov-Bohm effect. Let us continue a bit further the thinking in this example.

   (a) Consider again the path integral in the vicinity of the long thin solenoid. In particular, consider a path which starts at $\boldsymbol{x}$, loops around the solenoid, and returns to $\boldsymbol{x}$. Since the $\boldsymbol{B}$ and $\boldsymbol{E}$ fields are zero everywhere in the region of the path, the only effect on the particle's wave function in traversing this path is a phase shift, and the amount of phase shift depends on the magnetic flux in the solenoid, as we discussed in class. Suppose we are interested in a particle with charge of magnitude e (*e.g.*, an electron). Show that the magnetic flux Φ in the solenoid must be quantized, and give the possible values that Φ can have.

   (b) Wait a minute!!! Did you just show that there is no Aharonov-Bohm effect? We know from experiment that the effect is real. So, if you did what I expected you to do in part (a), there is a problem. Discuss!

   (c) The BCS theory for superconductivity assumes that the basic "charge carrier" in a superconductor is a pair of electrons (a "Cooper pair"). The Meissner effect for a (Type I) superconductor is that when such a material is placed in a magnetic field, and then cooled below a critical temperature, the magnetic field is excluded from the superconductor. Suppose that there is a small non-superconducting region traversing the superconductor, in which magnetic flux may be "trapped" as the material is cooled below the critical temperature. Ignoring part (b) above, what values do you expect to be possible for the trapped flux? [This effect has been experimentally observed.] What is the value (in Tesla-m^2) of the smallest non-zero flux value. You may find the following conversion constant handy:

$$1 = 0.3 \text{ Tesla-m/GeV}, \tag{21}$$

   where the "0.3" is more precisely the speed of light in nanometers/second.

   (d) How can we reconcile the answer to part (c), which turns out to be a correct result (even if the derivation might be flawed), with your discussion in part (b)? Let's examine the superconducting case more carefully. Let us suppose we have a ring of superconducting material. We assume a model for superconductivity in which the superconducting electrons are paired, and the resulting pairs are in a "Bose-condensate". Well, this precedes our discussion on identical particles, but we essentially mean that the pairs are all in the same quantum state. We may write our wave function for the superconducting pairs in the form:

$$\psi(\mathbf{x}) = \sqrt{\frac{\rho_s}{2}} e^{i\theta(\mathbf{x})}, \tag{22}$$

where ρ_s is the number density of superconducting electrons, and θ is a position-dependent phase. Note that we have normalized our wave function so that its absolute square gives the density of Cooper pairs. Find an expression for $\frac{\rho_s}{2}\mathbf{v}$, the Cooper pair number current density. Use this with the expression for the canonical momentum of a Cooper pair in a magnetic field (vector potential $\mathbf{A}$) to arrive at an expression for the electromagnetic current density of the superconducting electrons.

Now consider the following scenario: We apply an external magnetic field with the superconductor above its critical temperature (that is, not in a superconducting state). We then cool this system down below the critical temperature. We want to know what we can say about any magnetic flux which is trapped in the hole in the superconductor. Consider a contour in the interior of the superconductor, much further from the surfaces than any penetration depths. By considering an integral around this contour, see what you can say about the allowed values of flux through the hole.

(e) So far, no one has observed (at least not convincingly) a magnetic "charge", analogous to the electric charge. But there is nothing fundamental that seems to prevent us from modifying Maxwell's equations to accommodate the existence of such a "magnetic monopole". In particular, we may alter the divergence equation to:

$$\nabla \cdot \boldsymbol{B} = 4\pi\rho_M,$$

where ρ_M is the magnetic charge density.

Consider a magnetic monopole of strength e_M located at the origin. The $\boldsymbol{B}$-field due to this charge is simply:

$$\boldsymbol{B} = \frac{e_M}{r}\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction. The $\hat{\boldsymbol{r}}$-component of the curl of the vector potential is:

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}.$$

A solution, as you should quickly convince yourself, is a vector potential in the ϕ direction:

$$A_\phi = e_M \frac{1-\cos\theta}{r\sin\theta}.$$

Unfortunately(?), this is singular at $\theta = \pi$, *i.e.*, on the negative z-axis. We can fix this by using this form everywhere except in a cone about $\theta = \pi$, *i.e.*, for $\theta \leq \pi - \epsilon$, and use the alternate solution:

$$A'_\phi = e_M \frac{-1-\cos\theta}{r\sin\theta}$$

in the (overlapping) region $\theta \geq \epsilon$, thus covering the entire space. In the overlap region ($\epsilon \leq \theta \leq \pi - \epsilon$), either $\boldsymbol{A}$ or $\boldsymbol{A'}$ may be used, and must give the same result, *i.e.*, the two solutions are related by a gauge transformation – that is, they differ by the gradient of a scalar function.

Consider the effect of the vector potential on the wave function of an electron (charge $-e$). Invoke single-valuedness of the wave function, and determine the possible values of e_M that a magnetic charge can have. [This is sometimes called a "Dirac monopole".]