DUE: THURSDAY NOVEMBER 19, 2015

1. The Ricci tensor is defined as $R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = R^{\beta}{}_{\mu\beta\nu}$ [cf. eq.(13.74) on p. 314 of our textbook].

(a) Show that if any other pair of indices of $R_{\alpha\mu\beta\nu}$ are summed using the inverse metric tensor (e.g., $g^{\alpha\nu}R_{\alpha\mu\beta\nu}$), then the result is either zero or a multiple of the Ricci tensor.

(b) Starting with the definition of the Riemann curvature tensor, $R^{\mu}{}_{\lambda\alpha\beta}$, given in eq. (13.58) on p. 311 of our textbook, derive a formula for the Ricci tensor in terms of the connection coefficients and its derivatives. Using this formula, show that the Ricci tensor, $R_{\mu\nu}$, is symmetric under the interchange of μ and ν.

2. Dust is a fluid without internal stress or pressure. Its energy-momentum tensor is $T^{\mu\nu} = \rho u^{\mu}u^{\nu}$, where ρ is a scalar quantity (which may depend on x^{μ}) and $u^{\mu} \equiv dx^{\mu}/d\tau$ is the velocity four-vector. Show that $D_{\nu}T^{\mu\nu} = 0$ implies that the dust particles follow geodesics.

HINT: You will need to invoke the identity $g_{\alpha\beta}u^{\alpha}u^{\beta} = -c^2$. Taking the covariant derivative of this relation will also yield a useful identity.

3. In class, we derived the Schwarzschild metric as a static spherically symmetric solution to the vacuum Einstein equations, $R_{\mu\nu} = 0$.

(a) Assume that there is a non-zero cosmological constant. By solving the modified Einstein equations, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, determine the appropriate modification to the Schwarzschild metric.

(b) Using the metric obtained in part (a) for the case of $\Lambda \neq 0$, determine the orbit equation for a test particle in orbit around a spherically symmetric star of mass M.

(c) Using the new orbit equation derived in part (b), compute the perihelion advance of Mercury, assuming that the orbit is nearly circular, i.e., the eccentricity $|e| \ll 1$. Mercury makes 415 revolutions per century, has an eccentricity $e = 0.2056$ and a semi-major axis $a = 5.791 \times 10^{10}$ m. Using the observed data, set an upper bound on the value of Λ.

4. The Schwarzschild metric is given by:

$$ds^2 \equiv -c^2 d\tau^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \,.$$

Consider a spacecraft in a circular orbit around a Schwarzschild black hole under the assumption that the orbit is in the plane where $\theta = \pi/2$. Denote the two conserved quantities E and J by

$$E \equiv mc^2 \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} \,, \qquad J \equiv mr^2 \frac{d\phi}{d\tau} \,, \tag{1}$$

where $r_s \equiv 2GM/c^2$ and τ is the proper time.

(a) Using the Lagrangian method, write down the geodesic equations of motion for the variable r of the Schwarzschild metric. Noting that θ is constant and r is independent of τ for a circular orbit, derive a simple equation for $\dot{t}/\dot{\phi}$ (where $\dot{t} \equiv dt/d\tau$ and $\dot{\phi} \equiv d\phi/d\tau$) and then use eq. (1) to rewrite this equation in the following form,

$$\frac{E}{cJ} = \left(\tfrac{1}{2} r_s r\right)^{-1/2} \left(1 - \frac{r_s}{r}\right) . \tag{2}$$

(b) Recall that for a timelike geodesic, $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2$. Using this equation, show that for a circular orbit in the $\theta = \frac{1}{2}\pi$ plane,

$$E^2 \left(1 - \frac{r_s}{r}\right)^{-1} - \frac{c^2 J^2}{r^2} = m^2 c^4 \,. \tag{3}$$

(c) Obtain expressions for $dt/d\phi$ and $d\tau/d\phi$ in terms of r and r_s.

HINT: To derive an expression for $dt/d\phi$, use the chain rule,

$$\frac{dt}{d\phi} = \frac{dt}{d\tau}\frac{d\tau}{d\phi} \,,$$

to conclude that $dt/d\phi = \dot{t}/\dot{\phi}$, where $\dot{t}/\dot{\phi}$ is obtained in part (a). To derive the expression for $d\tau/d\phi = 1/\dot{\phi} = (1/\dot{t})dt/d\phi$, you will need to express $\dot{t}$ in terms of r and r_s. This can be achieved by using eqs. (2) and (3) to eliminate J, thereby obtaining an expression for E in terms of r and r_s. Then use eq. (1) to obtain the desired expression for $\dot{t}$.

5. Consider two twin astronauts, Alice and Bob, in the orbiting spacecraft at orbital radius $r = 2r_s$ around a Schwarzschild black hole. At coordinate time $t = 0$, Alice leaves the spacecraft and uses a rocket-pack to maintain a stationary position at radial distance $r = 2r_s$ and at fixed $\theta = \pi/2$ and $\phi = 0$. Bob stays behind inside the orbiting spacecraft. After one orbital revolution, Alice returns to the orbiting spacecraft. Determine which twin has aged more when Alice and Bob reunite.

HINT: Using the results of problem 4 part (c), first determine the value of the coordinate time t when the twins reunite by integrating $dt/d\phi$ from $\phi = 0$ to 2π. Then, compute the elapsed proper times for Alice and Bob during their respective journeys.