DUE: THURSDAY OCTOBER 22, 2015

1. Consider an inertial frame K' that is moving with velocity v along the x axis with respect to an inertial frame K. A photon of angular frequency ω' is omitted from a source located at the origin of K' and is received by an observer located at the origin of K. In frame K, the observer measures a photon of angular frequency ω. The photon is seen by the observer to be moving in the x–y plane in a direction that makes an angle θ with respect to the x-axis.

(a) By considering the Lorentz transformation properties of the four-vector k^μ that characterizes the photon, compute the angular frequency of the photon emitted from the source, ω', in terms of ω, v and θ. This computation yields the relativistic Doppler shift.

(b) In the special case where the photon is seen by the observer to move along the x-axis, determine whether ω is larger or smaller than ω'. Consider separately the cases where the source is either approaching ($\theta = 0$) or receding from the observer ($\theta = \pi$).

(c) Evaluate ω'/ω in the non-relativistic limit for the cases of $\theta = 0$, $\frac{1}{2}\pi$ and π. In each case, keep terms of $\mathcal{O}(v/c)$ but ignore any higher order terms.

2. (a) The line element of special relativity is given by $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$. Transform this line element from the usual $(ct; x, y, z)$ rectangular coordinates to new coordinates $(ct'; x', y', z')$ related by

$$
\begin{aligned}
t &= \left(\frac{c}{g} + \frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)\,, \\
x &= c\left(\frac{c}{g} + \frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}\,, \\
y &= y'\,, \\
z &= z'\,,
\end{aligned}
$$

for a constant g with the dimensions of acceleration.

(b) For $gt'/c \ll 1$, show that this corresponds to a transformation to a uniformly accelerated frame in Newtonian mechanics.

(c) Show that an at-rest clock in this frame at $x' = h$ runs fast compared to a clock at rest at $x' = 0$ by a factor of $(1 + gh/c^2)$. How is this related to the equivalence principle?

3. An atomic clock is placed in the basement of the Empire State Building, and another one is placed on the 102nd floor, 380 meters above the first clock.

(a) Which one runs faster because of the gravitational potential difference between them?

(b) How long will the clocks have to run before one of them gains 1 ns compared to the other because of the gravitational time dilation?

Consider two atomic clocks where one is located in an inertial frame on the axis of a rotating centrifuge, and an identical clock is located on the rim of the centrifuge. The linear speed of the rim of the centrifuge is v.

(c) Which clock runs faster?

(d) Assuming $v \ll c$, use the principle of equivalence to determine the fractional difference between the rates of the clocks.

HINT: For parts (c) and (d), replace the centrifugal acceleration by an equivalent gravitational field.

4. In a certain spacetime geometry, the metric is

$$ds^2 = (1 - Ar^2)^2(dr^2 - c^2dt^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

(a) Calculate the proper distance along a radial line from the center $r = 0$ to a coordinate radius $r = R$.

(b) Calculate the area of a sphere of coordinate radius $r = R$.

(c) Calculate the three-volume of a sphere of coordinate radius $r = R$.

(d) Calculate the four-volume of a four-dimensional tube bounded by a sphere of coordinate radius R and two $t =$constant planes separated by a time T.

5. The Schwarzschild metric is given by:

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

What is the coordinate velocity of light in the Schwarzschild metric as a function of r

(a) in the radial direction?

(b) in the transverse direction?