Parallel Transport and Curvature

1. Parallel transport along an infinitesimal curve

Consider a curved spacetime with a metric $g_{\mu\nu}$ and connection coefficients $\Gamma^\beta_{\mu\nu}$ that are symmetric under the interchange of their two lower indices. The mathematical expression for the parallel transport of vector components along a curve $x^\mu(\sigma)$ is[1]

$$\frac{dv^\beta}{d\sigma} + \Gamma^\beta_{\mu\nu} v^\mu \frac{dx^\nu}{d\sigma} = 0\,. \tag{1}$$

At some point P, corresponding to $\sigma = \sigma_P$, we specify the initial condition,

$$v^\beta_P \equiv v^\beta(\sigma_P)\,.$$

Using this initial condition, one can uniquely solve eq. (1) for $v^\beta(\sigma)$, the value of the parallel transported vector at any point σ along the curve $x^\mu(\sigma)$. The solution is

$$v^\beta(\sigma) = v^\beta_P - \int_{\sigma_P}^{\sigma} \Gamma^\beta_{\mu\nu}(\sigma') v^\mu(\sigma') \frac{dx^\nu}{d\sigma'}\, d\sigma'\,, \tag{2}$$

where we exhibit the σ' dependence of $\Gamma^\beta_{\mu\nu}$ and v^μ along the curve.

Suppose that the curve from σ_P to σ is infinitesimal in extent. Then, we can Taylor expand to first order around the point P,

$$v^\mu(\sigma') \simeq v^\mu_P + \left(\frac{\partial v^\mu}{\partial x^\alpha}\right)_P \left[x^\alpha(\sigma') - x^\alpha_P\right], \tag{3}$$

$$\Gamma^\beta_{\mu\nu}(\sigma') \simeq (\Gamma^\beta_{\mu\nu})_P + \left(\frac{\partial \Gamma^\beta_{\mu\nu}}{\partial x^\alpha}\right)_P \left[x^\alpha(\sigma') - x^\alpha_P\right]. \tag{4}$$

It is convenient to rewrite eq. (1) in the following form (by employing the chain rule),

$$\frac{dx^\nu}{d\sigma}\left(\frac{\partial v^\beta}{\partial x^\nu} + \Gamma^\beta_{\mu\nu} v^\mu\right) = 0\,. \tag{5}$$

We assume that the curve $x^\mu(\sigma)$ is parameterized such that one continuously moves along the curve as σ increases. This means that $dx^\mu/d\sigma \neq 0$ for all points along the curve, in which case one can divide eq. (5) by $dx^\mu/d\sigma$ to obtain,

$$\frac{\partial v^\beta}{\partial x^\nu} + \Gamma^\beta_{\mu\nu} v^\mu = 0\,. \tag{6}$$

[1] See eq. (13.47) of Ta-Pei Cheng, *Relativity, Gravitation and Cosmology: A Basic Introduction* (2nd edition).

Evaluating eq. (1) at the point P and relabeling the indices ($\beta \leftrightarrow \mu$ and $\nu \to \alpha$), yields,

$$\left(\frac{\partial v^\mu}{\partial x^\alpha}\right)_P = -(\Gamma^\mu_{\beta\alpha})_P v^\beta_P\,. \tag{7}$$

Inserting eq. (7) into eq. (3), we obtain,

$$v^\mu(\sigma') \simeq v^\mu_P - (\Gamma^\mu_{\beta\alpha})_P v^\beta_P\big[x^\alpha(\sigma') - x^\alpha_P\big]\,. \tag{8}$$

We now insert eqs. (4) and (8) into eq. (2) and discard terms that are quadratic (and higher order) in $x^\alpha - x^\alpha_P$. The end result is,

$$v^\beta(\sigma) \simeq v^\beta_P - (\Gamma^\beta_{\lambda\alpha})_P\, v^\lambda_P \int_{\sigma_P}^{\sigma} \frac{dx^\nu}{d\sigma'}\,d\sigma' - \left(\frac{\partial\Gamma^\beta_{\lambda\nu}}{\partial x^\alpha} - \Gamma^\beta_{\tau\nu}\Gamma^\tau_{\lambda\alpha}\right)_P v^\lambda_P \int_{\sigma_P}^{\sigma}\big[x^\alpha(\sigma') - x^\alpha_P\big]\frac{dx^\nu}{d\sigma'}\,d\sigma'\,. \tag{9}$$

2. Parallel transport around an infinitesimal closed loop and the Riemann curvature tensor

We now ask the following question. Suppose we parallel transport the vector $v^\beta(\sigma)$ along a path which starts and ends at the same point P. By assumption, the path in question is an infinitesimal closed path so that we can use the approximate formula given in eq. (9) We denote the vector after it traverses the path and returns to P by $v^\beta_{\parallel P}$. Then

$$v^\beta_{\parallel P} - v^\beta_P \simeq -(\Gamma^\beta_{\lambda\alpha})_P\, v^\lambda_P \oint dx^\nu - \left(\frac{\partial\Gamma^\beta_{\lambda\nu}}{\partial x^\alpha} - \Gamma^\beta_{\tau\nu}\Gamma^\tau_{\lambda\alpha}\right)_P v^\lambda_P \oint\big[x^\alpha(\sigma') - x^\alpha_P\big]\frac{dx^\nu}{d\sigma'}\,d\sigma'\,,$$

where we have used the chain rule to write $dx^\nu = (dx^\nu/d\sigma')d\sigma'$. For a closed loop,

$$\oint \frac{dx^\nu}{d\sigma'}\,d\sigma' = \oint dx^\nu = 0\,,$$

since the starting and ending point of the integral coincides. Hence, we are left with

$$v^\beta_{\parallel P} - v^\beta_P \simeq -\left(\frac{\partial\Gamma^\beta_{\lambda\nu}}{\partial x^\alpha} - \Gamma^\beta_{\tau\nu}\Gamma^\tau_{\lambda\alpha}\right)_P v^\lambda_P \oint x^\alpha(\sigma')\,\frac{dx^\nu}{d\sigma'}\,d\sigma'\,, \tag{10}$$

Since ν and α are dummy variables, we are free to relabel the indices such that $\nu \leftrightarrow \alpha$, which yields

$$v^\beta_{\parallel P} - v^\beta_P \simeq -\left(\frac{\partial\Gamma^\beta_{\lambda\alpha}}{\partial x^\nu} - \Gamma^\beta_{\tau\alpha}\Gamma^\tau_{\lambda\nu}\right)_P v^\lambda_P \oint x^\nu(\sigma')\,\frac{dx^\alpha}{d\sigma'}\,d\sigma'\,, \tag{11}$$

We now observe that

$$0 = \oint d(x^\nu x^\alpha) = \oint \left[x^\nu(\sigma')dx^\alpha + x^\alpha(\sigma')dx^\nu\right] = \oint \left(x^\nu(\sigma')\frac{dx^\alpha}{d\sigma'} + x^\alpha(\sigma')\frac{dx^\nu}{d\sigma'}\right) d\sigma' \,.$$

It follows that

$$\oint x^\nu(\sigma')\frac{dx^\alpha}{d\sigma'}\, d\sigma' = -\oint x^\alpha(\sigma')\frac{dx^\nu}{d\sigma'}\, d\sigma' \,.$$

Inserting this result into eq. (10) yields

$$v^\beta_{\parallel\, P} - v^\beta_P \simeq \left(\frac{\partial\Gamma^\beta_{\lambda\nu}}{\partial x^\alpha} - \Gamma^\beta_{\tau\nu}\Gamma^\tau_{\lambda\alpha}\right)_P v^\lambda_P \oint x^\nu(\sigma')\, \frac{dx^\alpha}{d\sigma'}\, d\sigma' \,, \tag{12}$$

If we add eqs. (11) and (12) and then divide by 2, we end up with

$$v^\beta_{\parallel\, P} - v^\beta_P \simeq -\frac{1}{2}\left(\frac{\partial\Gamma^\beta_{\lambda\alpha}}{\partial x^\nu} - \frac{\partial\Gamma^\beta_{\lambda\nu}}{\partial x^\alpha} + \Gamma^\beta_{\tau\nu}\Gamma^\tau_{\lambda\alpha} - \Gamma^\beta_{\tau\alpha}\Gamma^\tau_{\lambda\nu}\right)_P v^\lambda_P \oint x^\nu(\sigma')\, \frac{dx^\alpha}{d\sigma'}\, d\sigma' \,. \tag{13}$$

We recognize the appearance of the Riemann curvature tensor in eq. (13),

$$R^\beta{}_{\lambda\nu\alpha} = \frac{\partial\Gamma^\beta_{\lambda\alpha}}{\partial x^\nu} - \frac{\partial\Gamma^\beta_{\lambda\nu}}{\partial x^\alpha} + \Gamma^\beta_{\tau\nu}\Gamma^\tau_{\lambda\alpha} - \Gamma^\beta_{\tau\alpha}\Gamma^\tau_{\lambda\nu} \,. \tag{14}$$

Hence it follows that

$$\boxed{v^\beta_{\parallel\, P} - v^\beta_P \simeq -\tfrac{1}{2}\big(R^\beta{}_{\lambda\nu\alpha}\big)_P v^\lambda_P \oint x^\nu\, dx^\alpha \,.} \tag{15}$$

We see that in the absence of curvature, $v^\beta_{\parallel\, P} = v^\beta_P$. Thus, the presence of curvature at a point P can be detected by parallel transporting a vector around a closed infinitesimal loop that starts and ends at the point P and showing that the vector changes when it returns to its original position.

Ta-Pei Cheng performs the computation above on pp. 311–312 of our textbook (cf. Box 13.2) for the special case of an infinitesimal parallelogram. One can easily compute

$$\oint_C x^\nu\, dx^\alpha \,, \tag{16}$$

for this closed path C to derive eq. (13.66) on p. 312 of our textbook. However, it is instructive to examine eq. (16) more carefully to understand its geometric interpretation. To accomplish this, I propose to convert eq. (16) into a surface integral using a generalization of the Stokes integral theorem, which is evaluated over the two-dimensional surface bounded by C. We denote the surface by the equation $x^\alpha = x^\alpha(\xi,\eta)$, where ξ and η serve as coordinates defined on the surface. The infinitesimal surface area element can then be expressed in terms of the spacetime coordinates x^α via,

$$d\sigma^{\alpha\beta} = \left(\frac{\partial x^\alpha}{\partial\xi}\frac{\partial x^\beta}{\partial\eta} - \frac{\partial x^\beta}{\partial\xi}\frac{\partial x^\alpha}{\partial\eta}\right) d\xi\, d\eta \,, \tag{17}$$

where the expression inside the parenthesis is the Jacobian determinant that is needed to relate the area element expressed in terms of (ξ, η) and the area element expressed in terms of the x^α. Note that we do not take the absolute value of the Jacobian determinant, since it is convenient to allow $d\sigma^{\alpha\beta}$ to be either positive or negative.[2] The infinitesimal surface area element $d\sigma^{\alpha\beta}$ is an antisymmetric tensor under general coordinate transformations,[3]

$$d\sigma^{\alpha\beta} = -d\sigma^{\beta\alpha}\,. \tag{18}$$

We now invoke the Stokes integral theorem in curved spacetime,

$$\oint_C v_\beta\, dx^\beta = \int_S D_\rho v_\lambda\, d\sigma^{\rho\lambda} = \frac{1}{2}\int_S (D_\rho v_\lambda - D_\lambda v_\rho) d\sigma^{\rho\lambda}\,, \tag{19}$$

where v_β is an arbitrary covariant vector, D is the covariant derivative and S is any finite two-dimensional surface bounded by the closed curve C. In obtaining the final form for eq. (19) we have made use of eq. (18). Note that all terms appearing in eq. (19) transform as a scalar under general coordinate transformations, $x' = x'(x)$. However, as shown in problem 1(b) on Problem Set 3,

$$D_\rho v_\lambda - D_\lambda v_\rho = \partial_\rho v_\lambda - \partial_\lambda v_\rho\,.$$

Hence, eq. (19) can be rewritten as

$$\oint_C v_\beta\, dx^\beta = \frac{1}{2}\int_S (\partial_\rho v_\lambda - \partial_\lambda v_\rho) d\sigma^{\rho\lambda}\,, \tag{20}$$

Let us choose $v_\beta = \delta^\alpha_\beta x^\nu$ in eq. (20). Then,

$$\oint_C x^\nu\, dx^\alpha = \frac{1}{2}\int_S \left[\partial_\rho(\delta^\alpha_\lambda x^\nu) - \partial_\lambda(\delta^\alpha_\rho x^\nu)\right] d\sigma^{\rho\lambda} = \frac{1}{2}\int_S \left(\delta^\alpha_\lambda \delta^\nu_\rho - \delta^\alpha_\rho \delta^\nu_\lambda\right) d\sigma^{\rho\lambda} = \int d\sigma^{\nu\alpha}\,.$$

That is,

$$\int_S d\sigma^{\nu\alpha} = \oint_C x^\nu\, dx^\alpha\,. \tag{21}$$

Inserting eq. (21) into eq. (15) yields

$$\boxed{v^\beta_{\parallel\, P} - v^\beta_P \simeq -\tfrac{1}{2}\left(R^\beta{}_{\lambda\nu\alpha}\right)_P v^\lambda_P \int_S d\sigma^{\nu\alpha}\,.} \tag{22}$$

To make contact with eq. (13.66) on p. 312 of our textbook (which must be corrected by multiplying the right hand side by -1; cf. the errata linked to the class webpage), we should

[2]A negative Jacobian determinant arises if the old coordinate system is right-handed and the new coordinate system is left-handed (or vice versa).

[3]Some books denote $d\sigma^{\alpha\beta} = dx^\alpha \wedge dx^\beta$. However, this requires an introduction to differential forms, which is beyond the scope of this class.

identify the four-vectors a^ν and b^α. If we parameterize the surface S with coordinates (ξ, η), where the coordinate curves are tangent to a^ν and b^α, then we should identify

$$a^\nu = \frac{\partial x^\nu}{\partial \xi}\,, \qquad\qquad b^\alpha = \frac{\partial x^\alpha}{\partial \eta}\,.$$

Eq. (17) then yields

$$d\sigma^{\nu\alpha} = (a^\nu b^\alpha - a^\alpha b^\nu) d\xi\, d\eta\,.$$

Integrating over the parallelogram spanned by a^ν and b^α then yields

$$\int_S d\sigma^{\nu\alpha} = (a^\nu b^\alpha - a^\alpha b^\nu) \int_0^1 d\xi \int_0^1 d\eta = a^\nu b^\alpha - a^\alpha b^\nu\,.$$

Inserting this result back into eq. (22) yields

$$v^\beta_{\parallel\, P} - v^\beta_P \simeq -\tfrac{1}{2}\bigl(R^\beta{}_{\lambda\nu\alpha}\bigr)_P v^\lambda_P (a^\nu b^\alpha - a^\alpha b^\nu)\,.$$

Finally, using the fact that the Riemann curvature tensor is antisymmetric under the interchange of its last two indices, $R^\rho{}_{\lambda\nu\alpha} = -R^\rho{}_{\lambda\alpha\nu}$, it follows after some index relabeling that

$$v^\beta_{\parallel\, P} - v^\beta_P \simeq -\bigl(R^\beta{}_{\lambda\nu\alpha}\bigr)_P v^\lambda_P\, a^\nu b^\alpha\,,$$

which reproduces eq. (13.66) on p. 312 of our textbook once the omitted minus sign is correctly restored.

APPENDIX: A brief look at the Stokes Integral Theorem

Eq. (20) provides the relevant form for the Stokes integral theorem in four-dimensional spacetime. Let's check that we get the expected result in three-dimensional Euclidean space. In this case, we do not have to distinguish between upper and lower indices, so that

$$\oint_C v_i\, dx_i = \frac{1}{2}\int_S (\partial_j v_k - \partial_k v_j) d\sigma_{jk}\,, \tag{23}$$

In three-dimensional Euclidean space,

$$d\sigma_{jk} = \epsilon_{ijk} n_i dS\,,$$

where n_i are the coordinates of the outward normal to the surface S and dS is an infinitesimal surface element. Hence,

$$\int_S (\partial_j v_k - \partial_k v_j) d\sigma_{jk} = \epsilon_{ijk}\int_S n_i(\partial_j v_k - \partial_k v_j) dS = 2\epsilon_{ijk}\int_S n_i \partial_j v_k\, dS = 2\int_S \boldsymbol{\hat{n}}\boldsymbol{\cdot}(\boldsymbol{\vec{\nabla}}\times\boldsymbol{\vec{v}})\, dS\,,$$

where in the penultimate step we relabeled indices and used the fact that $\epsilon_{ijk} = -\epsilon_{ikj}$. Inserting the above result back into eq. (23) then yields the familiar form for the Stokes integral theorem for three-dimensional Euclidean space,

$$\oint_C \boldsymbol{\vec{v}}\boldsymbol{\cdot} \boldsymbol{d\vec{\ell}} = \int_S \boldsymbol{\hat{n}}\boldsymbol{\cdot}(\boldsymbol{\vec{\nabla}}\times\boldsymbol{\vec{v}})\, dS\,.$$