

DUE: THURSDAY NOVEMBER 19, 2015

1. The Ricci tensor is defined as  $R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = R^\beta_{\mu\beta\nu}$  [cf. eq.(13.74) on p. 314 of our textbook].

(a) Show that if any other pair of indices of  $R_{\alpha\mu\beta\nu}$  are summed using the inverse metric tensor (e.g.,  $g^{\alpha\nu}R_{\alpha\mu\beta\nu}$ ), then the result is either zero or a multiple of the Ricci tensor.

(b) Starting with the definition of the Riemann curvature tensor,  $R^\mu_{\lambda\alpha\beta}$ , given in eq. (13.58) on p. 311 of our textbook, derive a formula for the Ricci tensor in terms of the connection coefficients and its derivatives. Using this formula, show that the Ricci tensor,  $R_{\mu\nu}$ , is symmetric under the interchange of  $\mu$  and  $\nu$ .

2. Dust is a fluid without internal stress or pressure. Its energy-momentum tensor is  $T^{\mu\nu} = \rho u^\mu u^\nu$ , where  $\rho$  is a scalar quantity (which may depend on  $x^\mu$ ) and  $u^\mu \equiv dx^\mu/d\tau$  is the velocity four-vector. Show that  $D_\nu T^{\mu\nu} = 0$  implies that the dust particles follow geodesics.

*HINT:* You will need to invoke the identity  $g_{\alpha\beta}u^\alpha u^\beta = -c^2$ . Taking the covariant derivative of this relation will also yield a useful identity.

3. In class, we derived the Schwarzschild metric as a static spherically symmetric solution to the vacuum Einstein equations,  $R_{\mu\nu} = 0$ .

(a) Assume that there is a non-zero cosmological constant. By solving the modified Einstein equations,  $R_{\mu\nu} = \Lambda g_{\mu\nu}$ , determine the appropriate modification to the Schwarzschild metric.

(b) Using the metric obtained in part (a) for the case of  $\Lambda \neq 0$ , determine the orbit equation for a test particle in orbit around a spherically symmetric star of mass  $M$ .

(c) Using the new orbit equation derived in part (b), compute the perihelion advance of Mercury, assuming that the orbit is nearly circular, i.e., the eccentricity  $|e| \ll 1$ . Mercury makes 415 revolutions per century, has an eccentricity  $e = 0.2056$  and a semi-major axis  $a = 5.791 \times 10^{10}$  m. Using the observed data, set an upper bound on the value of  $\Lambda$ .

4. The Schwarzschild metric is given by:

$$ds^2 \equiv -c^2 d\tau^2 = - \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

Consider a spacecraft in a circular orbit around a Schwarzschild black hole under the assumption that the orbit is in the plane where  $\theta = \pi/2$ . Denote the two conserved quantities  $E$  and  $J$  by

$$E \equiv mc^2 \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau}, \quad J \equiv mr^2 \frac{d\phi}{d\tau}, \quad (1)$$

where  $r_s \equiv 2GM/c^2$  and  $\tau$  is the proper time.

(a) Using the Lagrangian method, write down the geodesic equations of motion for the variable  $r$  of the Schwarzschild metric. Noting that  $\theta$  is constant and  $r$  is independent of  $\tau$  for a circular orbit, derive a simple equation for  $\dot{t}/\dot{\phi}$  (where  $\dot{t} \equiv dt/d\tau$  and  $\dot{\phi} \equiv d\phi/d\tau$ ) and then use eq. (1) to rewrite this equation in the following form,

$$\frac{E}{cJ} = \left(\frac{1}{2}r_s r\right)^{-1/2} \left(1 - \frac{r_s}{r}\right). \quad (2)$$

(b) Recall that for a timelike geodesic,  $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -c^2$ . Using this equation, show that for a circular orbit in the  $\theta = \frac{1}{2}\pi$  plane,

$$E^2 \left(1 - \frac{r_s}{r}\right)^{-1} - \frac{c^2 J^2}{r^2} = m^2 c^4. \quad (3)$$

(c) Obtain expressions for  $dt/d\phi$  and  $d\tau/d\phi$  in terms of  $r$  and  $r_s$ .

*HINT:* To derive an expression for  $dt/d\phi$ , use the chain rule,

$$\frac{dt}{d\phi} = \frac{dt}{d\tau} \frac{d\tau}{d\phi},$$

to conclude that  $dt/d\phi = \dot{t}/\dot{\phi}$ , where  $\dot{t}/\dot{\phi}$  is obtained in part (a). To derive the expression for  $d\tau/d\phi = 1/\dot{\phi} = (1/\dot{t})dt/d\phi$ , you will need to express  $\dot{t}$  in terms of  $r$  and  $r_s$ . This can be achieved by using eqs. (2) and (3) to eliminate  $J$ , thereby obtaining an expression for  $E$  in terms of  $r$  and  $r_s$ . Then use eq. (1) to obtain the desired expression for  $\dot{t}$ .

5. Consider two twin astronauts, Alice and Bob, in the orbiting spacecraft at orbital radius  $r = 2r_s$  around a Schwarzschild black hole. At coordinate time  $t = 0$ , Alice leaves the spacecraft and uses a rocket-pack to maintain a stationary position at radial distance  $r = 2r_s$  and at fixed  $\theta = \pi/2$  and  $\phi = 0$ . Bob stays behind inside the orbiting spacecraft. After one orbital revolution, Alice returns to the orbiting spacecraft. Determine which twin has aged more when Alice and Bob reunite.

*HINT:* Using the results of problem 4 part (c), first determine the value of the coordinate time  $t$  when the twins reunite by integrating  $dt/d\phi$  from  $\phi = 0$  to  $2\pi$ . Then, compute the elapsed proper times for Alice and Bob during their respective journeys.