

DUE: THURSDAY NOVEMBER 19, 2015

1. The Ricci tensor is defined as $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^{\beta}_{\mu\beta\nu}$ [cf. eq.(13.74) on p. 314 of our textbook].

(a) Show that if any other pair of indices of $R_{\alpha\mu\beta\nu}$ are summed using the inverse metric tensor (e.g., $g^{\alpha\nu} R_{\alpha\mu\beta\nu}$), then the result is either zero or a multiple of the Ricci tensor.

(b) Starting with the definition of the Riemann curvature tensor, $R^{\mu}_{\lambda\alpha\beta}$, given in eq. (13.58) on p. 311 of our textbook, derive a formula for the Ricci tensor in terms of the connection coefficients and its derivatives. Using this formula, show that the Ricci tensor, $R_{\mu\nu}$, is symmetric under the interchange of μ and ν .

2. Dust is a fluid without internal stress or pressure. Its energy-momentum tensor is $T^{\mu\nu} = \rho u^{\mu} u^{\nu}$, where ρ is a scalar quantity (which may depend on x^{μ}) and $u^{\mu} \equiv dx^{\mu}/d\tau$ is the velocity four-vector. Show that $D_{\nu} T^{\mu\nu} = 0$ implies that the dust particles follow geodesics.

HINT: You will need to invoke the identity $g_{\alpha\beta} u^{\alpha} u^{\beta} = -c^2$. Taking the covariant derivative of this relation will also yield a useful identity.

3. In class, we derived the Schwarzschild metric as a static spherically symmetric solution to the vacuum Einstein equations, $R_{\mu\nu} = 0$.

(a) Assume that there is a non-zero cosmological constant. By solving the modified Einstein equations, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, determine the appropriate modification to the Schwarzschild metric.

(b) Using the metric obtained in part (a) for the case of $\Lambda \neq 0$, determine the orbit equation for a test particle in orbit around a spherically symmetric star of mass M .

(c) Using the new orbit equation derived in part (b), compute the perihelion advance of Mercury, assuming that the orbit is nearly circular, i.e., the eccentricity $|e| \ll 1$. Mercury makes 415 revolutions per century, has an eccentricity $e = 0.2056$ and a semi-major axis $a = 5.791 \times 10^{10}$ m. Using the observed data, set an upper bound on the value of Λ .

4. The Schwarzschild metric is given by:

$$ds^2 \equiv -c^2 d\tau^2 = - \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

Consider a spacecraft in a circular orbit around a Schwarzschild black hole under the assumption that the orbit is in the plane where $\theta = \pi/2$. Denote the two conserved quantities E and J by

$$E \equiv mc^2 \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau}, \quad J \equiv mr^2 \frac{d\phi}{d\tau}, \quad (1)$$

where $r_s \equiv 2GM/c^2$ and τ is the proper time.

(a) Using the Lagrangian method, write down the geodesic equations of motion for the variable r of the Schwarzschild metric. Noting that θ is constant and r is independent of τ for a circular orbit, derive a simple equation for $\dot{t}/\dot{\phi}$ (where $\dot{t} \equiv dt/d\tau$ and $\dot{\phi} \equiv d\phi/d\tau$) and then use eq. (1) to rewrite this equation in the following form,

$$\frac{E}{cJ} = \left(\frac{1}{2}r_s r\right)^{-1/2} \left(1 - \frac{r_s}{r}\right). \quad (2)$$

(b) Recall that for a timelike geodesic, $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -c^2$. Using this equation, show that for a circular orbit in the $\theta = \frac{1}{2}\pi$ plane,

$$E^2 \left(1 - \frac{r_s}{r}\right)^{-1} - \frac{c^2 J^2}{r^2} = m^2 c^4. \quad (3)$$

(c) Obtain expressions for $dt/d\phi$ and $d\tau/d\phi$ in terms of r and r_s .

HINT: To derive an expression for $dt/d\phi$, use the chain rule,

$$\frac{dt}{d\phi} = \frac{dt}{d\tau} \frac{d\tau}{d\phi},$$

to conclude that $dt/d\phi = \dot{t}/\dot{\phi}$, where $\dot{t}/\dot{\phi}$ is obtained in part (a). To derive the expression for $d\tau/d\phi = 1/\dot{\phi} = (1/\dot{t})dt/d\phi$, you will need to express \dot{t} in terms of r and r_s . This can be achieved by using eqs. (2) and (3) to eliminate J , thereby obtaining an expression for E in terms of r and r_s . Then use eq. (1) to obtain the desired expression for \dot{t} .

5. Consider two twin astronauts, Alice and Bob, in the orbiting spacecraft at orbital radius $r = 2r_s$ around a Schwarzschild black hole. At coordinate time $t = 0$, Alice leaves the spacecraft and uses a rocket-pack to maintain a stationary position at radial distance $r = 2r_s$ and at fixed $\theta = \pi/2$ and $\phi = 0$. Bob stays behind inside the orbiting spacecraft. After one orbital revolution, Alice returns to the orbiting spacecraft. Determine which twin has aged more when Alice and Bob reunite.

HINT: Using the results of problem 4 part (c), first determine the value of the coordinate time t when the twins reunite by integrating $dt/d\phi$ from $\phi = 0$ to 2π . Then, compute the elapsed proper times for Alice and Bob during their respective journeys.