MIDTERM ALERT: The midterm exam will be a take-home exam and will cover the first seven topics of the Physics 171 Course Outline. The exam will be posted to the course website on the evening of Friday November 6. Completed exams must be returned to my ISB mailbox no later than noon on Monday November 9. While working on the exam, you are permitted to consult with your class notes, any material provided on the course website, the textbook by Ta-Pei Cheng and one other relativity textbook of your choosing.

DUE: THURSDAY NOVEMBER 5, 2015

1. (a) Show that raising and lowering of indices commutes with covariant differentiation; *e.g.*, $D_\alpha A_\mu = D_\alpha(g_{\mu\nu}A^\nu) = g_{\mu\nu}D_\alpha A^\nu$.

(b) Suppose that A_μ is a covariant vector and $F_{\mu\nu}$ is an antisymmetric tensor. Prove that:

(i) $D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$

(ii) $D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} + D_\mu F_{\nu\rho} = \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho}\,.$

(c) Maxwell's equations in Minkowski space are given in Chapter 12.2 of Cheng. Using the principle of general covariance and the results of part (b), find the appropriate generalization of Maxwell's equations in curved spacetime.

(d) How should the equation for current conservation ($\partial_\mu J^\mu = 0$) be generalized in curved spacetime? [*EXTRA CREDIT:* Prove that this result is a consequence of Maxwell's equations in curved spacetime.]

2. (a) Suppose that the metric $g_{\mu\nu}$ is diagonal. Prove the following result for four-dimensional spacetime:

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2g}\frac{\partial g}{\partial x^\nu} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\nu}\,, \tag{1}$$

where $g \equiv \det\, g_{\mu\nu}$. Note the implicit sum over the index μ. [*EXTRA CREDIT:* Derive eq. (1) without assuming any special form for the metric.]

(b) The result of part (a) is valid for an arbitrary choice of metric. Making use of eq. (1), show that if A^μ is a contravariant vector and $F^{\mu\nu}$ is an antisymmetric tensor, then:

(i) $D_\nu A^\nu = \dfrac{1}{\sqrt{-g}}\dfrac{\partial}{\partial x^\nu}\left(\sqrt{-g}\,A^\nu\right),$

(ii) $D_\nu F^{\mu\nu} = \dfrac{1}{\sqrt{-g}}\dfrac{\partial}{\partial x^\nu}\left(\sqrt{-g}\,F^{\mu\nu}\right).$

3. Consider a three-dimensional spacetime with a metric that is given by:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\phi^2 \,.$$

(a) From the corresponding Lagrangian $L = g_{\mu\nu} q^\mu q^\nu$, where $q^\mu \equiv dx^\mu/ds$, write down the Euler-Lagrange equations which (as shown in class) are equivalent to the geodesic equations,

$$\frac{dq^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\, q^\alpha q^\beta = 0 \,.$$

Use this result to work out the non-vanishing connection coefficients.

(b) Check the connection coefficients obtained in part (a) by calculating them directly from the formula for $\Gamma^\mu_{\alpha\beta}$ in terms of the derivatives of the metric tensor.

4. The line element of flat spacetime in a frame with coordinates $(ct\,;\, x\,,\, y\,,\, z)$ that is rotating counterclockwise with an angular velocity ω about the z-axis of an inertial frame is

$$ds^2 = -[c^2 - \omega^2(x^2 + y^2)]dt^2 + 2\omega(x dy - y dx)dt + dx^2 + dy^2 + dz^2 \,. \tag{2}$$

(a) Show that by transforming to spherical coordinates (r, θ, ϕ), and then making the substitution $\phi' = \phi + \omega t$, the line element given in eq. (2) takes the following form

$$ds^2 = -c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi'^2)\,,$$

corresponding to the line element for Minkowski space in spherical coordinates (r, θ, ϕ').

(b) Find the geodesic equations in the rotating frame (i.e., written in terms of the variables x, y, z and t).

(c) Show that in the non-relativistic limit, the geodesic equations obtained in part (b) reduce to the usual equations of Newtonian mechanics for a free particle in a roatating frame exhibiting the centrifugal force and the Coriolis force.

5. The metric for the two-dimensional surface of a sphere of radius 1 is given by

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \,.$$

(a) Using the geodesic equations, show that all lines of longitude, corresponding to constant azimuthal angle ϕ on the surface of a sphere, are geodesics.

(b) *EXTRA CREDIT:* Solve the geodesic equations for the most general geodesic on the surface of a sphere.