The Cosmological Constant Problem

1. The numerical value of the vacuum energy density

The cosmological constant Λ was introduced by Albert Einstein into general relativity in 1917. Including the cosmological constant, Einstein's field equations are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}\,, \tag{1}$$

where G_N is Newton's gravitational constant. The cosmological constant can be interpreted as the energy density of the vacuum. Specifically, if we introduce

$$T^{\rm vac}_{\mu\nu} \equiv -\frac{c^4\Lambda}{8\pi G_N}g_{\mu\nu}\,, \tag{2}$$

then eq. (1) can be rewritten as

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \frac{8\pi G_N}{c^4}(T_{\mu\nu} + T^{\rm vac}_{\mu\nu})\,.$$

If we compare eq. (2) with the energy-momentum tensor of a perfect fluid,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu + p g_{\mu\nu}\,,$$

then we can conclude that the energy density of the vacuum is

$$c^2\rho_{\rm vac} = \frac{c^4\Lambda}{8\pi G_N}\,, \tag{3}$$

and the equation of state of the vacuum is

$$p_{\rm vac} = -\rho_{\rm vac}c^2\,. \tag{4}$$

The current astrophysical data can be interpreted as being consistent with a positive value of the cosmological constant, which implies that $\rho_{\rm vac} > 0$. In light of eq. (4), it follows that the pressure due to the vacuum energy is *negative*. Very strange stuff indeed!

To evaluate the numerical value of the energy density of the vacuum, we consult the latest data given in the table of Astrophysical constants and parameters in K.A. Olive *et al.* [Particle Data Group Collaboration], *Review of Particle Physics*, Chin. Phys. C **38**, 090001 (2014). This table includes the following two entries,

$$\frac{c^2}{3H_0^2} = 6.3 \pm 0.2 \times 10^{51}~{\rm m}^2\,, \tag{5}$$

$$\Omega_\Lambda \equiv \frac{\rho_{\rm vac}}{\rho_{c,0}} = 0.685^{+0.017}_{-0.016}\,, \tag{6}$$

where H_0 is the present day Hubble parameter and the so-called critical energy density today is given by

$$c^2 \rho_{c,0} \equiv \frac{3H_0^2 c^2}{8\pi G_N}\,. \tag{7}$$

Hence the ratio of eqs. (3) and (7) is

$$\Omega_\Lambda = \frac{\rho_{\rm vac}}{\rho_{c,0}} = \frac{\Lambda c^2}{3H_0^2}\,.$$

Employing the numbers given in eqs. (5) and (6), it follows that

$$\Lambda = \frac{3H_0^2}{c^2}\,\Omega_\Lambda = (1.09 \pm 0.04) \times 10^{-52}~{\rm m}^{-2}\,. \tag{8}$$

Using eq. (8), we obtain

$$\rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G_N} = \frac{(1.1\times 10^{-52}~{\rm m}^{-2})(3\times 10^8~{\rm m\,s}^{-1})^2}{8\pi(6.673\times 10^{-11}~{\rm m}^3\,{\rm kg}^{-1}\,{\rm s}^{-2})} = 5.9\times 10^{-27}~{\rm kg\,m}^{-3}\,.$$

Thus, the numerical value of the vacuum energy is

$$\rho_{\rm vac} c^2 = 5.31 \times 10^{-10}~{\rm J} = 3.32~{\rm GeV\,m}^{-3}\,, \tag{9}$$

after using the conversion 1 eV $= 1.6 \times 10^{-19}$ J and 1 GeV $= 10^9$ eV.

In order to see whether this vacuum energy is large or small, we need to invoke quantum mechanics. In quantum mechanics, there is a natural association between length scales and energy scales. The key conversion factor is

$$\hbar c = 197~{\rm MeV\,fm} = 1.97 \times 10^{-7}~{\rm eV\,m}\,,$$

where 1 fm $= 10^{-15}$ m. Thus,

$$1~{\rm m} = \hbar c\,(5.08 \times 10^{-6}~{\rm eV}^{-1}) \tag{10}$$

Using this conversion factor, we can write

$$\rho_{\rm vac} c^2 = \frac{3.32 \times 10^9~{\rm eV}}{(\hbar c)^3\,((5.08 \times 10^{-6}~{\rm eV}^{-1})^3} = \frac{(2.24 \times 10^{-3}~{\rm eV})^4}{(\hbar c)^3}\,. \tag{11}$$

2. The Planck scale

The Planck scale represents the energy scale at which classical general relativity must break down due to quantum mechanical effects. In order to estimate what this energy scale is, we first ask a simpler question. What is the minimum distance scale that makes sense to localize an electron of mass m? Let us try to localize the particle by using a beam of light of wavelength λ. The energy of the corresponding photons is $E = h\nu = hc/\lambda = 2\pi\hbar c/\lambda$. But, if $E \gtrsim 2mc^2$, then it is possible to convert the photons into e^+e^- pairs, and it no longer makes sense to say that you have isolated a single electron. Thus, we shall demand that $E < 2mc^2$, which yields

$$\frac{2\pi\hbar c}{\lambda} < 2mc^2 \,.$$

Ignoring constants of $\mathcal{O}(1)$, we conclude that

$$\lambda \gtrsim \frac{\hbar}{mc} \,,$$

where $\hbar/(mc)$ is the Compton wavelength of the electron. This argument applies to any particle of mass m, so we conclude that at best it is possible to localize a particle down to a distance scale equal to its Compton wavelength. This conclusion is a consequence of quantum mechanics and special relativity.

Consider now the gravitational potential energy of a particle of mass m,

$$\Phi \sim \frac{G_N m^2}{r} \,.$$

Using the argument just presented, the smallest value of r that makes sense is the Compton wavelength of the particle. For $r = \hbar/(mc)$, we have

$$\Phi \sim \frac{G_N m^2}{(\hbar/mc)} = \frac{G_N m^3 c}{\hbar} \,.$$

As above, we shall demand that this energy is below $2mc^2$, otherwise the energy of the gravitational field can create particle-antiparticle pairs, an inherently quantum mechanical effect. Surely, classical gravity must break down at this point. Again, we neglect $\mathcal{O}(1)$ constants, and require that

$$\frac{G_N m^3 c}{\hbar} \lesssim mc^2 \,. \qquad (12)$$

We now define the Planck mass $M_{\rm PL}$ via

$$M_{\rm PL}^2 \equiv \frac{\hbar c}{G_N} \,. \qquad (13)$$

In terms of the Planck mass, eq. (12) is equivalent to $m \lesssim M_{\rm PL}$. That is, a particle of mass $M_{\rm PL}$ has a gravitational energy equal to its rest mass at a distance equal to its Compton wavelength. For any mass above the Planck mass, quantum mechanical effects cannot be neglected, and thus classical general relativity must break down.

There is an equivalent characterization of the Planck mass. Namely, for a black hole of mass $M_{\rm PL}$, the value of the Compton wavelength is equal to the value of the Schwarzschild radius, $r_s \equiv 2G_N M/c^2$. Again neglecting $\mathcal{O}(1)$ constants,

$$\frac{G_N M}{c^2} \sim \frac{\hbar}{Mc} \quad \Longrightarrow M^2 \sim \frac{\hbar c}{G_N} = M_{\rm PL}^2 \,.$$

To be complete, we note the numerical value of the Planck mass. Actually, it is more common to quote the Planck energy, $M_{\rm PL}c^2$,

$$M_{\rm PL}c^2 \equiv \left(\frac{\hbar c^5}{G_N}\right)^{1/2} = 1.22 \times 10^{19}\ \text{GeV}\,.$$

Since energy scales and length scales are related via eq. (10), we can also define the Planck length,

$$L_{\rm PL} \equiv \left(\frac{\hbar G_N}{c^3}\right)^{1/2} = 1.62 \times 10^{-35}\ \text{m}\,.$$

It is an interesting fact that these are the unique energy and length scales that are made up of the fundamental constants $\hbar$, c and G_N.

3. The most horrendous fine-tuning in physics

What is the "expected" value of the cosmological constant? In quantum mechanics, the vacuum energy is not zero due to quantum fluctuations. Indeed, the ground state energy of the harmonic oscillator is $\frac{1}{2}\hbar\omega$ in contrast to the classical harmonic oscillator whose ground state energy is zero. Quantum fields can be described as an infinite collection of harmonic oscillators, so naively the vacuum energy, which would be a sum over all the harmonic oscillator ground state energies, should be infinite. But, in practice, we would expect the sum to be cut off at some energy scale above which the true (presently unknown) fundamental theory of nature must be invoked.

Given our lack of knowledge of the fundamental theory above the Planck energy scale, a reasonable first guess would be to cut off the vacuum energy sum at the Planck scale. Thus, the "prediction" of quantum mechanics is that the energy density of the vacuum due to vacuum fluctuations should be roughly given by

$$\rho_{\rm vac}^{\rm QM}c^2 \sim \frac{M_{\rm PL}c^2}{L_{\rm PL}^3} = \left(\frac{\hbar c^5}{G_N}\right)^{1/2}\left(\frac{c^3}{\hbar G_N}\right)c^2 = \left(\frac{\hbar c^5}{G_N}\right)^2\frac{1}{(\hbar c)^3} = \frac{(M_{\rm PL}c^2)^4}{(\hbar c)^3}\,.$$

Putting in the numbers,

$$\rho_{\rm vac}^{\rm QM}c^2 \sim \frac{(1.22\times 10^{19}\ \text{GeV})^4}{(\hbar c)^3} = \frac{(1.22\times 10^{28}\ \text{eV})^4}{(\hbar c)^3}\,. \tag{14}$$

Thus, the quantum mechanical prediction for the vacuum energy is given by eq. (14). How good is this prediction? Let us compare this to the observed vacuum energy given in eq. (11),

$$\frac{\rho_{\rm vac}c^2}{\rho_{\rm vac}^{\rm QM}c^2} = \left(\frac{2.24\times 10^{-3}\ \text{eV}}{1.22\times 10^{28}\ \text{eV}}\right)^4 = 1.13\times 10^{-123}\,.$$

The observed vacuum energy density is a factor of 10^{123} smaller than its predicted value! This is by far the worst prediction in the history of physics!!

So, how do we fix this? Presumably, there must be some contribution from the fundamental theory above the Planck energy scale which adds an additional contribution to the vacuum energy so that the observed vacuum energy is given by

$$\rho_{\rm vac}c^2 = \rho_{\rm vac}^{\rm QM}c^2 + \rho_{\rm vac}^{\rm new}c^2 \,.$$

However, if this is the case, value of $\rho_{\rm vac}^{\rm new}c^2$ must be so incredibly close to $\rho_{\rm vac}^{\rm QM}c^2$, such that a cancellation occurs that is accurate to 123 decimal places!! Such a cancellation would occur only if the value of $\rho_{\rm vac}^{\rm new}c^2$ were fine-tuned to unimaginable precision. Thus, it is often said that the cosmological constant problem is the most severe fine-tuning problem in all of physics.

Many physicists have tried to come up with clever mechanisms to "explain" this fine-tuning as a consequence of some presently unknown fundamental symmetry. Others have insisted that the solution must be anthropic. In this view, the number of vacuum states of some fundamental theory of physics (string theory?) is incredibly large, of $\mathcal{O}(10^{500})$ or even larger. Each vacuum state has a random value of the cosmological constant, so in very rare circumstances the vacuum energy will be 10^{123} times smaller than its "natural" value. If one can argue that the existence of galaxies, planets, human beings, etc. requires that the vacuum energy not be much larger than presently observed, then one would have an anthropic solution to the cosmological constant problem.

I will not conjecture here how the cosmological constant problem (i.e. the horrendous fine-tuning that seems to be required to explain the observed value) will ultimately be solved. Suffice it to say that it is still regarded as one of the most significant challenges for fundamental physics.