DUE: THURSDAY OCTOBER 8, 2015

1. Suppose that $\vec{a}$ and $\vec{b}$ are proper Euclidean three-vectors, whereas $\vec{c}$ and $\vec{d}$ are improper three-vectors (also called pseudovectors). Show that

(a) $\vec{a} \cdot \vec{b}$ and $\vec{c} \cdot \vec{d}$ are both scalar quantities, whereas $\vec{a} \cdot \vec{c}$ is a pseudoscalar.

(b) $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are both pseudovectors, whereas $\vec{a} \times \vec{c}$ is a proper vector.

HINT: In solving part (b), recall that the definition of the determinant of an arbitrary 3×3 matrix can be written as:

$$\epsilon_{ijk} R_{i\ell} R_{jm} R_{kn} = \epsilon_{\ell mn} \det R \,. \tag{1}$$

One can obtain a useful identity by multiplying both sides of eq. (1) by $R^{-1}_{\ell p}$ and summing over ℓ. Then, if R is an orthogonal matrix, show that $R^{-1}_{\ell p} = R_{p\ell}$, and verify that the end result (after an appropriate relabeling of indices) is: $\epsilon_{ijk} R_{jm} R_{kn} = \epsilon_{\ell mn} (\det R) R_{i\ell}$.

2. Consider a tensor T_{ij} in three dimensional Euclidean space. Under an arbitrary rotation of the three-dimensional coordinate space, the tensor is transformed. Show that:

(a) if T_{ij} is symmetric and traceless, then the transformed tensor is traceless and symmetric.

(b) if T_{ij} is antisymmetric, then the transformed tensor is antisymmetric.

State the analogous results for a tensor $T^{\mu\nu}$ in four-dimensional Minkowski space.

HINT: How should one define the trace of a second-rank tensor in Minkowski space such that the trace is a scalar?

3. The four-velocity vector is given by $U^\mu = dx^\mu / d\tau = (\gamma c\,;\, \gamma \vec{v})$, where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ and τ is the proper time. The four-momentum vector is $P^\mu = m U^\mu = (E/c\,;\, \vec{p})$, where m is the mass, $E = \gamma m c^2$ and $\vec{p} = \gamma m \vec{v}$.

(a) Determine the components of the four-acceleration vector, $A^\mu = dU^\mu / d\tau = (A^0\,;\, \vec{A})$ in terms of the three-velocity $\vec{v} = d\vec{x}/dt$ and the three-acceleration $\vec{a} = d\vec{v}/dt$.

HINT: Recall that $d\tau = \gamma^{-1} dt$. Using the chain rule, show that $c^2\, d\gamma/d\tau = \gamma^4\, \vec{v} \cdot \vec{a}$.

(b) Show that $U^\mu A_\mu = 0$.

(c) The relativistic generalization of Newton's second law is $F^\mu = m A^\mu$ where F^μ is the four-force vector. Using this result, show that the three-force $\vec{f} = d\vec{p}/dt$ is given by

$$\vec{f} = \gamma m \vec{a} + \frac{\gamma^3 m}{c^2} (\vec{v} \cdot \vec{a}) \vec{v} \,.$$

(d) Show that the three-force is proportional to the three-acceleration $\vec{\boldsymbol{a}}$ in two special cases: (i) linear motion where $\vec{\boldsymbol{a}}$ is parallel to $\vec{\boldsymbol{v}}$; and (ii) circular motion where $\vec{\boldsymbol{a}}$ is perpendicular to $\vec{\boldsymbol{v}}$. In cases (i) and (ii) where $\vec{\boldsymbol{f}} \propto \vec{\boldsymbol{a}}$, determine the corresponding proportionality constant. In light of these results, explain why it makes no sense to call γm the "relativistic mass."

(e) Show that the classical definition of power,

$$\frac{dE}{dt} = \vec{\boldsymbol{f}} \cdot \vec{\boldsymbol{v}}\,,$$

continues to hold in special relativity if E is identified with the relativistic energy.

4. Under a Lorentz transformation, a second-rank contravariant tensor transforms as follows:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}\,,$$

where Λ is the Lorentz transformation matrix.

(a) Consider an inertial frame K at rest, and a second inertial frame K' moving with velocity v along the x-direction with respect to K. Using the explicit result for Λ corresponding to the transformation between K and K', determine the electric and magnetic fields in frame K' in terms of the corresponding fields in frame K. Check that you correctly reproduce the results of eq. (2.16) on p. 21 of our textbook.

(b) Generalize the result of part (a) to the case where K' is moving with arbitrary velocity vector $\vec{\boldsymbol{v}}$ with respect to K.

5. Supernova 1987A was observed to occur at a distance of 170,000 light years from earth. Neutrinos and photons emitted from the supernova were observed by detectors on earth. The neutrinos (on average) had an energy of 10 MeV. Assume that the neutrinos have a rest mass-energy of $mc^2 = 1$ eV (whereas photons are massless).

(a) What is the elapsed time for one of the neutrinos to travel from the supernova to earth (as measured in the neutrino's frame)?

(b) Suppose the neutrino and photon were initially emitted from the supernova at the same time. How much later than the photon will the neutrino arrive at earth?

HINT: Since the neutrino is highly relativistic, you may assume in part (a) that the speed v of the neutrino is approximately equal to the speed of light c. However, the approximation $v = c$ is not sufficiently accurate to answer part (b).