§2.4 CONTINUUM MECHANICS (SOLIDS)

In this introduction to continuum mechanics we consider the basic equations describing the physical effects created by external forces acting upon solids and fluids. In addition to the basic equations that are applicable to all continua, there are equations which are constructed to take into account material characteristics. These equations are called constitutive equations. For example, in the study of solids the constitutive equations for a linear elastic material is a set of relations between stress and strain. In the study of fluids, the constitutive equations consists of a set of relations between stress and rate of strain. Constitutive equations are usually constructed from some basic axioms. The resulting equations have unknown material parameters which can be determined from experimental investigations.

One of the basic axioms, used in the study of elastic solids, is that of material invariance. This axiom requires that certain symmetry conditions of solids are to remain invariant under a set of orthogonal transformations and translations. This axiom is employed in the next section to simplify the constitutive equations for elasticity. We begin our study of continuum mechanics by investigating the development of constitutive equations for linear elastic solids.

Generalized Hooke's Law

If the continuum material is a linear elastic material, we introduce the generalized Hooke's law in Cartesian coordinates

$$\sigma_{ij} = c_{ijkl} e_{kl}, \quad i,j,k,l = 1,2,3. \tag{2.4.1}$$

The Hooke's law is a statement that the stress is proportional to the gradient of the deformation occurring in the material. These equations assume a linear relationship exists between the components of the stress tensor and strain tensor and we say stress is a linear function of strain. Such relations are referred to as a set of constitutive equations. Constitutive equations serve to describe the material properties of the medium when it is subjected to external forces.

Constitutive Equations

The equations (2.4.1) are constitutive equations which are applicable for materials exhibiting small deformations when subjected to external forces. The 81 constants c_{ijkl} are called the elastic stiffness of the material. The above relations can also be expressed in the form

$$e_{ij} = s_{ijkl} \sigma_{kl}, \quad i,j,k,l = 1,2,3 \tag{2.4.2}$$

where s_{ijkl} are constants called the elastic compliance of the material. Since the stress σ_{ij} and strain e_{ij} have been shown to be tensors we can conclude that both the elastic stiffness c_{ijkl} and elastic compliance s_{ijkl} are fourth order tensors. Due to the symmetry of the stress and strain tensors we find that the elastic stiffness and elastic compliance tensor must satisfy the relations

$$\begin{aligned} c_{ijkl} &= c_{jikl} = c_{ijlk} = c_{jilk} \\ s_{ijkl} &= s_{jikl} = s_{ijlk} = s_{jilk} \end{aligned} \tag{2.4.3}$$

and consequently only 36 of the 81 constants are actually independent. If all 36 of the material (crystal) constants are independent the material is called triclinic and there are no material symmetries.

Restrictions on Elastic Constants due to Symmetry

The equations (2.4.1) and (2.4.2) can be replaced by an equivalent set of equations which are easier to analyze. This is accomplished by defining the quantities

$$e_1, \quad e_2, \quad e_3, \quad e_4, \quad e_5, \quad e_6$$

$$\sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \sigma_4, \quad \sigma_5, \quad \sigma_6$$

where

$$\begin{pmatrix} e_1 & e_4 & e_5 \\ e_4 & e_2 & e_6 \\ e_5 & e_6 & e_3 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} \sigma_1 & \sigma_4 & \sigma_5 \\ \sigma_4 & \sigma_2 & \sigma_6 \\ \sigma_5 & \sigma_6 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}.$$

Then the generalized Hooke's law from the equations (2.4.1) and (2.4.2) can be represented in either of the forms

$$\sigma_i = c_{ij} e_j \quad \text{or} \quad e_i = s_{ij} \sigma_j \quad \text{where} \quad i, j = 1, \ldots, 6 \tag{2.4.4}$$

where c_{ij} are constants related to the elastic stiffness and s_{ij} are constants related to the elastic compliance. These constants satisfy the relation

$$s_{mi} c_{ij} = \delta_{mj} \quad \text{where} \quad i, m, j = 1, \ldots, 6 \tag{2.4.5}$$

Here

$$e_{ij} = \begin{cases} e_i, & i = j = 1, 2, 3 \\ e_{1+i+j}, & i \neq j, \text{ and } i = 1, \text{ or, } 2 \end{cases}$$

and similarly

$$\sigma_{ij} = \begin{cases} \sigma_i, & i = j = 1, 2, 3 \\ \sigma_{1+i+j}, & i \neq j, \text{ and } i = 1, \text{ or, } 2. \end{cases}$$

These relations show that the constants c_{ij} are related to the elastic stiffness coefficients c_{pqrs} by the relations

$$c_{m1} = c_{ij11} \qquad c_{m4} = 2c_{ij12}$$

$$c_{m2} = c_{ij22} \qquad c_{m5} = 2c_{ij13}$$

$$c_{m3} = c_{ij33} \qquad c_{m6} = 2c_{ij23}$$

where

$$m = \begin{cases} i, & \text{if } i = j = 1, 2, \text{ or } 3 \\ 1 + i + j, & \text{if } i \neq j \text{ and } i = 1 \text{ or } 2. \end{cases}$$

A similar type relation holds for the constants s_{ij} and s_{pqrs}. The above relations can be verified by expanding the equations (2.4.1) and (2.4.2) and comparing like terms with the expanded form of the equation (2.4.4).

The generalized Hooke's law can now be expressed in a form where the 36 independent constants can be examined in more detail under special material symmetries. We will examine the form

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}. \tag{2.4.6}$$

Alternatively, in the arguments that follow, one can examine the equivalent form

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}.$$

Material Symmetries

A material (crystal) with one plane of symmetry is called an aelotropic material. If we let the x_1-x_2 plane be a plane of symmetry then the equations (2.4.6) must remain invariant under the coordinate transformation

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.4.7}$$

which represents an inversion of the x_3 axis. That is, if the x_1-x_2 plane is a plane of symmetry we should be able to replace x_3 by $-x_3$ and the equations (2.4.6) should remain unchanged. This is equivalent to saying that a transformation of the type from equation (2.4.7) changes the Hooke's law to the form $\overline{e}_i = s_{ij}\overline{\sigma}_j$ where the s_{ij} remain unaltered because it is the same material. Employing the transformation equations

$$\overline{x}_1 = x_1, \qquad \overline{x}_2 = x_2, \qquad \overline{x}_3 = -x_3 \tag{2.4.8}$$

we examine the stress and strain transformation equations

$$\overline{\sigma}_{ij} = \sigma_{pq} \frac{\partial x_p}{\partial \overline{x}_i} \frac{\partial x_q}{\partial \overline{x}_j} \qquad \text{and} \qquad \overline{e}_{ij} = e_{pq} \frac{\partial x_p}{\partial \overline{x}_i} \frac{\partial x_q}{\partial \overline{x}_j}. \tag{2.4.9}$$

If we expand both of the equations (2.4.9) and substitute in the nonzero derivatives

$$\frac{\partial x_1}{\partial \overline{x}_1} = 1, \qquad \frac{\partial x_2}{\partial \overline{x}_2} = 1, \qquad \frac{\partial x_3}{\partial \overline{x}_3} = -1, \tag{2.4.10}$$

we obtain the relations

$$\begin{aligned} \overline{\sigma}_{11} &= \sigma_{11} & \overline{e}_{11} &= e_{11} \\ \overline{\sigma}_{22} &= \sigma_{22} & \overline{e}_{22} &= e_{22} \\ \overline{\sigma}_{33} &= \sigma_{33} & \overline{e}_{33} &= e_{33} \\ \overline{\sigma}_{21} &= \sigma_{21} & \overline{e}_{21} &= e_{21} \\ \overline{\sigma}_{31} &= -\sigma_{31} & \overline{e}_{31} &= -e_{31} \\ \overline{\sigma}_{23} &= -\sigma_{23} & \overline{e}_{23} &= -e_{23}. \end{aligned} \tag{2.4.11}$$

We conclude that if the material undergoes a strain, with the x_1-x_2 plane as a plane of symmetry then e_5 and e_6 change sign upon reversal of the x_3 axis and e_1, e_2, e_3, e_4 remain unchanged. Similarly, we find σ_5 and σ_6 change sign while $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ remain unchanged. The equation (2.4.6) then becomes

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ -e_5 \\ -e_6 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ -\sigma_5 \\ -\sigma_6 \end{pmatrix}. \tag{2.4.12}$$

If the stress strain relation for the new orientation of the x_3 axis is to have the same form as the old orientation, then the equations (2.4.6) and (2.4.12) must give the same results. Comparison of these equations we find that

$$\begin{gathered} s_{15} = s_{16} = 0 \\ s_{25} = s_{26} = 0 \\ s_{35} = s_{36} = 0 \\ s_{45} = s_{46} = 0 \\ s_{51} = s_{52} = s_{53} = s_{54} = 0 \\ s_{61} = s_{62} = s_{63} = s_{64} = 0. \end{gathered} \tag{2.4.13}$$

In summary, from an examination of the equations (2.4.6) and (2.4.12) we find that for an aelotropic material (crystal), with one plane of symmetry, the 36 constants s_{ij} reduce to 20 constants and the generalized Hooke's law (constitutive equation) has the form

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & 0 \\ s_{21} & s_{22} & s_{23} & s_{24} & 0 & 0 \\ s_{31} & s_{32} & s_{33} & s_{34} & 0 & 0 \\ s_{41} & s_{42} & s_{43} & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & s_{56} \\ 0 & 0 & 0 & 0 & s_{65} & s_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}. \tag{2.4.14}$$

Alternatively, the Hooke's law can be represented in the form

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}.$$

Additional Symmetries

If the material (crystal) is such that there is an additional plane of symmetry, say the x_2-x_3 plane, then reversal of the x_1 axis should leave the equations (2.4.14) unaltered. If there are two planes of symmetry then there will automatically be a third plane of symmetry. Such a material (crystal) is called orthotropic. Introducing the additional transformation

$$\overline{x}_1 = -x_1, \qquad \overline{x}_2 = x_2, \qquad \overline{x}_3 = x_3$$

which represents the reversal of the x_1 axes, the expanded form of equations (2.4.9) are used to calculate the effect of such a transformation upon the stress and strain tensor. We find $\sigma_1, \sigma_2, \sigma_3, \sigma_6, e_1, e_2, e_3, e_6$ remain unchanged while $\sigma_4, \sigma_5, e_4, e_5$ change sign. The equation (2.4.14) then becomes

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ -e_4 \\ -e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & 0 \\ s_{21} & s_{22} & s_{23} & s_{24} & 0 & 0 \\ s_{31} & s_{32} & s_{33} & s_{34} & 0 & 0 \\ s_{41} & s_{42} & s_{43} & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & s_{56} \\ 0 & 0 & 0 & 0 & s_{65} & s_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ -\sigma_4 \\ -\sigma_5 \\ \sigma_6 \end{pmatrix}. \tag{2.4.15}$$

Note that if the constitutive equations (2.4.14) and (2.4.15) are to produce the same results upon reversal of the x_1 axes, then we require that the following coefficients be equated to zero:

$$s_{14} = s_{24} = s_{34} = 0$$
$$s_{41} = s_{42} = s_{43} = 0$$
$$s_{56} = s_{65} = 0.$$

This then produces the constitutive equation

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \tag{2.4.16}$$

or its equivalent form

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$$

and the original 36 constants have been reduced to 12 constants. This is the constitutive equation for orthotropic material (crystals).

Axis of Symmetry

If in addition to three planes of symmetry there is an axis of symmetry then the material (crystal) is termed hexagonal. Assume that the x^1 axis is an axis of symmetry and consider the effect of the transformation

$$\overline{x}^1 = x^1, \qquad \overline{x}^2 = x^3 \qquad \overline{x}^3 = -x^2$$

upon the constitutive equations. It is left as an exercise to verify that the constitutive equations reduce to the form where there are 7 independent constants having either of the forms

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{21} & s_{23} & s_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}$$

or

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{21} & c_{23} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}.$$

Finally, if the material is completely symmetric, the x^2 axis is also an axis of symmetry and we can consider the effect of the transformation

$$\overline{x}^1 = -x^3, \qquad \overline{x}^2 = x^2, \qquad \overline{x}^3 = x^1$$

upon the constitutive equations. It can be verified that these transformations reduce the Hooke's law constitutive equation to the form

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{44} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}. \tag{2.4.17}$$

Materials (crystals) with atomic arrangements that exhibit the above symmetries are called isotropic materials. An equivalent form of (2.4.17) is the relation

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}.$$

The figure 2.4-1 lists values for the elastic stiffness associated with some metals which are isotropic[1]

[1]Additional constants are given in "International Tables of Selected Constants", Metals: Thermal and Mechanical Data, Vol. 16, Edited by S. Allard, Pergamon Press, 1969.

Metal	c_{11}	c_{12}	c_{44}
Na	0.074	0.062	0.042
Pb	0.495	0.423	0.149
Cu	1.684	1.214	0.754
Ni	2.508	1.500	1.235
Cr	3.500	0.678	1.008
Mo	4.630	1.610	1.090
W	5.233	2.045	1.607

Figure 2.4-1. Elastic stiffness coefficients for some metals which are cubic.
Constants are given in units of $10^{12}\, dynes/cm^2$

Under these conditions the stress strain constitutive relations can be written as

$$\begin{aligned}
\sigma_1 = \sigma_{11} &= (c_{11} - c_{12})e_{11} + c_{12}(e_{11} + e_{22} + e_{33}) \\
\sigma_2 = \sigma_{22} &= (c_{11} - c_{12})e_{22} + c_{12}(e_{11} + e_{22} + e_{33}) \\
\sigma_3 = \sigma_{33} &= (c_{11} - c_{12})e_{33} + c_{12}(e_{11} + e_{22} + e_{33}) \\
\sigma_4 = \sigma_{12} &= c_{44}e_{12} \\
\sigma_5 = \sigma_{13} &= c_{44}e_{13} \\
\sigma_6 = \sigma_{23} &= c_{44}e_{23}.
\end{aligned} \tag{2.4.18}$$

Isotropic Material

Materials (crystals) which are elastically the same in all directions are called isotropic. We have shown that for a cubic material which exhibits symmetry with respect to all axes and planes, the constitutive stress-strain relation reduces to the form found in equation (2.4.17). Define the quantities

$$s_{11} = \frac{1}{E}, \qquad s_{12} = -\frac{\nu}{E}, \qquad s_{44} = \frac{1}{2\mu}$$

where E is the Young's Modulus of elasticity, ν is the Poisson's ratio, and μ is the shear or rigidity modulus. For isotropic materials the three constants E, ν, μ are not independent as the following example demonstrates.

EXAMPLE 2.4-1. (Elastic constants) For an isotropic material, consider a cross section of material in the x^1-x^2 plane which is subjected to pure shearing so that $\sigma_4 = \sigma_{12}$ is the only nonzero stress as illustrated in the figure 2.4-2.

For the above conditions, the equation (2.4.17) reduces to the single equation

$$e_4 = e_{12} = s_{44}\sigma_4 = s_{44}\sigma_{12} \qquad \text{or} \qquad \mu = \frac{\sigma_{12}}{\gamma_{12}}$$

and so the shear modulus is the ratio of the shear stress to the shear angle. Now rotate the axes through a 45 degree angle to a barred system of coordinates where

$$x^1 = \overline{x}^1 \cos\alpha - \overline{x}^2 \sin\alpha \quad x^2 = \overline{x}^1 \sin\alpha + \overline{x}^2 \cos\alpha$$

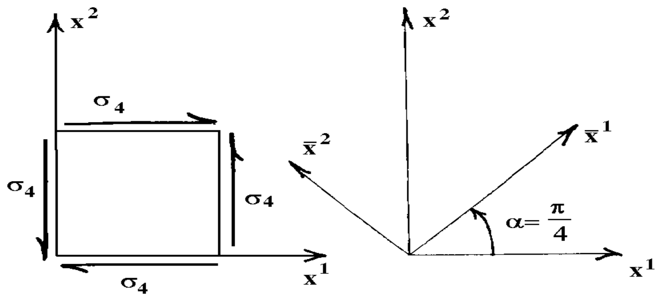

Figure 2.4-2. Element subjected to pure shearing

where $\alpha = \frac{\pi}{4}$. Expanding the transformation equations (2.4.9) we find that

$$\overline{\sigma}_1 = \overline{\sigma}_{11} = \cos\alpha \sin\alpha\, \sigma_{12} + \sin\alpha \cos\alpha\, \sigma_{21} = \sigma_{12} = \sigma_4$$
$$\overline{\sigma}_2 = \overline{\sigma}_{22} = -\sin\alpha \cos\alpha\, \sigma_{12} - \sin\alpha \cos\alpha\, \sigma_{21} = -\sigma_{12} = -\sigma_4,$$

and similarly

$$\overline{e}_1 = \overline{e}_{11} = e_4, \qquad \overline{e}_2 = \overline{e}_{22} = -e_4.$$

In the barred system, the Hooke's law becomes

$$\overline{e}_1 = s_{11}\overline{\sigma}_1 + s_{12}\overline{\sigma}_2 \qquad \text{or}$$
$$e_4 = s_{11}\sigma_4 - s_{12}\sigma_4 = s_{44}\sigma_4.$$

Hence, the constants s_{11}, s_{12}, s_{44} are related by the relation

$$s_{11} - s_{12} = s_{44} \qquad \text{or} \qquad \frac{1}{E} + \frac{\nu}{E} = \frac{1}{2\mu}. \tag{2.4.19}$$

This is an important relation connecting the elastic constants associated with isotropic materials. The above transformation can also be applied to triclinic, aelotropic, orthotropic, and hexagonal materials to find relationships between the elastic constants.

Observe also that some texts postulate the existence of a strain energy function U^* which has the property that $\sigma_{ij} = \frac{\partial U^*}{\partial e_{ij}}$. In this case the strain energy function, in the single index notation, is written $U^* = c_{ij}e_ie_j$ where c_{ij} and consequently s_{ij} are symmetric. In this case the previous discussed symmetries give the following results for the nonzero elastic compliances s_{ij} : 13 nonzero constants instead of 20 for aelotropic material, 9 nonzero constants instead of 12 for orthotropic material, and 6 nonzero constants instead of 7 for hexagonal material. This is because of the additional property that $s_{ij} = s_{ji}$ be symmetric.

■

The previous discussion has shown that for an isotropic material the generalized Hooke's law (constitutive equations) have the form

$$
\begin{aligned}
e_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right] \\
e_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{33} + \sigma_{11})\right] \\
e_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right] \\
e_{21} = e_{12} &= \frac{1+\nu}{E}\sigma_{12} \\
e_{32} = e_{23} &= \frac{1+\nu}{E}\sigma_{23} \\
e_{31} = e_{13} &= \frac{1+\nu}{E}\sigma_{13}
\end{aligned}, \tag{2.4.20}
$$

where equation (2.4.19) holds. These equations can be expressed in the indicial notation and have the form

$$
e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}, \tag{2.4.21}
$$

where $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$ is a stress invariant and δ_{ij} is the Kronecker delta. We can solve for the stress in terms of the strain by performing a contraction on i and j in equation (2.4.21). This gives the dilatation

$$
e_{ii} = \frac{1+\nu}{E}\sigma_{ii} - \frac{3\nu}{E}\sigma_{kk} = \frac{1-2\nu}{E}\sigma_{kk}.
$$

Note that from the result in equation (2.4.21) we are now able to solve for the stress in terms of the strain. We find

$$
\begin{aligned}
e_{ij} &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}e_{kk}\delta_{ij} \\
\frac{E}{1+\nu}e_{ij} &= \sigma_{ij} - \frac{\nu E}{(1+\nu)(1-2\nu)}e_{kk}\delta_{ij} \\
\text{or} \qquad \sigma_{ij} &= \frac{E}{1+\nu}e_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}e_{kk}\delta_{ij}.
\end{aligned} \tag{2.4.22}
$$

The tensor equation (2.4.22) represents the six scalar equations

$$
\begin{aligned}
\sigma_{11} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{11} + \nu(e_{22} + e_{33})\right] & \sigma_{12} &= \frac{E}{1+\nu}e_{12} \\
\sigma_{22} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{22} + \nu(e_{33} + e_{11})\right] & \sigma_{13} &= \frac{E}{1+\nu}e_{13} \\
\sigma_{33} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{33} + \nu(e_{22} + e_{11})\right] & \sigma_{23} &= \frac{E}{1+\nu}e_{23}.
\end{aligned}
$$

Alternative Approach to Constitutive Equations

The constitutive equation defined by Hooke's generalized law for isotropic materials can be approached from another point of view. Consider the generalized Hooke's law

$$\sigma_{ij} = c_{ijkl} e_{kl}, \qquad i,j,k,l = 1,2,3.$$

If we transform to a barred system of coordinates, we will have the new Hooke's law

$$\overline{\sigma}_{ij} = \overline{c}_{ijkl} \overline{e}_{kl}, \qquad i,j,k,l = 1,2,3.$$

For an isotropic material we require that

$$\overline{c}_{ijkl} = c_{ijkl}.$$

Tensors whose components are the same in all coordinate systems are called isotropic tensors. We have previously shown in Exercise 1.3, problem 18, that

$$c_{pqrs} = \lambda \delta_{pq} \delta_{rs} + \mu(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}) + \kappa(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr})$$

is an isotropic tensor when we consider affine type transformations. If we further require the symmetry conditions found in equations (2.4.3) be satisfied, we find that $\kappa = 0$ and consequently the generalized Hooke's law must have the form

$$\begin{aligned} \sigma_{pq} = c_{pqrs} e_{rs} &= [\lambda \delta_{pq}\delta_{rs} + \mu(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})]\, e_{rs} \\ \sigma_{pq} &= \lambda \delta_{pq} e_{rr} + \mu(e_{pq} + e_{qp}) \\ \text{or} \qquad \sigma_{pq} &= 2\mu e_{pq} + \lambda e_{rr} \delta_{pq}, \end{aligned} \tag{2.4.23}$$

where $e_{rr} = e_{11} + e_{22} + e_{33} = \Theta$ is the dilatation. The constants λ and μ are called Lame's constants. Comparing the equation (2.4.22) with equation (2.4.23) we find that the constants λ and μ satisfy the relations

$$\mu = \frac{E}{2(1+\nu)} \qquad\qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{2.4.24}$$

In addition to the constants E, ν, μ, λ, it is sometimes convenient to introduce the constant k, called the bulk modulus of elasticity, (Exercise 2.3, problem 23), defined by

$$k = \frac{E}{3(1-2\nu)}. \tag{2.4.25}$$

The stress-strain constitutive equation (2.4.23) was derived using Cartesian tensors. To generalize the equation (2.4.23) we consider a transformation from a Cartesian coordinate system y^i, $i = 1,2,3$ to a general coordinate system $\overline{x}^i$, $i = 1,2,3$. We employ the relations

$$\overline{g}_{ij} = \frac{\partial y^m}{\partial \overline{x}^i} \frac{\partial y^m}{\partial \overline{x}^j}, \qquad \overline{g}^{ij} = \frac{\partial \overline{x}^i}{\partial y^m} \frac{\partial \overline{x}^j}{\partial y^m}$$

and

$$\overline{\sigma}_{mn} = \sigma_{ij} \frac{\partial y^i}{\partial \overline{x}^m} \frac{\partial y^j}{\partial \overline{x}^n}, \qquad \overline{e}_{mn} = e_{ij} \frac{\partial y^i}{\partial \overline{x}^m} \frac{\partial y^j}{\partial \overline{x}^n}, \qquad \text{or} \qquad e_{rq} = \overline{e}_{ij} \frac{\partial \overline{x}^i}{\partial y^r} \frac{\partial \overline{x}^j}{\partial y^q}$$

and convert equation (2.4.23) to a more generalized form. Multiply equation (2.4.23) by $\frac{\partial y^p}{\partial \overline{x}^m}\frac{\partial y^q}{\partial \overline{x}^n}$ and verify the result

$$\overline{\sigma}_{mn} = \lambda \frac{\partial y^q}{\partial \overline{x}^m}\frac{\partial y^q}{\partial \overline{x}^n} e_{rr} + \mu \left(\overline{e}_{mn} + \overline{e}_{nm}\right),$$

which can be simplified to the form

$$\overline{\sigma}_{mn} = \lambda \overline{g}_{mn} \overline{e}_{ij} \overline{g}^{ij} + \mu \left(\overline{e}_{mn} + \overline{e}_{nm}\right).$$

Dropping the bar notation, we have

$$\sigma_{mn} = \lambda g_{mn} g^{ij} e_{ij} + \mu \left(e_{mn} + e_{nm}\right).$$

The contravariant form of this equation is

$$\sigma^{sr} = \lambda g^{sr} g^{ij} e_{ij} + \mu \left(g^{ms} g^{nr} + g^{ns} g^{mr}\right) e_{mn}.$$

Employing the equations (2.4.24) the above result can also be expressed in the form

$$\sigma^{rs} = \frac{E}{2(1+\nu)}\left(g^{ms}g^{nr} + g^{ns}g^{mr} + \frac{2\nu}{1-2\nu} g^{sr} g^{mn}\right) e_{mn}. \tag{2.4.26}$$

This is a more general form for the stress-strain constitutive equations which is valid in all coordinate systems. Multiplying by g_{sk} and employing the use of associative tensors, one can verify

$$\sigma^i_j = \frac{E}{1+\nu}\left(e^i_j + \frac{\nu}{1-2\nu} e^m_m \delta^i_j\right)$$

$$\text{or} \qquad \sigma^i_j = 2\mu e^i_j + \lambda e^m_m \delta^i_j,$$

are alternate forms for the equation (2.4.26). As an exercise, solve for the strains in terms of the stresses and show that

$$E e^i_j = (1+\nu)\sigma^i_j - \nu \sigma^m_m \delta^i_j.$$

EXAMPLE 2.4-2. (Hooke's law) Let us construct a simple example to test the results we have developed so far. Consider the tension in a cylindrical bar illustrated in the figure 2.4-3.

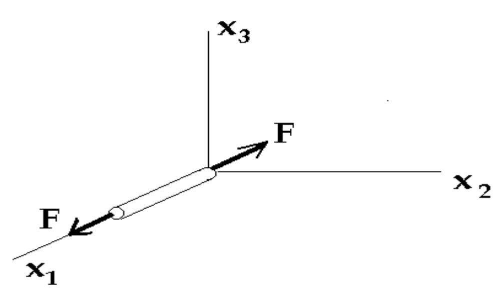

Figure 2.4-3. Stress in a cylindrical bar

Assume that

$$\sigma_{ij} = \begin{pmatrix} \frac{F}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where F is the constant applied force and A is the cross sectional area of the cylinder. Consequently, the generalized Hooke's law (2.4.21) produces the nonzero strains

$$\begin{aligned} e_{11} &= \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{\sigma_{11}}{E} \\ e_{22} &= \frac{-\nu}{E}\sigma_{11} \\ e_{33} &= \frac{-\nu}{E}\sigma_{11} \end{aligned}$$

From these equations we obtain:

The first part of Hooke's law

$$\sigma_{11} = Ee_{11} \text{ or } \frac{F}{A} = Ee_{11}.$$

The second part of Hooke's law

$$\frac{\text{lateral contraction}}{\text{longitudinal extension}} = \frac{-e_{22}}{e_{11}} = \frac{-e_{33}}{e_{11}} = \nu = \text{Poisson's ratio}.$$

This example demonstrates that the generalized Hooke's law for homogeneous and isotropic materials reduces to our previous one dimensional result given in (2.3.1) and (2.3.2).

■

Basic Equations of Elasticity

Assuming the density ϱ is constant, the basic equations of elasticity reduce to the equations representing conservation of linear momentum and angular momentum together with the strain-displacement relations and constitutive equations. In these equations the body forces are assumed known. These basic equations produce 15 equations in 15 unknowns and are a formidable set of equations to solve. Methods for solving these simultaneous equations are: 1) Express the linear momentum equations in terms of the displacements u_i and obtain a system of partial differential equations. Solve the system of partial differential equations for the displacements u_i and then calculate the corresponding strains. The strains can be used to calculate the stresses from the constitutive equations. 2) Solve for the stresses and from the stresses calculate the strains and from the strains calculate the displacements. This converse problem requires some additional considerations which will be addressed shortly.

Basic Equations of Linear Elasticity

- Conservation of linear momentum.

$$\sigma^{ij}_{\ ,i} + \varrho b^j = \varrho f^j \qquad j = 1,2,3. \tag{2.4.27(a)}$$

where σ^{ij} is the stress tensor, b^j is the body force per unit mass and f^j is the acceleration. If there is no motion, then $f^j = 0$ and these equations reduce to the equilibrium equations

$$\sigma^{ij}_{\ ,i} + \varrho b^j = 0 \qquad j = 1,2,3. \tag{2.4.27(b)}$$

- Conservation of angular momentum. $\sigma_{ij} = \sigma_{ji}$
- Strain tensor.

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.4.28}$$

where u_i denotes the displacement field.

- Constitutive equation. For a linear elastic isotropic material we have

$$\sigma^i_j = \frac{E}{1+\nu} e^i_j + \frac{E}{(1+\nu)(1-2\nu)} e^k_k \delta^i_j \qquad i,j = 1,2,3 \tag{2.4.29(a)}$$

or its equivalent form

$$\sigma^i_j = 2\mu e^i_j + \lambda e^r_r \delta^i_j \quad i,j = 1,2,3, \tag{2.4.29(b)}$$

where e^r_r is the dilatation. This produces 15 equations for the 15 unknowns

$$u_1, u_2, u_3, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}, e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33},$$

which represents 3 displacements, 6 strains and 6 stresses. In the above equations it is assumed that the body forces are known.

Navier's Equations

The equations (2.4.27) through (2.4.29) can be combined and written as one set of equations. The resulting equations are known as Navier's equations for the displacements u_i over the range $i = 1,2,3$. To derive the Navier's equations in Cartesian coordinates, we write the equations (2.4.27),(2.4.28) and (2.4.29) in Cartesian coordinates. We then calculate $\sigma_{ij,j}$ in terms of the displacements u_i and substitute the results into the momentum equation (2.4.27(a)). Differentiation of the constitutive equations (2.4.29(b)) produces

$$\sigma_{ij,j} = 2\mu e_{ij,j} + \lambda e_{kk,j} \delta_{ij}. \tag{2.4.30}$$

A contraction of the strain produces the dilatation

$$e_{rr} = \frac{1}{2}\left(u_{r,r} + u_{r,r}\right) = u_{r,r} \tag{2.4.31}$$

From the dilatation we calculate the covariant derivative

$$e_{kk,j} = u_{k,kj}. \tag{2.4.32}$$

Employing the strain relation from equation (2.4.28), we calculate the covariant derivative

$$e_{ij,j} = \frac{1}{2}(u_{i,jj} + u_{j,ij}). \tag{2.4.33}$$

These results allow us to express the covariant derivative of the stress in terms of the displacement field. We find

$$\begin{aligned} \sigma_{ij,j} &= \mu\left[u_{i,jj} + u_{j,ij}\right] + \lambda\delta_{ij}u_{k,kj} \\ \text{or} \qquad \sigma_{ij,j} &= (\lambda+\mu)u_{k,ki} + \mu u_{i,jj}. \end{aligned} \tag{2.4.34}$$

Substituting equation (2.4.34) into the linear momentum equation produces the Navier equations:

$$(\lambda+\mu)u_{k,ki} + \mu u_{i,jj} + \varrho b_i = \varrho f_i, \qquad i = 1,2,3. \tag{2.4.35}$$

In vector form these equations can be expressed

$$(\lambda+\mu)\nabla\left(\nabla\cdot\vec{u}\right) + \mu\nabla^2\vec{u} + \varrho\vec{b} = \varrho\vec{f}, \tag{2.4.36}$$

where $\vec{u}$ is the displacement vector, $\vec{b}$ is the body force per unit mass and $\vec{f}$ is the acceleration. In Cartesian coordinates these equations have the form:

$$(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1\partial x_i} + \frac{\partial^2 u_2}{\partial x_2\partial x_i} + \frac{\partial^2 u_3}{\partial x_3\partial x_i}\right) + \mu\nabla^2 u_i + \varrho b_i = \varrho\frac{\partial^2 u_i}{\partial t^2},$$

for $i = 1,2,3$, where

$$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_1{}^2} + \frac{\partial^2 u_i}{\partial x_2{}^2} + \frac{\partial^2 u_i}{\partial x_3{}^2}.$$

The Navier equations must be satisfied by a set of functions $u_i = u_i(x_1,x_2,x_3)$ which represent the displacement at each point inside some prescribed region R. Knowing the displacement field we can calculate the strain field directly using the equation (2.4.28). Knowledge of the strain field enables us to construct the corresponding stress field from the constitutive equations.

In the absence of body forces, such as gravity, the solution to equation (2.4.36) can be represented in the form $\vec{u} = \vec{u}^{(1)} + \vec{u}^{(2)}$, where $\vec{u}^{(1)}$ satisfies $\operatorname{div}\vec{u}^{(1)} = \nabla\cdot\vec{u}^{(1)} = 0$ and the vector $\vec{u}^{(2)}$ satisfies $\operatorname{curl}\vec{u}^{(2)} = \nabla\times\vec{u}^{(2)} = 0$. The vector field $\vec{u}^{(1)}$ is called a solenoidal field, while the vector field $\vec{u}^{(2)}$ is called an irrotational field. Substituting $\vec{u}$ into the equation (2.4.36) and setting $\vec{b} = 0$, we find in Cartesian coordinates that

$$\varrho\left(\frac{\partial^2\vec{u}^{(1)}}{\partial t^2} + \frac{\partial^2\vec{u}^{(2)}}{\partial t^2}\right) = (\lambda+\mu)\nabla\left(\nabla\cdot\vec{u}^{(2)}\right) + \mu\nabla^2\vec{u}^{(1)} + \mu\nabla^2\vec{u}^{(2)}. \tag{2.4.37}$$

The vector field $\vec{u}^{(1)}$ can be eliminated from equation (2.4.37) by taking the divergence of both sides of the equation. This produces

$$\varrho \frac{\partial^2 \nabla \cdot \vec{u}^{(2)}}{\partial t^2} = (\lambda + \mu)\nabla^2(\nabla \cdot \vec{u}^{(2)}) + \mu \nabla \cdot \nabla^2 \vec{u}^{(2)}.$$

The displacement field is assumed to be continuous and so we can interchange the order of the operators ∇^2 and ∇ and write

$$\nabla \cdot \left(\varrho \frac{\partial^2 \vec{u}^{(2)}}{\partial t^2} - (\lambda + 2\mu)\nabla^2 \vec{u}^{(2)} \right) = 0.$$

This last equation implies that

$$\varrho \frac{\partial^2 \vec{u}^{(2)}}{\partial t^2} = (\lambda + 2\mu)\nabla^2 \vec{u}^{(2)}$$

and consequently, $\vec{u}^{(2)}$ is a vector wave which moves with the speed $\sqrt{(\lambda + 2\mu)/\varrho}$. Similarly, when the vector field $\vec{u}^{(2)}$ is eliminated from the equation (2.4.37), by taking the curl of both sides, we find the vector $\vec{u}^{(1)}$ also satisfies a wave equation having the form

$$\varrho \frac{\partial^2 \vec{u}^{(1)}}{\partial t^2} = \mu \nabla^2 \vec{u}^{(1)}.$$

This later wave moves with the speed $\sqrt{\mu/\varrho}$. The vector $\vec{u}^{(2)}$ is a compressive wave, while the wave $u^{(1)}$ is a shearing wave.

The exercises 30 through 38 enable us to write the Navier's equations in Cartesian, cylindrical or spherical coordinates. In particular, we have for cartesian coordinates

$$\begin{aligned}
(\lambda+\mu)(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}) + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + \varrho b_x =& \varrho \frac{\partial^2 u}{\partial t^2} \\
(\lambda+\mu)(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z}) + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) + \varrho b_y =& \varrho \frac{\partial^2 v}{\partial t^2} \\
(\lambda+\mu)(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2}) + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) + \varrho b_z =& \varrho \frac{\partial^2 w}{\partial t^2}
\end{aligned}$$

and in cylindrical coordinates

$$\begin{aligned}
(\lambda+\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + \\
\mu(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}) + \varrho b_r =& \varrho \frac{\partial^2 u_r}{\partial t^2} \\
(\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + \\
\mu(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}) + \varrho b_\theta =& \varrho \frac{\partial^2 u_\theta}{\partial t^2} \\
(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + \\
\mu(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}) + \varrho b_z =& \varrho \frac{\partial^2 u_z}{\partial t^2}
\end{aligned}$$

and in spherical coordinates

$$
\begin{aligned}
(\lambda+\mu)\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2u_\rho)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)+\frac{1}{\rho\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)&+\\
\mu(\nabla^2u_\rho-\frac{2}{\rho^2}u_\rho-\frac{2}{\rho^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2u_\theta\cot\theta}{\rho^2}-\frac{2}{\rho^2\sin\theta}\frac{\partial u_\phi}{\partial\phi})+\varrho b_\rho&=\varrho\frac{\partial^2u_\rho}{\partial t^2}\\
(\lambda+\mu)\frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2u_\rho)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)+\frac{1}{\rho\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)&+\\
\mu(\nabla^2u_\theta+\frac{2}{\rho^2}\frac{\partial u_\rho}{\partial\theta}-\frac{u_\theta}{\rho^2\sin^2\theta}-\frac{2}{\rho^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\phi}{\partial\phi})+\varrho b_\theta&=\varrho\frac{\partial^2u_\theta}{\partial t^2}\\
(\lambda+\mu)\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2u_\rho)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)+\frac{1}{\rho\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)&+\\
\mu(\nabla^2u_\phi-\frac{1}{\rho^2\sin^2\theta}u_\phi+\frac{2}{\rho^2\sin\theta}\frac{\partial u_\rho}{\partial\phi}+\frac{2\cos\theta}{\rho^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi})+\varrho b_\phi&=\varrho\frac{\partial^2u_\phi}{\partial t^2}
\end{aligned}
$$

where ∇^2 is determined from either equation (2.1.12) or (2.1.13).

Boundary Conditions

In elasticity the body forces per unit mass ($b_i, i = 1,2,3$) are assumed known. In addition one of the following type of boundary conditions is usually prescribed:

- The displacements u_i, $\quad i = 1,2,3$ are prescribed on the boundary of the region R over which a solution is desired.
- The stresses (surface tractions) are prescribed on the boundary of the region R over which a solution is desired.
- The displacements $u_i, i = 1,2,3$ are given over one portion of the boundary and stresses (surface tractions) are specified over the remaining portion of the boundary. This type of boundary condition is known as a mixed boundary condition.

General Solution of Navier's Equations

There has been derived a general solution to the Navier's equations. It is known as the Papkovich-Neuber solution. In the case of a solid in equilibrium one must solve the equilibrium equations

$$
\begin{aligned}
(\lambda+\mu)\nabla\left(\nabla\cdot\vec{u}\right)+\mu\nabla^2\vec{u}+\varrho\vec{b}=0 \quad \text{or}\\
\nabla^2\vec{u}+\frac{1}{1-2\nu}\nabla(\nabla\cdot\vec{u})+\frac{\varrho}{\mu}\vec{b}=0 \quad (\nu\neq\frac{1}{2})
\end{aligned}
\tag{2.4.38}
$$

THEOREM A general elastostatic solution of the equation (2.4.38) in terms of harmonic potentials $\phi,\vec{\psi}$ is

$$\vec{u} = \text{grad}\,(\phi + \vec{r}\cdot\vec{\psi}) - 4(1-\nu)\vec{\psi} \tag{2.4.39}$$

where ϕ and $\vec{\psi}$ are continuous solutions of the equations

$$\nabla^2\phi = \frac{-\varrho\vec{r}\cdot\vec{b}}{4\mu(1-\nu)} \qquad \text{and} \qquad \nabla^2\vec{\psi} = \frac{\varrho\vec{b}}{4\mu(1-\nu)} \tag{2.4.40}$$

with $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ a position vector to a general point (x,y,z) within the continuum.

Proof: First we write equation (2.4.38) in the tensor form

$$u_{i,kk} + \frac{1}{1-2\nu}(u_{j,j})_{,i} + \frac{\varrho}{\mu}b_i = 0 \tag{2.4.41}$$

Now our problem is to show that equation (2.4.39), in tensor form,

$$u_i = \phi_{,i} + (x_j\psi_j)_{,i} - 4(1-\nu)\psi_i \tag{2.4.42}$$

is a solution of equation (2.4.41). Toward this purpose, we differentiate equation (2.4.42)

$$u_{i,k} = \phi_{,ik} + (x_j\psi_j)_{,ik} - 4(1-\nu)\psi_{i,k} \tag{2.4.43}$$

and then contract on i and k giving

$$u_{i,i} = \phi_{,ii} + (x_j\psi_j)_{,ii} - 4(1-\nu)\psi_{i,i}. \tag{2.4.44}$$

Employing the identity $(x_j\psi_j)_{,ii} = 2\psi_{i,i} + x_i\psi_{i,kk}$ the equation (2.4.44) becomes

$$u_{i,i} = \phi_{,ii} + 2\psi_{i,i} + x_i\psi_{i,kk} - 4(1-\nu)\psi_{i,i}. \tag{2.4.45}$$

By differentiating equation (2.4.43) we establish that

$$\begin{aligned} u_{i,kk} &= \phi_{,ikk} + (x_j\psi_j)_{,ikk} - 4(1-\nu)\psi_{i,kk} \\ &= (\phi_{,kk})_{,i} + ((x_j\psi_j)_{,kk})_{,i} - 4(1-\nu)\psi_{i,kk} \\ &= [\phi_{,kk} + 2\psi_{j,j} + x_j\psi_{j,kk}]_{,i} - 4(1-\nu)\psi_{i,kk}. \end{aligned} \tag{2.4.46}$$

We use the hypothesis

$$\phi_{,kk} = \frac{-\varrho x_j F_j}{4\mu(1-\nu)} \qquad \text{and} \qquad \psi_{j,kk} = \frac{\varrho F_j}{4\mu(1-\nu)},$$

and simplify the equation (2.4.46) to the form

$$u_{i,kk} = 2\psi_{j,ji} - 4(1-\nu)\psi_{i,kk}. \tag{2.4.47}$$

Also by differentiating (2.4.45) one can establish that

$$\begin{aligned} u_{j,ji} &= (\phi_{,jj})_{,i} + 2\psi_{j,ji} + (x_j\psi_{j,kk})_{,i} - 4(1-\nu)\psi_{j,ji} \\ &= \left(\frac{-\varrho x_j F_j}{4\mu(1-\nu)}\right)_{,i} + 2\psi_{j,ji} + \left(\frac{\varrho x_j F_j}{4\mu(1-\nu)}\right)_{,i} - 4(1-\nu)\psi_{j,ji} \\ &= -2(1-2\nu)\psi_{j,ji}. \end{aligned} \tag{2.4.48}$$

Finally, from the equations (2.4.47) and (2.4.48) we obtain the desired result that

$$u_{i,kk} + \frac{1}{1-2\nu} u_{j,ji} + \frac{\varrho F_i}{\mu} = 0.$$

Consequently, the equation (2.4.39) is a solution of equation (2.4.38).

As a special case of the above theorem, note that when the body forces are zero, the equations (2.4.40) become

$$\nabla^2 \phi = 0 \qquad \text{and} \qquad \nabla^2 \vec{\psi} = \vec{0}.$$

In this case, we find that equation (2.4.39) is a solution of equation (2.4.38) provided ϕ and each component of $\vec{\psi}$ are harmonic functions. The Papkovich-Neuber potentials are used together with complex variable theory to solve various two-dimensional elastostatic problems of elasticity. Note also that the Papkovich-Neuber potentials are not unique as different combinations of ϕ and $\vec{\psi}$ can produce the same value for $\vec{u}$.

Compatibility Equations

If we know or can derive the displacement field $u_i, i = 1, 2, 3$ we can then calculate the components of the strain tensor

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.4.49}$$

Knowing the strain components, the stress is found using the constitutive relations.

Consider the converse problem where the strain tensor is given or implied due to the assigned stress field and we are asked to determine the displacement field $u_i, i = 1, 2, 3$. Is this a realistic request? Is it even possible to solve for three displacements given six strain components? It turns out that certain mathematical restrictions must be placed upon the strain components in order that the inverse problem have a solution. These mathematical restrictions are known as compatibility equations. That is, we cannot arbitrarily assign six strain components e_{ij} and expect to find a displacement field $u_i, i = 1, 2, 3$ with three components which satisfies the strain relation as given in equation (2.4.49).

EXAMPLE 2.4-3. Suppose we are given the two partial differential equations,

$$\frac{\partial u}{\partial x} = x + y \quad \text{and} \quad \frac{\partial u}{\partial y} = x^3.$$

Can we solve for $u = u(x, y)$? The answer to this question is "no", because the given equations are inconsistent. The inconsistency is illustrated if we calculate the mixed second derivatives from each equation. We find from the first equation that $\frac{\partial^2 u}{\partial x \partial y} = 1$ and from the second equation we calculate $\frac{\partial^2 u}{\partial y \partial x} = 3x^2$. These mixed second partial derivatives are unequal for all x different from $\sqrt{3}/3$. In general, if we have two first order partial differential equations $\frac{\partial u}{\partial x} = f(x, y)$ and $\frac{\partial u}{\partial y} = g(x, y)$, then for consistency (integrability of the equations) we require that the mixed partial derivatives

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial f}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial g}{\partial x}$$

be equal to one another for all x and y values over the domain for which the solution is desired. This is an example of a compatibility equation.

A similar situation occurs in two dimensions for a material in a state of strain where $e_{zz} = e_{zx} = e_{zy} = 0$, called plane strain. In this case, are we allowed to arbitrarily assign values to the strains e_{xx}, e_{yy} and e_{xy} and from these strains determine the displacement field $u = u(x, y)$ and $v = v(x, y)$ in the $x-$ and $y-$directions? Let us try to answer this question. Assume a state of plane strain where $e_{zz} = e_{zx} = e_{zy} = 0$. Further, let us assign 3 arbitrary functional values f, g, h such that

$$e_{xx} = \frac{\partial u}{\partial x} = f(x, y), \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = g(x, y), \quad e_{yy} = \frac{\partial v}{\partial y} = h(x, y).$$

We must now decide whether these equations are consistent. That is, will we be able to solve for the displacement field $u = u(x, y)$ and $v = v(x, y)$? To answer this question, let us derive a compatibility equation (integrability condition). From the given equations we can calculate the following partial derivatives

$$\begin{aligned}
\frac{\partial^2 e_{xx}}{\partial y^2} &= \frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial^2 f}{\partial y^2} \\
\frac{\partial^2 e_{yy}}{\partial x^2} &= \frac{\partial^3 v}{\partial y \partial x^2} = \frac{\partial^2 h}{\partial x^2} \\
2\frac{\partial^2 e_{xy}}{\partial x \partial y} &= \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial y \partial x^2} = 2\frac{\partial^2 g}{\partial x \partial y}.
\end{aligned}$$

This last equation gives us the compatibility equation

$$2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2}$$

or the functions g, f, h must satisfy the relation

$$2\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 h}{\partial x^2}.$$

■

Cartesian Derivation of Compatibility Equations

If the displacement field $u_i, i = 1, 2, 3$ is known we can derive the strain and rotation tensors

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{and} \qquad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}). \tag{2.4.50}$$

Now work backwards. Assume the strain and rotation tensors are given and ask the question, "Is it possible to solve for the displacement field $u_i, i = 1, 2, 3$?" If we view the equation (2.4.50) as a system of equations with unknowns e_{ij}, ω_{ij} and u_i and if by some means we can eliminate the unknowns ω_{ij} and u_i then we will be left with equations which must be satisfied by the strains e_{ij}. These equations are known as the compatibility equations and they represent conditions which the strain components must satisfy in order that a displacement function exist and the equations (2.4.37) are satisfied. Let us see if we can operate upon the equations (2.4.50) to eliminate the quantities u_i and ω_{ij} and hence derive the compatibility equations.

Addition of the equations (2.4.50) produces

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} = e_{ij} + \omega_{ij}. \tag{2.4.51}$$

Differentiate this expression with respect to x_k and verify the result

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial e_{ij}}{\partial x_k} + \frac{\partial \omega_{ij}}{\partial x_k}. \tag{2.4.52}$$

We further assume that the displacement field is continuous so that the mixed partial derivatives are equal and

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial^2 u_i}{\partial x_k \partial x_j}. \tag{2.4.53}$$

Interchanging j and k in equation (2.4.52) gives us

$$\frac{\partial^2 u_i}{\partial x_k \partial x_j} = \frac{\partial e_{ik}}{\partial x_j} + \frac{\partial \omega_{ik}}{\partial x_j}. \tag{2.4.54}$$

Equating the second derivatives from equations (2.4.54) and (2.4.52) and rearranging terms produces the result

$$\frac{\partial e_{ij}}{\partial x_k} - \frac{\partial e_{ik}}{\partial x_j} = \frac{\partial \omega_{ik}}{\partial x_j} - \frac{\partial \omega_{ij}}{\partial x_k} \tag{2.4.55}$$

Making the observation that ω_{ij} satisfies $\frac{\partial \omega_{ik}}{\partial x_j} - \frac{\partial \omega_{ij}}{\partial x_k} = \frac{\partial \omega_{jk}}{\partial x_i}$, the equation (2.4.55) simplifies to the form

$$\frac{\partial e_{ij}}{\partial x_k} - \frac{\partial e_{ik}}{\partial x_j} = \frac{\partial \omega_{jk}}{\partial x_i}. \tag{2.4.56}$$

The term involving ω_{jk} can be eliminated by using the mixed partial derivative relation

$$\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_m} = \frac{\partial^2 \omega_{jk}}{\partial x_m \partial x_i}. \tag{2.4.57}$$

To derive the compatibility equations we differentiate equation (2.4.56) with respect to x_m and then interchanging the indices i and m and substitute the results into equation (2.4.57). This will produce the compatibility equations

$$\frac{\partial^2 e_{ij}}{\partial x_m \partial x_k} + \frac{\partial^2 e_{mk}}{\partial x_i \partial x_j} - \frac{\partial^2 e_{ik}}{\partial x_m \partial x_j} - \frac{\partial^2 e_{mj}}{\partial x_i \partial x_k} = 0. \tag{2.4.58}$$

This is a set of 81 partial differential equations which must be satisfied by the strain components. Fortunately, due to symmetry considerations only 6 of these 81 equations are distinct. These 6 distinct equations are known as the St. Venant's compatibility equations and can be written as

$$\begin{aligned}
\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} &= \frac{\partial^2 e_{12}}{\partial x_1 \partial x_3} - \frac{\partial^2 e_{23}}{\partial {x_1}^2} + \frac{\partial^2 e_{31}}{\partial x_1 \partial x_2} \\
\frac{\partial^2 e_{22}}{\partial x_1 \partial x_3} &= \frac{\partial^2 e_{23}}{\partial x_2 \partial x_1} - \frac{\partial^2 e_{31}}{\partial {x_2}^2} + \frac{\partial^2 e_{12}}{\partial x_2 \partial x_3} \\
\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} &= \frac{\partial^2 e_{31}}{\partial x_3 \partial x_2} - \frac{\partial^2 e_{12}}{\partial {x_3}^2} + \frac{\partial^2 e_{23}}{\partial x_3 \partial x_1} \\
2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} &= \frac{\partial^2 e_{11}}{\partial {x_2}^2} + \frac{\partial^2 e_{22}}{\partial {x_1}^2} \\
2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3} &= \frac{\partial^2 e_{22}}{\partial {x_3}^2} + \frac{\partial^2 e_{33}}{\partial {x_2}^2} \\
2\frac{\partial^2 e_{31}}{\partial x_3 \partial x_1} &= \frac{\partial^2 e_{33}}{\partial {x_1}^2} + \frac{\partial^2 e_{11}}{\partial {x_3}^2}.
\end{aligned} \tag{2.4.59}$$

Observe that the fourth compatibility equation is the same as that derived in the example 2.4-3.

These compatibility equations can also be expressed in the indicial form

$$e_{ij,km} + e_{mk,ji} - e_{ik,jm} - e_{mj,ki} = 0. \tag{2.4.60}$$

Compatibility Equations in Terms of Stress

In the generalized Hooke's law, equation (2.4.29), we can solve for the strain in terms of stress. This in turn will give rise to a representation of the compatibility equations in terms of stress. The resulting equations are known as the Beltrami-Michell equations. Utilizing the strain-stress relation

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

we substitute for the strain in the equations (2.4.60) and rearrange terms to produce the result

$$\begin{aligned}\sigma_{ij,km} + \sigma_{mk,ji} - \sigma_{ik,jm} - \sigma_{mj,ki} = \\ \frac{\nu}{1+\nu}\left[\delta_{ij}\sigma_{nn,km} + \delta_{mk}\sigma_{nn,ji} - \delta_{ik}\sigma_{nn,jm} - \delta_{mj}\sigma_{nn,ki}\right].\end{aligned} \tag{2.4.61}$$

Now only 6 of these 81 equations are linearly independent. It can be shown that the 6 linearly independent equations are equivalent to the equations obtained by setting $k = m$ and summing over the repeated indices. We then obtain the equations

$$\sigma_{ij,mm} + \sigma_{mm,ij} - \left(\sigma_{im,m}\right)_{,j} - \left(\sigma_{mj,m}\right)_{,i} = \frac{\nu}{1+\nu}\left[\delta_{ij}\sigma_{nn,mm} + \sigma_{nn,ij}\right].$$

Employing the equilibrium equation $\sigma_{ij,i} + \varrho b_j = 0$ the above result can be written in the form

$$\sigma_{ij,mm} + \frac{1}{1+\nu}\sigma_{kk,ij} - \frac{\nu}{1+\nu}\delta_{ij}\sigma_{nn,mm} = -(\varrho b_i)_{,j} - (\varrho b_j)_{,i}$$

or

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} - \frac{\nu}{1+\nu}\delta_{ij}\sigma_{nn,mm} = -(\varrho b_i)_{,j} - (\varrho b_j)_{,i}.$$

This result can be further simplified by observing that a contraction on the indices k and i in equation (2.4.61) followed by a contraction on the indices m and j produces the result

$$\sigma_{ij,ij} = \frac{1-\nu}{1+\nu}\sigma_{nn,jj}.$$

Consequently, the Beltrami-Michell equations can be written in the form

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{pp,ij} = -\frac{\nu}{1-\nu}\delta_{ij}\left(\varrho b_k\right)_{,k} - \left(\varrho b_i\right)_{,j} - \left(\varrho b_j\right)_{,i}. \tag{2.4.62}$$

Their derivation is left as an exercise. The Beltrami-Michell equations together with the linear momentum (equilibrium) equations $\sigma_{ij,i} + \varrho b_j = 0$ represent 9 equations in six unknown stresses. This combinations of equations is difficult to handle. An easier combination of equations in terms of stress functions will be developed shortly.

The Navier equations with boundary conditions are difficult to solve in general. Let us take the momentum equations (2.4.27(a)), the strain relations (2.4.28) and constitutive equations (Hooke's law) (2.4.29) and make simplifying assumptions so that a more tractable systems results.

Plane Strain

The plane strain assumption usually is applied in situations where there is a cylindrical shaped body whose axis is parallel to the z axis and loads are applied along the $z-$direction. In any x-y plane we assume that the surface tractions and body forces are independent of z. We set all strains with a subscript z equal to zero. Further, all solutions for the stresses, strains and displacements are assumed to be only functions of x and y and independent of z. Note that in plane strain the stress σ_{zz} is different from zero.

In Cartesian coordinates the strain tensor is expressible in terms of its physical components which can be represented in the matrix form

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix}.$$

If we assume that all strains which contain a subscript z are zero and the remaining strain components are functions of only x and y, we obtain a state of plane strain. For a state of plane strain, the stress components are obtained from the constitutive equations. The condition of plane strain reduces the constitutive equations to the form:

$$\begin{aligned}
e_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\
e_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] \\
0 &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \\
e_{xy} = e_{yx} &= \frac{1+\nu}{E}\sigma_{xy} \\
e_{zy} = e_{yz} &= \frac{1+\nu}{E}\sigma_{yz} = 0 \\
e_{zx} = e_{xz} &= \frac{1+\nu}{E}\sigma_{xz} = 0
\end{aligned}
\qquad
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{xx} + \nu e_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{yy} + \nu e_{xx}\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu(e_{yy} + e_{xx})\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}e_{xy} \\
\sigma_{xz} &= 0 \\
\sigma_{yz} &= 0
\end{aligned}
\tag{2.4.63}$$

where $\sigma_{xx},\quad \sigma_{yy},\quad \sigma_{zz},\quad \sigma_{xy},\quad \sigma_{xz},\quad \sigma_{yz}$ are the physical components of the stress. The above constitutive equations imply that for a state of plane strain we will have

$$\begin{aligned}
\sigma_{zz} &= \nu(\sigma_{xx} + \sigma_{yy}) \\
e_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \\
e_{yy} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] \\
e_{xy} &= \frac{1+\nu}{E}\sigma_{xy}.
\end{aligned}$$

Also under these conditions the compatibility equations reduce to

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}.$$

Plane Stress

An assumption of plane stress is usually applied to thin flat plates. The plate thinness is assumed to be in the $z-$direction and loads are applied perpendicular to z. Under these conditions all stress components with a subscript z are assumed to be zero. The remaining stress components are then treated as functions of x and y.

In Cartesian coordinates the stress tensor is expressible in terms of its physical components and can be represented by the matrix

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}.$$

If we assume that all the stresses with a subscript z are zero and the remaining stresses are only functions of x and y we obtain a state of plane stress. The constitutive equations simplify if we assume a state of plane stress. These simplified equations are

$$\begin{aligned}
e_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} & \sigma_{xx} &= \frac{E}{1-\nu^2}\left[e_{xx} + \nu e_{yy}\right] \\
e_{yy} &= \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx} & \sigma_{yy} &= \frac{E}{1-\nu^2}\left[e_{yy} + \nu e_{xx}\right] \\
e_{zz} &= -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) & \sigma_{zz} &= 0 = (1-\nu)e_{zz} + \nu(e_{xx} + e_{yy}) \\
e_{xy} &= \frac{1+\nu}{E}\sigma_{xy} & \sigma_{xy} &= \frac{E}{1+\nu}e_{xy} \\
e_{xz} &= 0 & \sigma_{yz} &= 0 \\
e_{yz} &= 0. & \sigma_{xz} &= 0
\end{aligned} \tag{2.4.64}$$

For a state of plane stress the compatibility equations reduce to

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \tag{2.4.65}$$

and the three additional equations

$$\frac{\partial^2 e_{zz}}{\partial x^2} = 0, \qquad \frac{\partial^2 e_{zz}}{\partial y^2} = 0, \qquad \frac{\partial^2 e_{zz}}{\partial x \partial y} = 0.$$

These three additional equations complicate the plane stress problem.

Airy Stress Function

In Cartesian coordinates we examine the equilibrium equations (2.4.25(b)) under the conditions of plane strain. In terms of physical components we find that these equations reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \varrho b_x = 0, \qquad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \varrho b_y = 0, \qquad \frac{\partial \sigma_{zz}}{\partial z} = 0.$$

The last equation is satisfied since σ_{zz} is a function of x and y. If we further assume that the body forces are conservative and derivable from a potential function V by the operation $\varrho \vec{b} = -\text{grad}\, V$ or $\varrho b_i = -V_{,i}$ we can express the above equilibrium equations in the form:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial V}{\partial x} &= 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial V}{\partial y} &= 0
\end{aligned} \tag{2.4.66}$$

We will consider these equations together with the compatibility equations (2.4.65). The equations (2.4.66) will be automatically satisfied if we introduce a scalar function $\phi = \phi(x, y)$ and assume that the stresses are derivable from this function and the potential function V according to the rules:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V. \tag{2.4.67}$$

The function $\phi = \phi(x, y)$ is called the Airy stress function after the English astronomer and mathematician Sir George Airy (1801–1892). Since the equations (2.4.67) satisfy the equilibrium equations we need only consider the compatibility equation(s).

For a state of plane strain we substitute the relations (2.4.63) into the compatibility equation (2.4.65) and write the compatibility equation in terms of stresses. We then substitute the relations (2.4.67) and express the compatibility equation in terms of the Airy stress function ϕ. These substitutions are left as exercises. After all these substitutions the compatibility equation, for a state of plane strain, reduces to the form

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} + \frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) = 0. \tag{2.4.68}$$

In the special case where there are no body forces we have $V = 0$ and equation (2.4.68) is further simplified to the biharmonic equation.

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0. \tag{2.4.69}$$

In polar coordinates the biharmonic equation is written

$$\nabla^4 \phi = \nabla^2(\nabla^2 \phi) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0.$$

For conditions of plane stress, we can again introduce an Airy stress function using the equations (2.4.67). However, an exact solution of the plane stress problem which satisfies all the compatibility equations is difficult to obtain. By removing the assumptions that $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are independent of z, and neglecting body forces, it can be shown that for symmetrically distributed external loads the stress function ϕ can be represented in the form

$$\phi = \psi - \frac{\nu z^2}{2(1+\nu)}\nabla^2 \psi \tag{2.4.70}$$

where ψ is a solution of the biharmonic equation $\nabla^4 \psi = 0$. Observe that if z is very small, (the condition of a thin plate), then equation (2.4.70) gives the approximation $\phi \approx \psi$. Under these conditions, we obtain the approximate solution by using only the compatibility equation (2.4.65) together with the stress function defined by equations (2.4.67) with $V = 0$. Note that the solution we obtain from equation (2.4.69) does not satisfy all the compatibility equations, however, it does give an excellent first approximation to the solution in the case where the plate is very thin.

In general, for plane strain or plane stress problems, the equation (2.4.68) or (2.4.69) must be solved for the Airy stress function ϕ which is defined over some region R. In addition to specifying a region of the x, y plane, there are certain boundary conditions which must be satisfied. The boundary conditions specified for the stress will translate through the equations (2.4.67) to boundary conditions being specified for ϕ. In the special case where there are no body forces, both the problems for plane stress and plane strain are governed by the biharmonic differential equation with appropriate boundary conditions.

EXAMPLE 2.4-4 Assume there exist a state of plane strain with zero body forces. For F_{11}, F_{12}, F_{22} constants, consider the function defined by

$$\phi = \phi(x,y) = \frac{1}{2}\left(F_{22}\,x^2 - 2F_{12}\,xy + F_{11}\,y^2\right).$$

This function is an Airy stress function because it satisfies the biharmonic equation $\nabla^4\phi = 0$. The resulting stress field is

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = F_{11} \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = F_{22} \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = F_{12}.$$

This example, corresponds to stresses on an infinite flat plate and illustrates a situation where all the stress components are constants for all values of x and y. In this case, we have $\sigma_{zz} = \nu(F_{11}+F_{22})$. The corresponding strain field is obtained from the constitutive equations. We find these strains are

$$e_{xx} = \frac{1+\nu}{E}\left[(1-\nu)F_{11} - \nu F_{22}\right] \qquad e_{yy} = \frac{1+\nu}{E}\left[(1-\nu)F_{22} - \nu F_{11}\right] \qquad e_{xy} = \frac{1+\nu}{E}F_{12}.$$

The displacement field is found to be

$$u = u(x,y) = \frac{1+\nu}{E}\left[(1-\nu)F_{11} - \nu F_{22}\right]x + \left(\frac{1+\nu}{E}\right)F_{12}y + c_1 y + c_2$$
$$v = v(x,y) = \frac{1+\nu}{E}\left[(1-\nu)F_{22} - \nu F_{11}\right]y + \left(\frac{1+\nu}{E}\right)F_{12}x - c_1 x + c_3,$$

with c_1, c_2, c_3 constants, and is obtained by integrating the strain displacement equations given in Exercise 2.3, problem 2.

EXAMPLE 2.4-5. A special case from the previous example is obtained by setting $F_{22} = F_{12} = 0$. This is the situation of an infinite plate with only tension in the $x-$direction. In this special case we have $\phi = \frac{1}{2}F_{11}y^2$. Changing to polar coordinates we write

$$\phi = \phi(r,\theta) = \frac{F_{11}}{2}r^2\sin^2\theta = \frac{F_{11}}{4}r^2(1-\cos 2\theta).$$

The Exercise 2.4, problem 20, suggests we utilize the Airy equations in polar coordinates and calculate the stresses

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = F_{11}\cos^2\theta = \frac{F_{11}}{2}(1+\cos 2\theta)$$
$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = F_{11}\sin^2\theta = \frac{F_{11}}{2}(1-\cos 2\theta)$$
$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{F_{11}}{2}\sin 2\theta.$$

EXAMPLE 2.4-6. We now consider an infinite plate with a circular hole $x^2+y^2=a^2$ which is traction free. Assume the plate has boundary conditions at infinity defined by $\sigma_{xx}=F_{11}, \quad \sigma_{yy}=0, \quad \sigma_{xy}=0$. Find the stress field.

Solution:

The traction boundary condition at $r=a$ is $t_i=\sigma_{mi}n_m$ or

$$t_1=\sigma_{11}n_1+\sigma_{12}n_2 \qquad \text{and} \qquad t_2=\sigma_{12}n_1+\sigma_{22}n_2.$$

For polar coordinates we have $n_1=n_r=1$, $n_2=n_\theta=0$ and so the traction free boundary conditions at the surface of the hole are written $\sigma_{rr}|_{r=a}=0 \quad \text{and} \quad \sigma_{r\theta}|_{r=a}=0$. The results from the previous example are used as the boundary conditions at infinity.

Our problem is now to solve for the Airy stress function $\phi=\phi(r,\theta)$ which is a solution of the biharmonic equation. The previous example 2.4-5 and the form of the boundary conditions at infinity suggests that we assume a solution to the biharmonic equation which has the form $\phi=\phi(r,\theta)=f_1(r)+f_2(r)\cos 2\theta$, where f_1, f_2 are unknown functions to be determined. Substituting the assumed solution into the biharmonic equation produces the equation

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(f_1''+\frac{1}{r}f_1'\right)+\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(f_2''+\frac{1}{r}f_2'-4\frac{f_2}{r^2}\right)\cos 2\theta=0.$$

We therefore require that f_1, f_2 be chosen to satisfy the equations

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(f_1''+\frac{1}{r}f_1'\right)=0 \qquad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(f_2''+\frac{1}{r}f_2'-4\frac{f_2}{r^2}\right)=0$$

$$\text{or} \qquad r^4f_1^{(iv)}+2r^3f_1'''-r^2f_1''+rf_1'=0 \qquad r^4f_2^{(iv)}+2r^3f_2'''-9r^2f_2''+9rf_2'=0$$

These equations are Cauchy type equations. Their solutions are obtained by assuming a solution of the form $f_1=r^\lambda$ and $f_2=r^m$ and then solving for the constants λ and m. We find the general solutions of the above equations are

$$f_1=c_1r^2\ln r+c_2r^2+c_3\ln r+c_4 \quad \text{and} \quad f_2=c_5r^2+c_6r^4+\frac{c_7}{r^2}+c_8.$$

The constants $c_i, i=1,\dots,8$ are now determined from the boundary conditions. The constant c_4 can be arbitrary since the derivative of a constant is zero. The remaining constants are determined from the stress conditions. Using the results from Exercise 2.4, problem 20, we calculate the stresses

$$\begin{aligned}
\sigma_{rr}&=c_1(1+2\ln r)+2c_2+\frac{c_3}{r^2}-\left(2c_5+6\frac{c_7}{r^4}+4\frac{c_8}{r^2}\right)\cos 2\theta\\
\sigma_{\theta\theta}&=c_1(3+2\ln r)+2c_2-\frac{c_3}{r^2}+\left(2c_5+12c_6r^2+6\frac{c_7}{r^4}\right)\cos 2\theta\\
\sigma_{r\theta}&=\left(2c_5+6c_6r^2-6\frac{c_7}{r^4}-2\frac{c_8}{r^2}\right)\sin 2\theta.
\end{aligned}$$

The stresses are to remain bounded for all values of r and consequently we require c_1 and c_6 to be zero to avoid infinite stresses for large values of r. The stress $\sigma_{rr}|_{r=a} = 0$ requires that

$$2c_2 + \frac{c_3}{a^2} = 0 \quad \text{and} \quad 2c_5 + 6\frac{c_7}{a^4} + 4\frac{c_8}{a^2} = 0.$$

The stress $\sigma_{r\theta}|_{r=a} = 0$ requires that

$$2c_5 - 6\frac{c_7}{a^4} - 2\frac{c_8}{a^2} = 0.$$

In the limit as $r \to \infty$ we require that the stresses must satisfy the boundary conditions from the previous example 2.4-5. This leads to the equations $2c_2 = \frac{F_{11}}{2}$ and $2c_5 = -\frac{F_{11}}{2}$. Solving the above system of equations produces the Airy stress function

$$\phi = \phi(r,\theta) = \frac{F_{11}}{4} + \frac{F_{11}}{4}r^2 - \frac{a^2}{2}F_{11}\ln r + c_4 + \left(\frac{F_{11}a^2}{2} - \frac{F_{11}}{4}r^2 - \frac{F_{11}a^4}{4r^2}\right)\cos 2\theta$$

and the corresponding stress field is

$$\begin{aligned}
\sigma_{rr} &= \frac{F_{11}}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{F_{11}}{2}\left(1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right)\cos 2\theta \\
\sigma_{r\theta} &= -\frac{F_{11}}{2}\left(1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right)\sin 2\theta \\
\sigma_{\theta\theta} &= \frac{F_{11}}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{F_{11}}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta.
\end{aligned}$$

There is a maximum stress $\sigma_{\theta\theta} = 3F_{11}$ at $\theta = \pi/2, 3\pi/2$ and a minimum stress $\sigma_{\theta\theta} = -F_{11}$ at $\theta = 0, \pi$. The effect of the circular hole has been to magnify the applied stress. The factor of 3 is known as a stress concentration factor. In general, sharp corners and unusually shaped boundaries produce much higher stress concentration factors than rounded boundaries.

EXAMPLE 2.4-7. Consider an infinite cylindrical tube, with inner radius R_1 and the outer radius R_0, which is subjected to an internal pressure P_1 and an external pressure P_0 as illustrated in the figure 2.4-7. Find the stress and displacement fields.

Solution: Let u_r, u_θ, u_z denote the displacement field. We assume that $u_\theta = 0$ and $u_z = 0$ since the cylindrical surface r equal to a constant does not move in the θ or z directions. The displacement $u_r = u_r(r)$ is assumed to depend only upon the radial distance r. Under these conditions the Navier equations become

$$(\lambda + 2\mu)\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_r)\right) = 0.$$

This equation has the solution $u_r = c_1\frac{r}{2} + \frac{c_2}{r}$ and the strain components are found from the relations

$$e_{rr} = \frac{du_r}{dr}, \quad e_{\theta\theta} = \frac{u_r}{r}, \quad e_{zz} = e_{r\theta} = e_{rz} = e_{z\theta} = 0.$$

The stresses are determined from Hooke's law (the constitutive equations) and we write

$$\sigma_{ij} = \lambda\delta_{ij}\Theta + 2\mu e_{ij},$$

where

$$\Theta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r)$$

is the dilatation. These stresses are found to be

$$\sigma_{rr} = (\lambda+\mu)c_1 - \frac{2\mu}{r^2}c_2 \quad \sigma_{\theta\theta} = (\lambda+\mu)c_1 + \frac{2\mu}{r^2}c_2 \quad \sigma_{zz} = \lambda c_1 \quad \sigma_{r\theta} = \sigma_{rz} = \sigma_{z\theta} = 0.$$

We now apply the boundary conditions

$$\sigma_{rr}|_{r=R_1} n_r = -\left[(\lambda+\mu)c_1 - \frac{2\mu}{R_1^2}c_2\right] = +P_1 \quad \text{and} \quad \sigma_{rr}|_{r=R_0} n_r = \left[(\lambda+\mu)c_1 - \frac{2\mu}{R_0^2}c_2\right] = -P_0.$$

Solving for the constants c_1 and c_2 we find

$$c_1 = \frac{R_1^2 P_1 - R_0^2 P_0}{(\lambda+\mu)(R_0^2 - R_1^2)}, \qquad c_2 = \frac{R_1^2 R_0^2 (P_1 - P_0)}{2\mu(R_0^2 - R_1^2)}.$$

This produces the displacement field

$$u_r = \frac{R_1^2 P_1}{2(R_0^2 - R_1^2)}\left(\frac{r}{\lambda+\mu} + \frac{R_0^2}{\mu r}\right) - \frac{R_0^2 P_0}{2(R_0^2 - R_1^2)}\left(\frac{r}{\lambda+\mu} + \frac{R_1^2}{\mu r}\right), \qquad u_\theta = 0, \qquad u_z = 0,$$

and stress fields

$$\begin{aligned}
\sigma_{rr} &= \frac{R_1^2 P_1}{R_0^2 - R_1^2}\left(1 - \frac{R_0^2}{r^2}\right) - \frac{R_0^2 P_0}{R_0^2 - R_1^2}\left(1 - \frac{R_1^2}{r^2}\right)\\
\sigma_{\theta\theta} &= \frac{R_1^2 P_1}{R_0^2 - R_1^2}\left(1 + \frac{R_0^2}{r^2}\right) - \frac{R_0^2 P_0}{R_0^2 - R_1^2}\left(1 + \frac{R_1^2}{r^2}\right)\\
\sigma_{zz} &= \left(\frac{\lambda}{\lambda+\mu}\right)\frac{R_1^2 P_1 - R_0^2 P_0}{R_0^2 - R_1^2}\\
\sigma_{rz} &= \sigma_{z\theta} = \sigma_{r\theta} = 0
\end{aligned}$$

EXAMPLE 2.4-8. By making simplifying assumptions the Navier equations can be reduced to a more tractable form. For example, we can reduce the Navier equations to a one dimensional problem by making the following assumptions

1. Cartesian coordinates $x_1 = x, \quad x_2 = y, \quad x_3 = z$
2. $u_1 = u_1(x,t), \quad u_2 = u_3 = 0.$
3. There are no body forces.
4. Initial conditions of $\quad u_1(x,0) = 0 \quad$ and $\quad \dfrac{\partial u_1(x,0)}{\partial t} = 0$
5. Boundary conditions of the displacement type $\quad u_1(0,t) = f(t),$

where $f(t)$ is a specified function. These assumptions reduce the Navier equations to the single one dimensional wave equation

$$\frac{\partial^2 u_1}{\partial t^2} = \alpha^2 \frac{\partial^2 u_1}{\partial x^2}, \qquad \alpha^2 = \frac{\lambda + 2\mu}{\rho}.$$

The solution of this equation is

$$u_1(x,t) = \begin{cases} f(t - x/\alpha), & x \le \alpha t \\ 0, & x > \alpha t \end{cases}.$$

The solution represents a longitudinal elastic wave propagating in the $x-$direction with speed α. The stress wave associated with this displacement is determined from the constitutive equations. We find

$$\sigma_{xx} = (\lambda + \mu)e_{xx} = (\lambda + \mu)\frac{\partial u_1}{\partial x}.$$

This produces the stress wave

$$\sigma_{xx} = \begin{cases} -\frac{(\lambda+\mu)}{\alpha} f'(t - x/\alpha), & x \le \alpha t \\ 0, & x > \alpha t \end{cases}.$$

Here there is a discontinuity in the stress wave front at $x = \alpha t$.

Summary of Basic Equations of Elasticity

The equilibrium equations for a continuum have been shown to have the form $\sigma^{ij}_{\ ,j} + \varrho b^i = 0$, where b^i are the body forces per unit mass and σ^{ij} is the stress tensor. In addition to the above equations we have the constitutive equations $\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$ which is a generalized Hooke's law relating stress to strain for a linear elastic isotropic material. The strain tensor is related to the displacement field u_i by the strain equations $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$. These equations can be combined to obtain the Navier equations $\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \varrho b_i = 0$.

The above equations must be satisfied at all interior points of the material body. A boundary value problem results when conditions on the displacement of the boundary are specified. That is, the Navier equations must be solved subject to the prescribed displacement boundary conditions. If conditions on the stress at the boundary are specified, then these prescribed stresses are called surface tractions and must satisfy the relations $t^{i\,(n)} = \sigma^{ij}n_j$, where n_i is a unit outward normal vector to the boundary. For surface tractions, we need to use the compatibility equations combined with the constitutive equations and equilibrium equations. This gives rise to the Beltrami-Michell equations of compatibility

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + \varrho(b_{i,j} + b_{j,i}) + \frac{\nu}{1-\nu}\varrho b_{k,k} = 0.$$

Here we must solve for the stress components throughout the continuum where the above equations hold subject to the surface traction boundary conditions. Note that if an elasticity problem is formed in terms of the displacement functions, then the compatibility equations can be ignored.

For mixed boundary value problems we must solve a system of equations consisting of the equilibrium equations, constitutive equations, and strain displacement equations. We must solve these equations subject to conditions where the displacements u_i are prescribed on some portion(s) of the boundary and stresses are prescribed on the remaining portion(s) of the boundary. Mixed boundary value problems are more difficult to solve.

For elastodynamic problems, the equilibrium equations are replaced by equations of motion. In this case we need a set of initial conditions as well as boundary conditions before attempting to solve our basic system of equations.

EXERCISE 2.4

► **1.** Verify the generalized Hooke's law constitutive equations for hexagonal materials.

In the following problems the Young's modulus E, Poisson's ratio ν, the shear modulus or modulus of rigidity μ (sometimes denoted by G in Engineering texts), Lame's constant λ and the bulk modulus of elasticity k are assumed to satisfy the equations (2.4.19), (2.4.24) and (2.4.25). Show that these relations imply the additional relations given in the problems 2 through 6.

► **2.**

$$E = \frac{\mu(3\lambda + 2\mu)}{\mu + \lambda} \qquad E = \frac{9k(k-\lambda)}{3k-\lambda} \qquad E = \frac{9k\mu}{\mu + 3k}$$
$$E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} \qquad E = 2\mu(1+\nu) \qquad E = 3(1-2\nu)k$$

► **3.**

$$\nu = \frac{3k-E}{6k} \qquad \nu = \frac{\sqrt{(E+\lambda)^2 + 8\lambda^2} - (E+\lambda)}{4\lambda} \qquad \nu = \frac{E-2\mu}{2\mu}$$
$$\nu = \frac{\lambda}{2(\mu+\lambda)} \qquad \nu = \frac{3k-2\mu}{2(\mu+3k)} \qquad \nu = \frac{\lambda}{3k-\lambda}$$

► **4.**

$$k = \frac{\sqrt{(E+\lambda)^2 + 8\lambda^2} + (E+3\lambda)}{6} \qquad k = \frac{E}{3(1-2\nu)} \qquad k = \frac{2\mu(1+\nu)}{3(1-2\nu)}$$
$$k = \frac{2\mu + 3\lambda}{3} \qquad k = \frac{\mu E}{3(3\mu - E)} \qquad k = \frac{\lambda(1+\nu)}{3\nu}$$

► **5.**

$$\mu = \frac{3(k-\lambda)}{2} \qquad \mu = \frac{3k(1-2\nu)}{2(1+\nu)} \qquad \mu = \frac{\sqrt{(E+\lambda)^2 + 8\lambda^2} + (E-3\lambda)}{4}$$
$$\mu = \frac{\lambda(1-2\nu)}{2\nu} \qquad \mu = \frac{3Ek}{9k-E} \qquad \mu = \frac{E}{2(1+\nu)}$$

► **6.**

$$\lambda = \frac{3k\nu}{1+\nu} \qquad \lambda = \frac{3k-2\mu}{3} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$
$$\lambda = \frac{\mu(2\mu - E)}{E - 3\mu} \qquad \lambda = \frac{3k(3k-E)}{9k-E} \qquad \lambda = \frac{2\mu\nu}{1-2\nu}$$

► **7.** The previous exercises 2 through 6 imply that the generalized Hooke's law

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}$$

is expressible in a variety of forms. From the set of constants (μ,λ,ν,E,k) we can select any two constants and then express Hooke's law in terms of these constants.

(a) Express the above Hooke's law in terms of the constants E and ν.
(b) Express the above Hooke's law in terms of the constants k and E.
(c) Express the above Hooke's law in terms of physical components. Hint: The quantity e_{kk} is an invariant hence all you need to know is how second order tensors are represented in terms of physical components. See also problems 10,11,12.

▶ **8.** Verify the equations defining the stress for plane strain in Cartesian coordinates are

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{xx} + \nu e_{yy}] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{yy} + \nu e_{xx}] \\
\sigma_{zz} &= \frac{E\nu}{(1+\nu)(1-2\nu)}[e_{xx} + e_{yy}] \\
\sigma_{xy} &= \frac{E}{1+\nu}e_{xy} \\
\sigma_{yz} = \sigma_{xz} &= 0
\end{aligned}
$$

▶ **9.** Verify the equations defining the stress for plane strain in polar coordinates are

$$
\begin{aligned}
\sigma_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{rr} + \nu e_{\theta\theta}] \\
\sigma_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{\theta\theta} + \nu e_{rr}] \\
\sigma_{zz} &= \frac{\nu E}{(1+\nu)(1-2\nu)}[e_{rr} + e_{\theta\theta}] \\
\sigma_{r\theta} &= \frac{E}{1+\nu}e_{r\theta} \\
\sigma_{rz} = \sigma_{\theta z} &= 0
\end{aligned}
$$

▶ **10.** Write out the independent components of Hooke's generalized law for strain in terms of stress, and stress in terms of strain, in Cartesian coordinates. Express your results using the parameters ν and E. (Assume a linear elastic, homogeneous, isotropic material.)

▶ **11.** Write out the independent components of Hooke's generalized law for strain in terms of stress, and stress in terms of strain, in cylindrical coordinates. Express your results using the parameters ν and E. (Assume a linear elastic, homogeneous, isotropic material.)

▶ **12.** Write out the independent components of Hooke's generalized law for strain in terms of stress, and stress in terms of strain in spherical coordinates. Express your results using the parameters ν and E. (Assume a linear elastic, homogeneous, isotropic material.)

▶ **13.** For a linear elastic, homogeneous, isotropic material assume there exists a state of plane strain in Cartesian coordinates. Verify the equilibrium equations are

$$
\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \varrho b_x &= 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \varrho b_y &= 0 \\
\frac{\partial \sigma_{zz}}{\partial z} + \varrho b_z &= 0
\end{aligned}
$$

Hint: See problem 14, Exercise 2.3.

▶ **14 .** For a linear elastic, homogeneous, isotropic material assume there exists a state of plane strain in polar coordinates. Verify the equilibrium equations are

$$\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \varrho b_r &= 0 \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r}\sigma_{r\theta} + \varrho b_\theta &= 0 \\
\frac{\partial \sigma_{zz}}{\partial z} + \varrho b_z &= 0
\end{aligned}$$

Hint: See problem 15, Exercise 2.3.

▶ **15.** For a linear elastic, homogeneous, isotropic material assume there exists a state of plane stress in Cartesian coordinates. Verify the equilibrium equations are

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \varrho b_x &= 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \varrho b_y &= 0
\end{aligned}$$

▶ **16.** Determine the compatibility equations in terms of the Airy stress function ϕ when there exists a state of plane stress. Assume the body forces are derivable from a potential function V.

▶ **17.** For a linear elastic, homogeneous, isotropic material assume there exists a state of plane stress in polar coordinates. Verify the equilibrium equations are

$$\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \varrho b_r &= 0 \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r}\sigma_{r\theta} + \varrho b_\theta &= 0
\end{aligned}$$

► **18.** Figure 2.4-4 illustrates the state of equilibrium on an element in polar coordinates assumed to be of unit length in the z-direction. Verify the stresses given in the figure and then sum the forces in the r and θ directions to derive the same equilibrium laws developed in the previous exercise.

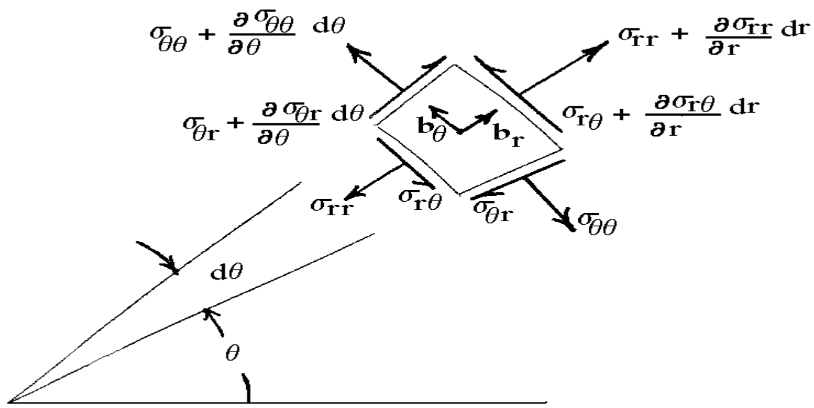

Figure 2.4-4. Polar element in equilibrium.

Hint: Resolve the stresses into components in the r and θ directions. Use the results that $\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$ and $\cos\frac{d\theta}{2} \approx 1$ for small values of $d\theta$. Sum forces and then divide by $r dr\, d\theta$ and take the limit as $dr \to 0$ and $d\theta \to 0$.

► **19.** Express each of the physical components of plane stress in polar coordinates, σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$ in terms of the physical components of stress in Cartesian coordinates σ_{xx}, σ_{yy}, σ_{xy}. Hint: Consider the transformation law $\overline{\sigma}_{ij} = \sigma_{ab}\dfrac{\partial x^a}{\partial \overline{x}^i}\dfrac{\partial x^b}{\partial \overline{x}^j}$.

► **20.** Use the results from problem 19 and assume the stresses are derivable from the relations

$$\sigma_{xx} = V + \frac{\partial^2\phi}{\partial y^2}, \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}, \quad \sigma_{yy} = V + \frac{\partial^2\phi}{\partial x^2}$$

where V is a potential function and ϕ is the Airy stress function. Show that upon changing to polar coordinates the Airy equations for stress become

$$\sigma_{rr} = V + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \qquad \sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}, \qquad \sigma_{\theta\theta} = V + \frac{\partial^2\phi}{\partial r^2}.$$

► **21.** Verify that the Airy stress equations in polar coordinates, given in problem 20, satisfy the equilibrium equations in polar coordinates derived in problem 17.

▶ **22.** In Cartesian coordinates show that the traction boundary conditions, equations (2.3.11), can be written in terms of the constants λ and μ as

$$T_1 = \lambda n_1 e_{kk} + \mu \left[2n_1 \frac{\partial u_1}{\partial x^1} + n_2 \left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1} \right) + n_3 \left(\frac{\partial u_1}{\partial x^3} + \frac{\partial u_3}{\partial x^1} \right) \right]$$
$$T_2 = \lambda n_2 e_{kk} + \mu \left[n_1 \left(\frac{\partial u_2}{\partial x^1} + \frac{\partial u_1}{\partial x^2} \right) + 2n_2 \frac{\partial u_2}{\partial x^2} + n_3 \left(\frac{\partial u_2}{\partial x^3} + \frac{\partial u_3}{\partial x^2} \right) \right]$$
$$T_3 = \lambda n_3 e_{kk} + \mu \left[n_1 \left(\frac{\partial u_3}{\partial x^1} + \frac{\partial u_1}{\partial x^3} \right) + n_2 \left(\frac{\partial u_3}{\partial x^2} + \frac{\partial u_2}{\partial x^3} \right) + 2n_3 \frac{\partial u_3}{\partial x^3} \right]$$

where (n_1, n_2, n_3) are the direction cosines of the unit normal to the surface, u_1, u_2, u_3 are the components of the displacements and T_1, T_2, T_3 are the surface tractions.

▶ **23.** Consider an infinite plane subject to tension in the $x-$direction only. Assume a state of plane strain and let $\sigma_{xx} = T$ with $\sigma_{xy} = \sigma_{yy} = 0$. Find the strain components e_{xx}, e_{yy} and e_{xy}. Also find the displacement field $u = u(x, y)$ and $v = v(x, y)$.

▶ **24.** Consider an infinite plane subject to tension in the y-direction only. Assume a state of plane strain and let $\sigma_{yy} = T$ with $\sigma_{xx} = \sigma_{xy} = 0$. Find the strain components e_{xx}, e_{yy} and e_{xy}. Also find the displacement field $u = u(x, y)$ and $v = v(x, y)$.

▶ **25.** Consider an infinite plane subject to tension in both the x and y directions. Assume a state of plane strain and let $\sigma_{xx} = T$, $\sigma_{yy} = T$ and $\sigma_{xy} = 0$. Find the strain components e_{xx}, e_{yy} and e_{xy}. Also find the displacement field $u = u(x, y)$ and $v = v(x, y)$.

▶ **26.** An infinite cylindrical rod of radius R_0 has an external pressure P_0 as illustrated in figure 2.5-5. Find the stress and displacement fields.

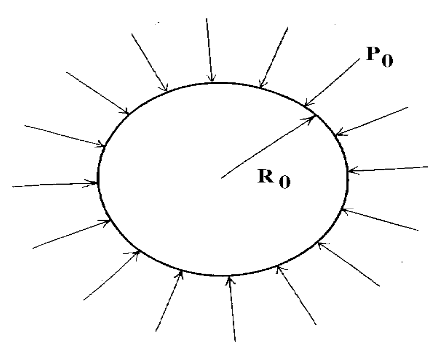

Figure 2.4-5. External pressure on a rod.

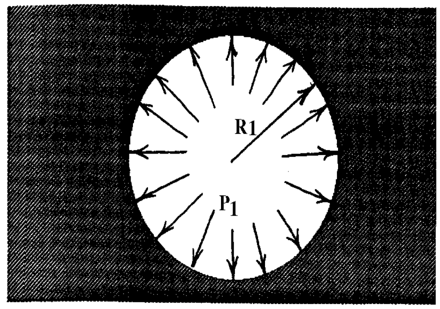

Figure 2.4-6. Internal pressure on circular hole.

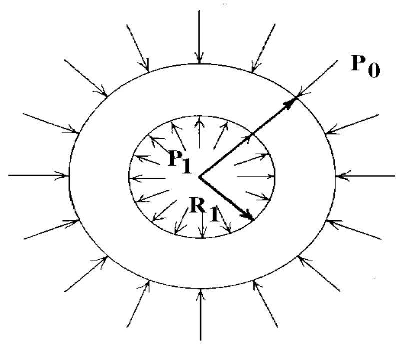

Figure 2.4-7. Tube with internal and external pressure.

► **27.** An infinite plane has a circular hole of radius R_1 with an internal pressure P_1 as illustrated in the figure 2.4-6. Find the stress and displacement fields.

► **28.** A tube of inner radius R_1 and outer radius R_0 has an internal pressure of P_1 and an external pressure of P_0 as illustrated in the figure 2.4-7. Verify the stress and displacement fields derived in example 2.4-7.

► **29.** Use Cartesian tensors and combine the equations of equilibrium $\sigma_{ij,j} + \varrho b_i = 0$, Hooke's law $\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$ and the strain tensor $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and derive the Navier equations of equilibrium

$$\sigma_{ij,j} + \varrho b_i = (\lambda + \mu)\frac{\partial \Theta}{\partial x^i} + \mu \frac{\partial^2 u_i}{\partial x^k \partial x^k} + \varrho b_i = 0,$$

where $\Theta = e_{11} + e_{22} + e_{33}$ is the dilatation.

► **30.** Show the Navier equations in problem 29 can be written in the tensor form

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \varrho b_i = 0$$

or the vector form

$$\mu \nabla^2 \vec{u} + (\lambda + \mu)\nabla\,(\nabla \cdot \vec{u}) + \varrho \vec{b} = \vec{0}.$$

▶ **31.** Show that in an orthogonal coordinate system the components of $\nabla(\nabla\cdot\vec{u})$ can be expressed in terms of physical components by the relation

$$[\nabla(\nabla\cdot\vec{u})]_i = \frac{1}{h_i}\frac{\partial}{\partial x^i}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3u(1))}{\partial x^1}+\frac{\partial(h_1h_3u(2))}{\partial x^2}+\frac{\partial(h_1h_2u(3))}{\partial x^3}\right]\right\}$$

▶ **32.** Show that in orthogonal coordinates the components of $\nabla^2\vec{u}$ can be written

$$\left[\nabla^2\vec{u}\right]_i = g^{jk}u_{i,jk} = A_i$$

and in terms of physical components one can write

$$\begin{aligned} h_iA(i) = \sum_{j=1}^{3}\frac{1}{h_j^2}\Bigg[&\frac{\partial^2(h_iu(i))}{\partial x^j\partial x^j} - 2\sum_{m=1}^{3}\left\{ {m \atop i\,j} \right\}\frac{\partial(h_mu(m))}{\partial x^j} - \sum_{m=1}^{3}\left\{ {m \atop j\,j} \right\}\frac{\partial(h_iu(i))}{\partial x^m} \\ &- \sum_{m=1}^{3}h_mu(m)\left(\frac{\partial}{\partial x^j}\left\{ {m \atop i\,j} \right\} - \sum_{p=1}^{3}\left\{ {m \atop i\,p} \right\}\left\{ {p \atop j\,j} \right\} - \sum_{p=1}^{3}\left\{ {m \atop j\,p} \right\}\left\{ {p \atop i\,j} \right\}\right)\Bigg] \end{aligned}$$

▶ **33.** Use the results in problem 32 to show in Cartesian coordinates the physical components of $[\nabla^2\vec{u}]_i = A_i$ can be represented

$$\begin{aligned} \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_1 &= A(1) = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} \\ \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_2 &= A(2) = \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} \\ \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_3 &= A(3) = \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2} \end{aligned}$$

where (u,v,w) are the components of the displacement vector $\vec{u}$.

▶ **34.** Use the results in problem 32 to show in cylindrical coordinates the physical components of $[\nabla^2\vec{u}]_i = A_i$ can be represented

$$\begin{aligned} \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_r &= A(1) = \nabla^2u_r - \frac{1}{r^2}u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} \\ \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_\theta &= A(2) = \nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{1}{r^2}u_\theta \\ \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_z &= A(3) = \nabla^2u_z \end{aligned}$$

where u_r, u_θ, u_z are the physical components of $\vec{u}$ and $\nabla^2\alpha = \frac{\partial^2\alpha}{\partial r^2}+\frac{1}{r}\frac{\partial\alpha}{\partial r}+\frac{1}{r^2}\frac{\partial^2\alpha}{\partial\theta^2}+\frac{\partial^2\alpha}{\partial z^2}$

▶ **35.** Use the results in problem 32 to show in spherical coordinates the physical components of $[\nabla^2\vec{u}]_i = A_i$ can be represented

$$\begin{aligned} \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_\rho &= A(1) = \nabla^2u_\rho - \frac{2}{\rho^2}u_\rho - \frac{2}{\rho^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2\cot\theta}{\rho^2}u_\theta - \frac{2}{\rho^2\sin\theta}\frac{\partial u_\phi}{\partial\phi} \\ \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_\theta &= A(2) = \nabla^2u_\theta + \frac{2}{\rho^2}\frac{\partial u_\rho}{\partial\theta} - \frac{1}{\rho^2\sin\theta}u_\theta - \frac{2\cos\theta}{\rho^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} \\ \left[\nabla^2\vec{u}\right]\cdot\hat{\mathbf{e}}_\phi &= A(3) = \nabla^2u_\phi - \frac{1}{\rho^2\sin^2\theta}u_\phi + \frac{2}{\rho^2\sin\theta}\frac{\partial u_\rho}{\partial\phi} + \frac{2\cos\theta}{\rho^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} \end{aligned}$$

where u_ρ, u_θ, u_ϕ are the physical components of $\vec{u}$ and where

$$\nabla^2\alpha = \frac{\partial^2\alpha}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial\alpha}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\alpha}{\partial\theta^2}+\frac{\cot\theta}{\rho^2}\frac{\partial\alpha}{\partial\theta}+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2\alpha}{\partial\phi^2}$$

▶ **36.** Combine the results from problems 30,31,32 and 33 and write the Navier equations of equilibrium in Cartesian coordinates. Alternatively, write the stress-strain relations (2.4.29(b)) in terms of physical components and then use these results, together with the results from Exercise 2.3, problems 2 and 14, to derive the Navier equations.

▶ **37.** Combine the results from problems 30,31,32 and 34 and write the Navier equations of equilibrium in cylindrical coordinates. Alternatively, write the stress-strain relations (2.4.29(b)) in terms of physical components and then use these results, together with the results from Exercise 2.3, problems 3 and 15, to derive the Navier equations.

▶ **38.** Combine the results from problems 30,31,32 and 35 and write the Navier equations of equilibrium in spherical coordinates. Alternatively, write the stress-strain relations (2.4.29(b)) in terms of physical components and then use these results, together with the results from Exercise 2.3, problems 4 and 16, to derive the Navier equations.

▶ **39.** Assume $\varrho\vec{b} = -\text{grad}\, V$ and let ϕ denote the Airy stress function defined by

$$\begin{aligned}\sigma_{xx} =& V + \frac{\partial^2\phi}{\partial y^2}\\ \sigma_{yy} =& V + \frac{\partial^2\phi}{\partial x^2}\\ \sigma_{xy} =& -\frac{\partial^2\phi}{\partial x\partial y}\end{aligned}$$

(a) Show that for conditions of plane strain the equilibrium equations in two dimensions are satisfied by the above definitions. (b) Express the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x\partial y}$$

in terms of ϕ and V and show that

$$\nabla^4\phi + \frac{1-2\nu}{1-\nu}\nabla^2 V = 0.$$

▶ **40.** Consider the case where the body forces are conservative and derivable from a scalar potential function such that $\varrho b_i = -V_{,i}$. Show that under conditions of plane strain in rectangular Cartesian coordinates the compatibility equation $e_{11,22} + e_{22,11} = 2e_{12,12}$ can be reduced to the form $\nabla^2\sigma_{ii} = \frac{1}{1-\nu}\nabla^2 V \qquad , i = 1,2$ involving the stresses and the potential. Hint: Differentiate the equilibrium equations.

▶ **41.** Use the relation $\sigma^i_j = 2\mu e^i_j + \lambda e^m_m \delta^i_j$ and solve for the strain in terms of the stress.

▶ **42.** Derive the equation (2.4.26) from the equation (2.4.23).

▶ **43.** In two dimensions assume that the body forces are derivable from a potential function V and $\varrho b^i = -g^{ij}V_{,j}$. Also assume that the stress is derivable from the Airy stress function and the potential function by employing the relations $\sigma^{ij} = \epsilon^{im}\epsilon^{jn}\, u_{m,n} + g^{ij}V \qquad i,j,m,n = 1,2$ where $u_m = \phi_{,m}$ and ϵ^{pq} is the two dimensional epsilon permutation symbol and all indices have the range 1,2.

(a) Show that $\epsilon^{im}\epsilon^{jn}\,(\phi_m)_{,nj} = 0$.

(b) Show that $\sigma^{ij}_{\ ,j} = -\varrho b^i$.

(c) Verify the stress laws for cylindrical and Cartesian coordinates given in problem 20 by using the above expression for σ^{ij}. Hint: Expand the contravariant derivative and convert all terms to physical components. Also recall that $\epsilon^{ij} = \frac{1}{\sqrt{g}}e^{ij}$.

► **44.** Consider a material with body forces per unit volume $\rho F^i, i = 1, 2, 3$ and surface tractions denoted by $\sigma^r = \sigma^{rj} n_j$, where n_j is a unit surface normal. Further, let δu_i denote a small displacement vector associated with a small variation in the strain δe_{ij}.

(a) Show the work done during a small variation in strain is $\delta W = \delta W_B + \delta W_S$ where $\delta W_B = \int_V \rho F^i \delta u_i \, d\tau$ is a volume integral representing the work done by the body forces and $\delta W_S = \int_S \sigma^r \delta u_r \, dS$ is a surface integral representing the work done by the surface forces.

(b) Using the Gauss divergence theorem show that the work done can be represented as

$$\delta W = \frac{1}{2} \int_V c^{ijmn} \delta [e_{mn} e_{ij}] \, d\tau \quad \text{or} \quad W = \frac{1}{2} \int_V \sigma^{ij} e_{ij} \, d\tau.$$

The scalar quantity $\frac{1}{2} \sigma^{ij} e_{ij}$ is called the strain energy density or strain energy per unit volume. Hint: Interchange subscripts, add terms and calculate $2W = \int_V \sigma^{ij} [\delta u_{i,j} + \delta u_{j,i}] \, d\tau$.

► **45.** Consider a spherical shell subjected to an internal pressure p_i and external pressure p_o. Let a denote the inner radius and b the outer radius of the spherical shell. Find the displacement and stress fields in spherical coordinates (ρ, θ, ϕ).
Hint: Assume symmetry in the θ and ϕ directions and let the physical components of displacements satisfy the relations $u_\rho = u_\rho(\rho), \quad u_\theta = u_\phi = 0$.

► **46.** (a) Verify the average normal stress is proportional to the dilatation, where the proportionality constant is the bulk modulus of elasticity. i.e. Show that $\frac{1}{3}\sigma^i_i = \frac{E}{1-2\nu} \frac{1}{3} e^i_i = k e^i_i$ where k is the bulk modulus of elasticity.

(b) Define the quantities of strain deviation and stress deviation in terms of the average normal stress $s = \frac{1}{3}\sigma^i_i$ and average cubic dilatation $e = \frac{1}{3} e^i_i$ as follows

$$\text{strain deviator} \qquad \varepsilon^i_j = e^i_j - e\delta^i_j$$

$$\text{stress deviator} \qquad s^i_j = \sigma^i_j - s\delta^i_j$$

Show that zero results when a contraction is performed on the stress and strain deviators. (The above definitions are used to split the strain tensor into two parts. One part represents pure dilatation and the other part represents pure distortion.)

(c) Show that $(1 - 2\nu)s = Ee \quad \text{or} \quad s = (3\lambda + 2\mu)e$

(d) Express Hooke's law in terms of the strain and stress deviator and show

$$E(\varepsilon^i_j + e\delta^i_j) = (1 + \nu)s^i_j + (1 - 2\nu)s\delta^i_j$$

which simplifies to $s^i_j = 2\mu\varepsilon^i_j$.

► **47.** Show the strain energy density (problem 44) can be written in terms of the stress and strain deviators (problem 46) and

$$W = \frac{1}{2} \int_V \sigma^{ij} e_{ij} \, d\tau = \frac{1}{2} \int_V (3se + s^{ij} \varepsilon_{ij}) \, d\tau$$

and from Hooke's law

$$W = \frac{3}{2} \int_V ((3\lambda + 2\mu)e^2 + \frac{2\mu}{3} \varepsilon^{ij} \varepsilon_{ij}) \, d\tau.$$

▶ **48.** Find the stress σ_{rr},$\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ in an infinite plate with a small circular hole, which is traction free, when the plate is subjected to a pure shearing force F_{12}. Determine the maximum stress.

▶ **49.** Show that in terms of E and ν

$$C_{1111} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \qquad C_{1122} = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad C_{1212} = \frac{E}{2(1+\nu)}$$

▶ **50.** Show that in Cartesian coordinates the quantity

$$S = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - (\sigma_{xy})^2 - (\sigma_{yz})^2 - (\sigma_{xz})^2$$

is a stress invariant. Hint: First verify that in tensor form $S = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})$.

▶ **51.** Show that in Cartesian coordinates for a state of plane strain where the displacements are given by $u = u(x,y)$,$v = v(x,y)$ and $w = 0$, the stress components must satisfy the equations

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \varrho b_x =& 0\\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \varrho b_y =& 0\\
\nabla^2(\sigma_{xx} + \sigma_{yy}) =& \frac{-\varrho}{1-\nu}\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)
\end{aligned}$$

▶ **52.** Show that in Cartesian coordinates for a state of plane stress where $\sigma_{xx} = \sigma_{xx}(x,y)$, $\sigma_{yy} = \sigma_{yy}(x,y)$, $\sigma_{xy} = \sigma_{xy}(x,y)$ and $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ the stress components must satisfy

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \varrho b_x =& 0\\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \varrho b_y =& 0\\
\nabla^2(\sigma_{xx} + \sigma_{yy}) =& -\varrho(\nu+1)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)
\end{aligned}$$