§2.6 ELECTRIC AND MAGNETIC FIELDS

Introduction

In electromagnetic theory the mks system of units and the Gaussian system of units are the ones most often encountered. In this section the equations will be given in the mks system of units. If you want the equations in the Gaussian system of units make the replacements given in the column 3 of Table 1.

Table 1. MKS AND GAUSSIAN UNITS

MKS symbol	MKS units	Replacement symbol	GAUSSIAN units
$\vec{E}$ (Electric field)	volt/m	$\vec{E}$	statvolt/cm
$\vec{B}$ (Magnetic field)	weber/m^2	$\frac{\vec{B}}{c}$	gauss
$\vec{D}$ (Displacement field)	coulomb/m^2	$\frac{\vec{D}}{4\pi}$	statcoulomb/cm^2
$\vec{H}$ (Auxiliary Magnetic field)	ampere/m	$\frac{c\vec{H}}{4\pi}$	oersted
$\vec{J}$ (Current density)	ampere/m^2	$\vec{J}$	statampere/cm^2
$\vec{A}$ (Vector potential)	weber/m	$\frac{\vec{A}}{c}$	gauss-cm
$\mathcal{V}$ (Electric potential)	volt	$\mathcal{V}$	statvolt
ϵ (Dielectric constant)		$\frac{\epsilon}{4\pi}$	
μ (Magnetic permeability)		$\frac{4\pi\mu}{c^2}$	

Electrostatics

A basic problem in electrostatic theory is to determine the force $\vec{F}$ on a charge Q placed a distance r from another charge q. The solution to this problem is Coulomb's law

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\,\widehat{\mathbf{e}}_r \tag{2.6.1}$$

where q, Q are measured in coulombs, $\epsilon_0 = 8.85 \times 10^{-12}$ coulomb^2/N $\cdot$ m^2 is called the permittivity in a vacuum, r is in meters, $[\vec{F}]$ has units of Newtons and $\widehat{\mathbf{e}}_r$ is a unit vector pointing from q to Q if q, Q have the same sign or pointing from Q to q if q, Q are of opposite sign. The quantity $\vec{E} = \vec{F}/Q$ is called the electric field produced by the charges. In the special case $Q = 1$, we have $\vec{E} = \vec{F}$ and so $Q = 1$ is called a test charge. This tells us that the electric field at a point P can be viewed as the force per unit charge exerted on a test charge Q placed at the point P. The test charge Q is always positive and so is repulsed if q is positive and attracted if q is negative.

The electric field associated with many charges is obtained by the principal of superposition. For example, let $q_1, q_2, \ldots, q_n$ denote n-charges having respectively the distances $r_1, r_2, \ldots, r_n$ from a test charge Q placed at a point P. The force exerted on Q is

$$\begin{aligned}\vec{F} =&\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n\\ \vec{F} =&\frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{r_1^2}\,\widehat{\mathbf{e}}_{r_1} + \frac{q_2 Q}{r_2^2}\,\widehat{\mathbf{e}}_{r_2} + \cdots + \frac{q_n Q}{r_n^2}\,\widehat{\mathbf{e}}_{r_n}\right)\\ \text{or}\quad \vec{E} = \vec{E}(P) =&\frac{\vec{F}}{Q} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\,\widehat{\mathbf{e}}_{r_i}\end{aligned} \tag{2.6.2}$$

where $\vec{E} = \vec{E}(P)$ is the electric field associated with the system of charges. The equation (2.6.2) can be generalized to other situations by defining other types of charge distributions. We introduce a line charge density λ^*, (coulomb/m), a surface charge density μ^*, (coulomb/m^2), a volume charge density ρ^*, (coulomb/m^3), then we can calculate the electric field associated with these other types of charge distributions. For example, if there is a charge distribution $\lambda^* = \lambda^*(s)$ along a curve C, where s is an arc length parameter, then we would have

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_C \frac{\widehat{\mathbf{e}}_r}{r^2}\,\lambda^* ds \tag{2.6.3}$$

as the electric field at a point P due to this charge distribution. The integral in equation (2.6.3) being a line integral along the curve C and where ds is an element of arc length. Here equation (2.6.3) represents a continuous summation of the charges along the curve C. For a continuous charge distribution over a surface S, the electric field at a point P is

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int\int_S \frac{\widehat{\mathbf{e}}_r}{r^2}\mu^* d\sigma \tag{2.6.4}$$

where $d\sigma$ represents an element of surface area on S. Similarly, if ρ^* represents a continuous charge distribution throughout a volume V, then the electric field is represented

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int\int\int_V \frac{\widehat{\mathbf{e}}_r}{r^2}\rho^*\, d\tau \tag{2.6.5}$$

where $d\tau$ is an element of volume. In the equations (2.6.3), (2.6.4), (2.6.5) we let (x,y,z) denote the position of the test charge and let (x',y',z') denote a point on the line, on the surface or within the volume, then

$$\vec{r} = (x-x')\,\widehat{\mathbf{e}}_1 + (y-y')\,\widehat{\mathbf{e}}_2 + (z-z')\,\widehat{\mathbf{e}}_3 \tag{2.6.6}$$

represents the distance from the point P to an element of charge $\lambda^* ds$, $\mu^* d\sigma$ or $\rho^* d\tau$ with $r = |\vec{r}|$ and $\widehat{\mathbf{e}}_r = \dfrac{\vec{r}}{r}$.

If the electric field is conservative, then $\nabla\times\vec{E} = 0$, and so it is derivable from a potential function $\mathcal{V}$ by taking the negative of the gradient of $\mathcal{V}$ and

$$\vec{E} = -\nabla\mathcal{V}. \tag{2.6.7}$$

For these conditions note that $\nabla\mathcal{V}\cdot d\vec{r} = -\vec{E}\cdot d\vec{r}$ is an exact differential so that the potential function can be represented by the line integral

$$\mathcal{V} = \mathcal{V}(P) = -\int_\alpha^P \vec{E}\cdot d\vec{r} \tag{2.6.8}$$

where α is some reference point (usually infinity, where $\mathcal{V}(\infty) = 0$). For a conservative electric field the line integral will be independent of the path connecting any two points a and b so that

$$\mathcal{V}(b) - \mathcal{V}(a) = -\int_\alpha^b \vec{E}\cdot d\vec{r} - \left(-\int_\alpha^a \vec{E}\cdot d\vec{r}\right) = -\int_a^b \vec{E}\cdot d\vec{r} = \int_a^b \nabla\mathcal{V}\cdot d\vec{r}. \tag{2.6.9}$$

Let $\alpha = \infty$ in equation (2.6.8), then the potential function associated with a point charge moving in the radial direction $\widehat{\mathbf{e}}_r$ is

$$\mathcal{V}(r) = -\int_\infty^r \vec{E}\cdot d\vec{r} = \frac{-q}{4\pi\epsilon_0}\int_\infty^r \frac{1}{r^2}dr = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\,\Big|_\infty^r = \frac{q}{4\pi\epsilon_0 r}.$$

By superposition, the potential at a point P for a continuous volume distribution of charges is given by $\mathcal{V}(P) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho^*}{r}\,d\tau$ and for a surface distribution of charges $\mathcal{V}(P) = \frac{1}{4\pi\epsilon_0}\iint_S \frac{\mu^*}{r}\,d\sigma$ and for a line distribution of charges $\mathcal{V}(P) = \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda^*}{r}\,ds$; and for a discrete distribution of point charges $\mathcal{V}(P) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} \frac{q_i}{r_i}$. When the potential functions are defined from a common reference point, then the principal of superposition applies.

The potential function $\mathcal{V}$ is related to the work done W in moving a charge within the electric field. The work done in moving a test charge Q from point a to point b is an integral of the force times distance moved. The electric force on a test charge Q is $\vec{F} = Q\vec{E}$ and so the force $\vec{F} = -Q\vec{E}$ is in opposition to this force as you move the test charge. The work done is

$$W = \int_a^b \vec{F}\cdot d\vec{r} = \int_a^b -Q\vec{E}\cdot d\vec{r} = Q\int_a^b \nabla V\cdot d\vec{r} = Q[\mathcal{V}(b) - \mathcal{V}(a)]. \tag{2.6.10}$$

The work done is independent of the path joining the two points and depends only on the end points and the change in the potential. If one moves Q from infinity to point b, then the above becomes $W = QV(b)$.

An electric field $\vec{E} = \vec{E}(P)$ is a vector field which can be represented graphically by constructing vectors at various selected points in the space. Such a plot is called a vector field plot. A field line associated with a vector field is a curve such that the tangent vector to a point on the curve has the same direction as the vector field at that point. Field lines are used as an aid for visualization of an electric field and vector fields in general. The tangent to a field line at a point has the same direction as the vector field $\vec{E}$ at that point. For example, in two dimensions let $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2$ denote the position vector to a point on a field line. The tangent vector to this point has the direction $d\vec{r} = dx\,\widehat{\mathbf{e}}_1 + dy\,\widehat{\mathbf{e}}_2$. If $\vec{E} = \vec{E}(x,y) = -N(x,y)\,\widehat{\mathbf{e}}_1 + M(x,y)\,\widehat{\mathbf{e}}_2$ is the vector field constructed at the same point, then $\vec{E}$ and $d\vec{r}$ must be colinear. Thus, for each point (x,y) on a field line we require that $d\vec{r} = K\vec{E}$ for some constant K. Equating like components we find that the field lines must satisfy the differential relation.

$$\begin{aligned} \frac{dx}{-N(x,y)} = \frac{dy}{M(x,y)} &= K \\ \text{or} \qquad M(x,y)\,dx + N(x,y)\,dy &= 0. \end{aligned} \tag{2.6.11}$$

In two dimensions, the family of equipotential curves $\mathcal{V}(x,y) = C_1$ =constant, are orthogonal to the family of field lines and are described by solutions of the differential equation

$$N(x,y)\,dx - M(x,y)\,dy = 0$$

obtained from equation (2.6.11) by taking the negative reciprocal of the slope. The field lines are perpendicular to the equipotential curves because at each point on the curve $\mathcal{V} = C_1$ we have $\nabla\mathcal{V}$ being perpendicular to the curve $\mathcal{V} = C_1$ and so it is colinear with $\vec{E}$ at this same point. Field lines associated with electric fields are called electric lines of force. The density of the field lines drawn per unit cross sectional area are proportional to the magnitude of the vector field through that area.

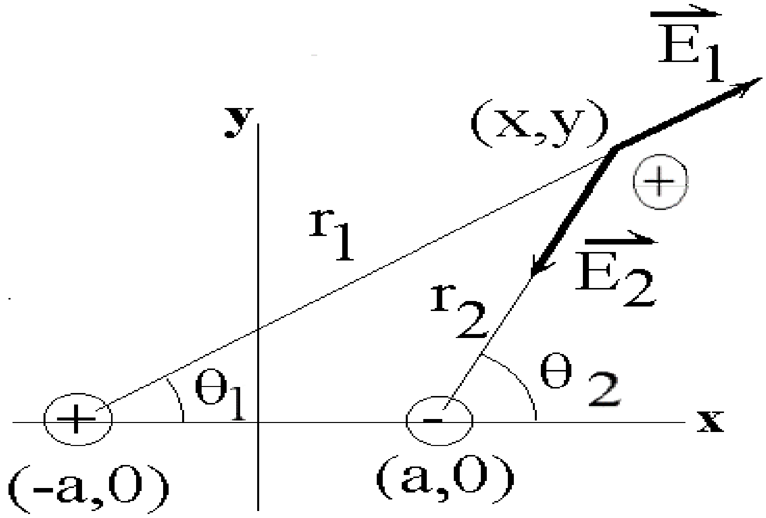

Figure 2.6-1. Electric forces due to a positive charge at $(-a,0)$ and negative charge at $(a,0)$.

EXAMPLE 2.6-1.

Find the field lines and equipotential curves associated with a positive charge q located at the point $(-a,0)$ and a negative charge $-q$ located at the point $(a,0)$.

Solution: With reference to the figure 2.6-1, the total electric force $\vec{E}$ on a test charge $Q=1$ place at a general point (x,y) is, by superposition, the sum of the forces from each of the isolated charges and is $\vec{E}=\vec{E}_1+\vec{E}_2$. The electric force vectors due to each individual charge are

$$\begin{aligned}\vec{E}_1 =&\frac{kq(x+a)\,\widehat{\mathbf{e}}_1+kqy\,\widehat{\mathbf{e}}_2}{r_1^3} \quad \text{with} \quad r_1^2=(x+a)^2+y^2\\ \vec{E}_2 =&\frac{-kq(x-a)\,\widehat{\mathbf{e}}_1-kqy\,\widehat{\mathbf{e}}_2}{r_2^3} \quad \text{with} \quad r_2^2=(x-a)^2+y^2\end{aligned} \tag{2.6.12}$$

where $k=\dfrac{1}{4\pi\epsilon_0}$ is a constant. This gives

$$\vec{E}=\vec{E}_1+\vec{E}_2=\left[\frac{kq(x+a)}{r_1^3}-\frac{kq(x-a)}{r_2^3}\right]\widehat{\mathbf{e}}_1+\left[\frac{kqy}{r_1^3}-\frac{kqy}{r_2^3}\right]\widehat{\mathbf{e}}_2.$$

This determines the differential equation of the field lines

$$\frac{dx}{\frac{kq(x+a)}{r_1^3}-\frac{kq(x-a)}{r_2^3}}=\frac{dy}{\frac{kqy}{r_1^3}-\frac{kqy}{r_2^3}}. \tag{2.6.13}$$

To solve this differential equation we make the substitutions

$$\cos\theta_1=\frac{x+a}{r_1} \quad \text{and} \quad \cos\theta_2=\frac{x-a}{r_2} \tag{2.6.14}$$

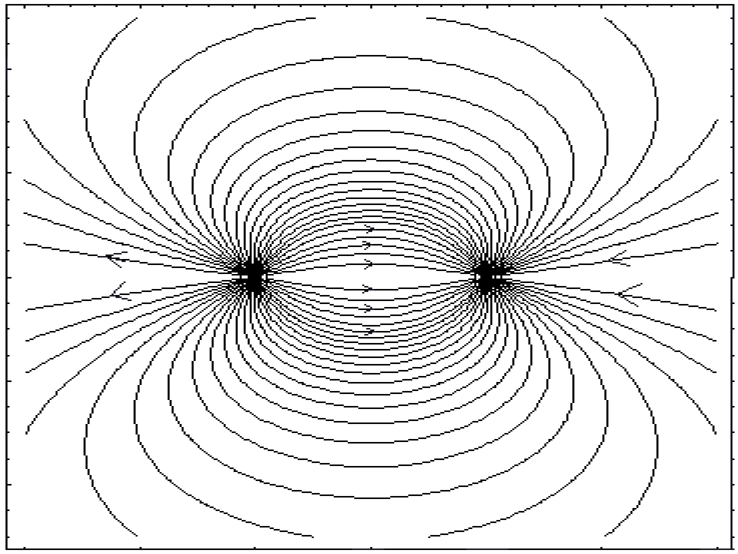

Figure 2.6-2. Lines of electric force between two opposite sign charges.

as suggested by the geometry from figure 2.6-1. From the equations (2.6.12) and (2.6.14) we obtain the relations

$$\begin{aligned}
-\sin\theta_1\, d\theta_1 =&\frac{r_1 dx - (x+a)\, dr_1}{r_1^2} \\
2r_1 dr_1 =&2(x+a)\, dx + 2y dy \\
-\sin\theta_2\, d\theta_2 =&\frac{r_2\, dx - (x-a) dr_2}{r_2^2} \\
2r_2\, dr_2 =&2(x-a)\, dx + 2y\, dy
\end{aligned}$$

which implies that

$$\begin{aligned}
-\sin\theta_1\, d\theta_1 &= -\frac{(x+a) y\, dy}{r_1^3} + \frac{y^2\, dx}{r_1^3} \\
-\sin\theta_2\, d\theta_2 &= -\frac{(x-a) y\, dy}{r_2^3} + \frac{y^2\, dx}{r_2^3}
\end{aligned} \tag{2.6.15}$$

Now compare the results from equation (2.6.15) with the differential equation (2.6.13) and determine that y is an integrating factor of equation (2.6.13) . This shows that the differential equation (2.6.13) can be written in the much simpler form of the exact differential equation

$$-\sin\theta_1\, d\theta_1 + \sin\theta_2\, d\theta_2 = 0 \tag{2.6.16}$$

in terms of the variables θ_1 and θ_2. The equation (2.6.16) is easily integrated to obtain

$$\cos\theta_1 - \cos\theta_2 = C \tag{2.6.17}$$

where C is a constant of integration. In terms of x, y the solution can be written

$$\frac{x+a}{\sqrt{(x+a)^2 + y^2}} - \frac{x-a}{\sqrt{(x-a)^2 + y^2}} = C. \tag{2.6.18}$$

These field lines are illustrated in the figure 2.6-2.

The differential equation for the equipotential curves is obtained by taking the negative reciprocal of the slope of the field lines. This gives

$$\frac{dy}{dx} = \frac{\frac{kq(x-a)}{r_2^3} - \frac{kq(x+a)}{r_1^3}}{\frac{kqy}{r_1^3} - \frac{kqy}{r_2^3}}.$$

This result can be written in the form

$$-\left[\frac{(x+a)dx + ydy}{r_1^3}\right] + \left[\frac{(x-a)dx + ydy}{r_2^3}\right] = 0$$

which simplifies to the easily integrable form

$$-\frac{dr_1}{r_1^2} + \frac{dr_2}{r_2^2} = 0$$

in terms of the new variables r_1 and r_2. An integration produces the equipotential curves

$$\begin{aligned} \frac{1}{r_1} - \frac{1}{r_2} &= C_2 \\ \text{or} \quad \frac{1}{\sqrt{(x+a)^2+y^2}} - \frac{1}{\sqrt{(x-a)^2+y^2}} &= C_2. \end{aligned}$$

The potential function for this problem can be interpreted as a superposition of the potential functions $\mathcal{V}_1 = -\frac{kq}{r_1}$ and $\mathcal{V}_2 = \frac{kq}{r_2}$ associated with the isolated point charges at the points $(-a, 0)$ and $(a, 0)$. ■

Observe that the electric lines of force move from positive charges to negative charges and they do not cross one another. Where field lines are close together the field is strong and where the lines are far apart the field is weak. If the field lines are almost parallel and equidistant from one another the field is said to be uniform. The arrows on the field lines show the direction of the electric field $\vec{E}$. If one moves along a field line in the direction of the arrows the electric potential is decreasing and they cross the equipotential curves at right angles. Also, when the electric field is conservative we will have $\nabla \times \vec{E} = 0$.

In three dimensions the situation is analogous to what has been done in two dimensions. If the electric field is $\vec{E} = \vec{E}(x, y, z) = P(x, y, z)\,\widehat{\mathbf{e}}_1 + Q(x, y, z)\,\widehat{\mathbf{e}}_2 + R(x, y, z)\,\widehat{\mathbf{e}}_3$ and $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3$ is the position vector to a variable point (x, y, z) on a field line, then at this point $d\vec{r}$ and $\vec{E}$ must be colinear so that $d\vec{r} = K\vec{E}$ for some constant K. Equating like coefficients gives the system of equations

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)} = K. \tag{2.6.19}$$

From this system of equations one must try to obtain two independent integrals, call them $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$. These integrals represent one-parameter families of surfaces. When any two of these surfaces intersect, the result is a curve which represents a field line associated with the vector field $\vec{E}$. These type of field lines in three dimensions are more difficult to illustrate.

The electric flux ϕ_E of an electric field $\vec{E}$ over a surface S is defined as the summation of the normal component of $\vec{E}$ over the surface and is represented

$$\phi_E = \iint_S \vec{E} \cdot \hat{\mathbf{n}}\, d\sigma \quad \text{with units of } \frac{\mathrm{N\,m^2}}{C} \tag{2.6.20}$$

where $\hat{\mathbf{n}}$ is a unit normal to the surface. The flux ϕ_E can be thought of as being proportional to the number of electric field lines passing through an element of surface area. If the surface is a closed surface we have by the divergence theorem of Gauss

$$\phi_E = \iiint_V \nabla \cdot \vec{E}\, d\tau = \iint_S \vec{E} \cdot \hat{\mathbf{n}}\, d\sigma$$

where V is the volume enclosed by S.

Gauss Law

Let $d\sigma$ denote an element of surface area on a surface S. A cone is formed if all points on the boundary of $d\sigma$ are connected by straight lines to the origin. The cone need not be a right circular cone. The situation is illustrated in the figure 2.6-3.

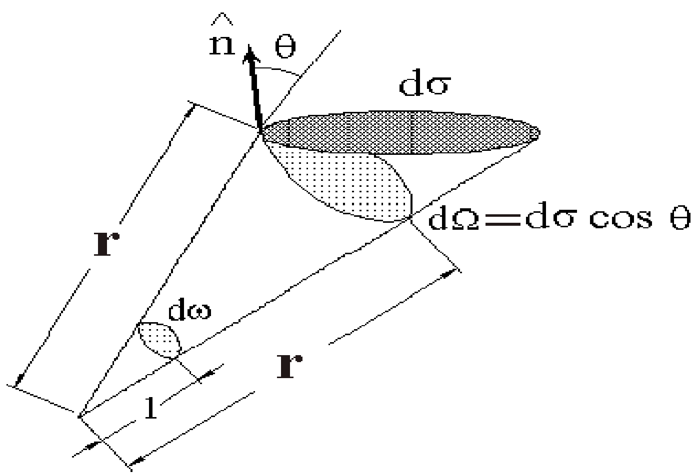

Figure 2.6-3. Solid angle subtended by element of area.

We let $\vec{r}$ denote a position vector from the origin to a point on the boundary of $d\sigma$ and let $\hat{\mathbf{n}}$ denote a unit outward normal to the surface at this point. We then have $\hat{\mathbf{n}} \cdot \vec{r} = r\cos\theta$ where $r = |\vec{r}|$ and θ is the angle between the vectors $\hat{\mathbf{n}}$ and $\vec{r}$. Construct a sphere, centered at the origin, having radius r. This sphere intersects the cone in an element of area $d\Omega$. The solid angle subtended by $d\sigma$ is defined as $d\omega = \dfrac{d\Omega}{r^2}$. Note that this is equivalent to constructing a unit sphere at the origin which intersect the cone in an element of area $d\omega$. Solid angles are measured in steradians. The total solid angle about a point equals the area of the sphere divided by its radius squared or 4π steradians. The element of area $d\Omega$ is the projection of $d\sigma$ on the constructed sphere and $d\Omega = d\sigma\, \cos\theta = \dfrac{\hat{\mathbf{n}} \cdot \vec{r}}{r} d\sigma$ so that $d\omega = \dfrac{\hat{\mathbf{n}} \cdot \vec{r}}{r^3} d\sigma = \dfrac{d\Omega}{r^2}$. Observe that sometimes the dot product $\hat{\mathbf{n}} \cdot \vec{r}$ is negative, the sign depending upon which of the normals to the surface is constructed. (i.e. the inner or outer normal.)

The Gauss law for electrostatics in a vacuum states that the flux through any surface enclosing many charges is the total charge enclosed by the surface divided by ϵ_0. The Gauss law is written

$$\iint_S \vec{E} \cdot \hat{\mathbf{n}}\, d\sigma = \begin{cases} \frac{Q_e}{\epsilon_0} & \text{for charges inside } S \\ 0 & \text{for charges outside } S \end{cases} \tag{2.6.21}$$

where Q_e represents the total charge enclosed by the surface S with $\hat{\mathbf{n}}$ the unit outward normal to the surface. The proof of Gauss's theorem follows. Consider a single charge q within the closed surface S. The electric field at a point on the surface S due to the charge q within S is represented $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\widehat{\mathbf{e}}_r$ and so the flux integral is

$$\phi_E = \iint_S \vec{E}\cdot\hat{\mathbf{n}}\,d\sigma = \iint_S \frac{q}{4\pi\epsilon_0}\frac{\widehat{\mathbf{e}}_r\cdot\hat{\mathbf{n}}}{r^2}\,d\sigma = \frac{q}{4\pi\epsilon_0}\iint_S \frac{d\Omega}{r^2} = \frac{q}{\epsilon_0} \tag{2.6.22}$$

since $\frac{\widehat{\mathbf{e}}_r\cdot\hat{\mathbf{n}}}{r^2} = \frac{\cos\theta\,d\sigma}{r^2} = \frac{d\Omega}{r^2} = d\omega$ and $\iint_S d\omega = 4\pi$. By superposition of the charges, we obtain a similar result for each of the charges within the surface. Adding these results gives $Q_e = \sum_{i=1}^{n} q_i$. For a continuous distribution of charge inside the volume we can write $Q_e = \iiint_V \rho^*\,d\tau$, where ρ^* is the charge distribution per unit volume. Note that charges outside of the closed surface do not contribute to the total flux across the surface. This is because the field lines go in one side of the surface and go out the other side. In this case $\iint_S \vec{E}\cdot\hat{\mathbf{n}}\,d\sigma = 0$ for charges outside the surface. Also the position of the charge or charges within the volume does not effect the Gauss law.

The equation (2.6.21) is the Gauss law in integral form. We can put this law in differential form as follows. Using the Gauss divergence theorem we can write for an arbitrary volume that

$$\iint_S \vec{E}\cdot\hat{\mathbf{n}}\,d\sigma = \iiint_V \nabla\cdot\vec{E}\,d\tau = \iiint_V \frac{\rho^*}{\epsilon_0}\,d\tau = \frac{Q_e}{\epsilon_0} = \frac{1}{\epsilon_0}\iiint_V \rho^*\,d\tau$$

which for an arbitrary volume implies

$$\nabla\cdot\vec{E} = \frac{\rho^*}{\epsilon_0}. \tag{2.6.23}$$

The equations (2.6.23) and (2.6.7) can be combined so that the Gauss law can also be written in the form $\nabla^2\mathcal{V} = -\frac{\rho^*}{\epsilon_0}$ which is called Poisson's equation.

EXAMPLE 2.6-2

Find the electric field associated with an infinite plane sheet of positive charge.

Solution: Assume there exists a uniform surface charge μ^* and draw a circle at some point on the plane surface. Now move the circle perpendicular to the surface to form a small cylinder which extends equal distances above and below the plane surface. We calculate the electric flux over this small cylinder in the limit as the height of the cylinder goes to zero. The charge inside the cylinder is $\mu^* A$ where A is the area of the circle. We find that the Gauss law requires that

$$\iint_S \vec{E}\cdot\hat{\mathbf{n}}\,d\sigma = \frac{Q_e}{\epsilon_0} = \frac{\mu^* A}{\epsilon_0} \tag{2.6.24}$$

where $\hat{\mathbf{n}}$ is the outward normal to the cylinder as we move over the surface S. By the symmetry of the situation the electric force vector is uniform and must point away from both sides to the plane surface in the direction of the normals to both sides of the surface. Denote the plane surface normals by $\widehat{\mathbf{e}}_n$ and $-\widehat{\mathbf{e}}_n$ and assume that $\vec{E} = \beta\,\widehat{\mathbf{e}}_n$ on one side of the surface and $\vec{E} = -\beta\,\widehat{\mathbf{e}}_n$ on the other side of the surface for some constant β. Substituting this result into the equation (2.6.24) produces

$$\iint_S \vec{E}\cdot\hat{\mathbf{n}}\,d\sigma = 2\beta A \tag{2.6.25}$$

since only the ends of the cylinder contribute to the above surface integral. On the sides of the cylinder we will have $\hat{\mathbf{n}} \cdot \pm \widehat{\mathbf{e}}_n = 0$ and so the surface integral over the sides of the cylinder is zero. By equating the results from equations (2.6.24) and (2.6.25) we obtain the result that $\beta = \dfrac{\mu^*}{2\epsilon_0}$ and consequently we can write $\vec{E} = \dfrac{\mu^*}{2\epsilon_0}\widehat{\mathbf{e}}_n$ where $\widehat{\mathbf{e}}_n$ represents one of the normals to the surface. ■

Note an electric field will always undergo a jump discontinuity when crossing a surface charge μ^*. As in the above example we have $\vec{E}_{up} = \dfrac{\mu^*}{2\epsilon_0}\widehat{\mathbf{e}}_n$ and $\vec{E}_{down} = -\dfrac{\mu^*}{2\epsilon}\widehat{\mathbf{e}}_n$ so that the difference is

$$\vec{E}_{up} - \vec{E}_{down} = \frac{\mu^*}{\epsilon_0}\widehat{\mathbf{e}}_n \qquad \text{or} \qquad E^i n_i^{(1)} + E^i n_i^{(2)} + \frac{\mu^*}{\epsilon_0} = 0. \tag{2.6.26}$$

It is this difference which causes the jump discontinuity.

EXAMPLE 2.6-3.

Calculate the electric field associated with a uniformly charged sphere of radius a.

Solution: We proceed as in the previous example. Let μ^* denote the uniform charge distribution over the surface of the sphere and let $\widehat{\mathbf{e}}_r$ denote the unit normal to the sphere. The total charge then is written as $q = \iint_{S_a} \mu^* d\sigma = 4\pi a^2 \mu^*$. If we construct a sphere of radius $r > a$ around the charged sphere, then we have by the Gauss theorem

$$\iiint_{S_r} \vec{E} \cdot \widehat{\mathbf{e}}_r \, d\sigma = \frac{Q_e}{\epsilon_0} = \frac{q}{\epsilon_0}. \tag{2.6.27}$$

Again, we can assume symmetry for $\vec{E}$ and assume that it points radially outward in the direction of the surface normal $\widehat{\mathbf{e}}_r$ and has the form $\vec{E} = \beta\widehat{\mathbf{e}}_r$ for some constant β. Substituting this value for $\vec{E}$ into the equation (2.6.27) we find that

$$\iint_{S_r} \vec{E} \cdot \widehat{\mathbf{e}}_r \, d\sigma = \beta \iint_{S_r} d\sigma = 4\pi\beta r^2 = \frac{q}{\epsilon_0}. \tag{2.6.28}$$

This gives $\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\widehat{\mathbf{e}}_r$ where $\widehat{\mathbf{e}}_r$ is the outward normal to the sphere. This shows that the electric field outside the sphere is the same as if all the charge were situated at the origin. ■

For S a piecewise closed surface enclosing a volume V and $F^i = F^i(x^1, x^2, x^3)$ $i = 1, 2, 3$, a continuous vector field with continuous derivatives the Gauss divergence theorem enables us to replace a flux integral of F^i over S by a volume integral of the divergence of F^i over the volume V such that

$$\iint_S F^i n_i \, d\sigma = \iiint_V F^i_{,i} \, d\tau \quad \text{or} \quad \iint_S \vec{F} \cdot \hat{\mathbf{n}} \, d\sigma = \iiint_V div\vec{F} \, d\tau. \tag{2.6.29}$$

If V contains a simple closed surface Σ where F^i is discontinuous we must modify the above Gauss divergence theorem.

EXAMPLE 2.6-4.

We examine the modification of the Gauss divergence theorem for spheres in order to illustrate the concepts. Let V have surface area S which encloses a surface Σ. Consider the figure 2.6-4 where the volume V enclosed by S and containing Σ has been cut in half.

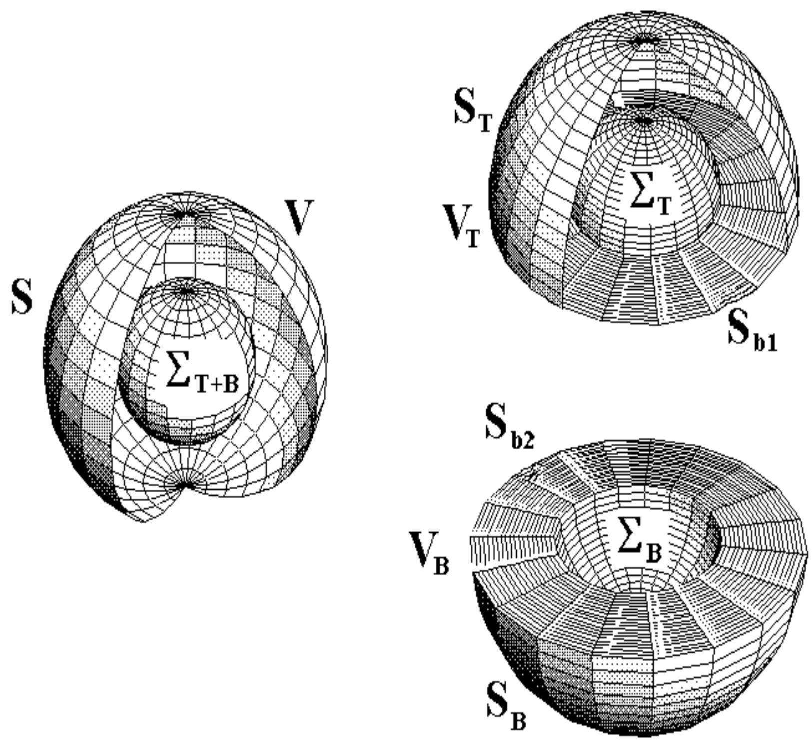

Figure 2.6-4. Sphere S containing sphere Σ.

Applying the Gauss divergence theorem to the top half of figure 2.6-4 gives

$$\iint_{S_T} F^i n_i^T \, d\sigma + \iint_{S_{b1}} F^i n_i^{b_T} \, d\sigma + \iint_{\Sigma_T} F^i n_i^{\Sigma_T} \, d\sigma = \iiint_{V_T} F^i_{,i} \, d\tau \tag{2.6.30}$$

where the n_i are the unit outward normals to the respective surfaces S_T, S_{b1} and Σ_T. Applying the Gauss divergence theorem to the bottom half of the sphere in figure 2.6-4 gives

$$\iint_{S_B} F^i n_i^B \, d\sigma + \iint_{S_{b2}} F^i n_i^{b_B} \, d\sigma + \iint_{\Sigma_B} F^i n_i^{\Sigma_B} \, d\sigma = \iiint_{V_B} F^i_{,i} \, d\tau \tag{2.6.31}$$

Observe that the unit normals to the surfaces S_{b1} and S_{b2} are equal and opposite in sign so that adding the equations (2.6.30) and (2.6.31) we obtain

$$\iint_S F^i n_i \, d\sigma + \iint_\Sigma F^i n_i^{(1)} \, d\sigma = \iiint_{V_T+V_B} F^i_{,i} \, d\tau \tag{2.6.32}$$

where $S = S_T + S_B$ is the total surface area of the outside sphere and $\Sigma = \Sigma_T + \Sigma_B$ is the total surface area of the inside sphere, and $n_i^{(1)}$ is the inward normal to the sphere Σ when the top and bottom volumes are combined. Applying the Gauss divergence theorem to just the isolated small sphere Σ we find

$$\iint_\Sigma F^i n_i^{(2)}\, d\sigma = \iiint_{V_\Sigma} F^i_{\ ,i}\, d\tau \tag{2.6.33}$$

where $n_i^{(2)}$ is the outward normal to Σ. By adding the equations (2.6.33) and (2.6.32) we find that

$$\iint_S F^i n_i\, d\sigma + \iint_\Sigma \left(F^i n_i^{(1)} + F^i n_i^{(2)}\right)\, d\sigma = \iiint_V F^i_{\ ,i}\, d\tau \tag{2.6.34}$$

where $V = V_T + V_B + V_\Sigma$. The equation (2.6.34) can also be written as

$$\iint_S F^i n_i\, d\sigma = \iiint_V F^i_{\ ,i}\, d\tau - \iint_\Sigma \left(F^i n_i^{(1)} + F^i n_i^{(2)}\right)\, d\sigma. \tag{2.6.35}$$

In the case that V contains a surface Σ the total electric charge inside S is

$$Q_e = \iiint_V \rho^*\, d\tau + \iint_\Sigma \mu^*\, d\sigma \tag{2.6.36}$$

where μ^* is the surface charge density on Σ and ρ^* is the volume charge density throughout V. The Gauss theorem requires that

$$\iint_S E^i n_i\, d\sigma = \frac{Q_e}{\epsilon_0} = \frac{1}{\epsilon_0}\iiint_V \rho^*\, d\tau + \frac{1}{\epsilon_0}\iint_\Sigma \mu^*\, d\sigma. \tag{2.6.37}$$

In the case of a jump discontinuity across the surface Σ we use the results of equation (2.6.34) and write

$$\iint_S E^i n_i\, d\sigma = \iiint_V E^i_{\ ,i}\, d\tau - \iint_\Sigma \left(E^i n_i^{(1)} + E^i n_i^{(2)}\right)\, d\sigma. \tag{2.6.38}$$

Subtracting the equation (2.6.37) from the equation (2.6.38) gives

$$\iiint_V \left(E^i_{\ ,i} - \frac{\rho^*}{\epsilon_0}\right)\, d\tau - \iint_\Sigma \left(E^i n_i^{(1)} + E^i n_i^{(2)} + \frac{\mu^*}{\epsilon_0}\right)\, d\sigma = 0. \tag{2.6.39}$$

For arbitrary surfaces S and Σ, this equation implies the differential form of the Gauss law

$$E^i_{\ ,i} = \frac{\rho^*}{\epsilon_0}. \tag{2.6.40}$$

Further, on the surface Σ, where there is a surface charge distribution we have

$$E^i n_i^{(1)} + E^i n_i^{(2)} + \frac{\mu^*}{\epsilon_0} = 0 \tag{2.6.41}$$

which shows the electric field undergoes a discontinuity when you cross a surface charge μ^*. ∎

Electrostatic Fields in Materials

When charges are introduced into materials it spreads itself throughout the material. Materials in which the spreading occurs quickly are called conductors, while materials in which the spreading takes a long time are called nonconductors or dielectrics. Another electrical property of materials is the ability to hold local charges which do not come into contact with other charges. This property is called induction. For example, consider a single atom within the material. It has a positively charged nucleus and negatively charged electron cloud surrounding it. When this atom experiences an electric field $\vec{E}$ the negative cloud moves opposite to $\vec{E}$ while the positively charged nucleus moves in the direction of $\vec{E}$. If $\vec{E}$ is large enough it can ionize the atom by pulling the electrons away from the nucleus. For moderately sized electric fields the atom achieves an equilibrium position where the positive and negative charges are offset. In this situation the atom is said to be polarized and have a dipole moment $\vec{p}$.

Definition: When a pair of charges $+q$ and $-q$ are separated by a distance $2\vec{d}$ the electric dipole moment is defined by $\vec{p} = 2\vec{d}q$, where $\vec{p}$ has dimensions of [C m].

In the special case where $\vec{d}$ has the same direction as $\vec{E}$ and the material is symmetric we say that $\vec{p}$ is proportional to $\vec{E}$ and write $\vec{p} = \alpha\vec{E}$, where α is called the atomic polarizability. If in a material subject to an electric field their results many such dipoles throughout the material then the dielectric is said to be polarized. The vector quantity $\vec{P}$ is introduced to represent this effect. The vector $\vec{P}$ is called the polarization vector having units of [C/m^2], and represents an average dipole moment per unit volume of material. The vectors P_i and E_i are related through the displacement vector D_i such that

$$P_i = D_i - \epsilon_0 E_i. \tag{2.6.42}$$

For an anisotropic material (crystal)

$$D_i = \epsilon_i^j E_j \qquad \text{and} \qquad P_i = \alpha_i^j E_j \tag{2.6.43}$$

where ϵ_i^j is called the dielectric tensor and α_i^j is called the electric susceptibility tensor. Consequently,

$$P_i = \alpha_i^j E_j = \epsilon_i^j E_j - \epsilon_0 E_i = (\epsilon_i^j - \epsilon_0 \delta_i^j) E_j \quad \text{so that} \qquad \alpha_i^j = \epsilon_i^j - \epsilon_0 \delta_i^j. \tag{2.6.44}$$

A dielectric material is called homogeneous if the electric force and displacement vector are the same for any two points within the medium. This requires that the electric force and displacement vectors be constant parallel vector fields. It is left as an exercise to show that the condition for homogeneity is that $\epsilon_{i,k}^j = 0$. A dielectric material is called isotropic if the electric force vector and displacement vector have the same direction. This requires that $\epsilon_i^j = \epsilon\delta_j^i$ where δ_j^i is the Kronecker delta. The term $\epsilon = \epsilon_0 K_e$ is called the dielectric constant of the medium. The constant $\epsilon_0 = 8.85(10)^{-12}\,\text{coul}^2/\text{N}\cdot\text{m}^2$ is the permittivity of free space and the quantity $k_e = \frac{\epsilon}{\epsilon_0}$ is called the relative dielectric constant (relative to ϵ_0). For free space $k_e = 1$. Similarly for an isotropic material we have $\alpha_i^j = \epsilon_0 \alpha_e \delta_i^j$ where α_e is called the electric susceptibility. For a linear medium the vectors $\vec{P}$, $\vec{D}$ and $\vec{E}$ are related by

$$D_i = \epsilon_0 E_i + P_i = \epsilon_0 E_i + \epsilon_0 \alpha_e E_i = \epsilon_0 (1 + \alpha_e) E_i = \epsilon_0 K_e E_i = \epsilon E_i \tag{2.6.45}$$

where $K_e = 1 + \alpha_e$ is the relative dielectric constant. The equation (2.6.45) are constitutive equations for dielectric materials.

The effect of polarization is to produce regions of bound charges ρ_b within the material and bound surface charges μ_b together with free charges ρ_f which are not a result of the polarization. Within dielectrics we have $\nabla \cdot \vec{P} = \rho_b$ for bound volume charges and $\vec{P} \cdot \widehat{\mathbf{e}}_n = \mu_b$ for bound surface charges, where $\widehat{\mathbf{e}}_n$ is a unit normal to the bounding surface of the volume. In these circumstances the expression for the potential function is written

$$\mathcal{V} = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho_b}{r}\, d\tau + \frac{1}{4\pi\epsilon_0} \iint_S \frac{\mu_b}{r}\, d\sigma \tag{2.6.46}$$

and the Gauss law becomes

$$\epsilon_0 \nabla \cdot \vec{E} = \rho^* = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f \qquad \text{or} \qquad \nabla(\epsilon_0 \vec{E} + \vec{P}) = \rho_f. \tag{2.6.47}$$

Since $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ the Gauss law can also be written in the form

$$\nabla \cdot \vec{D} = \rho_f \qquad \text{or} \qquad D^i_{,i} = \rho_f. \tag{2.6.48}$$

When no confusion arises we replace ρ_f by ρ. In integral form the Gauss law for dielectrics is written

$$\iint_S \vec{D} \cdot \hat{\mathbf{n}}\, d\sigma = Q_{fe} \tag{2.6.49}$$

where Q_{fe} is the total free charge density within the enclosing surface.

Magnetostatics

A stationary charge generates an electric field $\vec{E}$ while a moving charge generates a magnetic field $\vec{B}$. Magnetic field lines associated with a steady current moving in a wire form closed loops as illustrated in the figure 2.6-5.

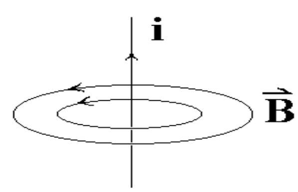

Figure 2.6-5. Magnetic field lines.

The direction of the magnetic force is determined by the right hand rule where the thumb of the right hand points in the direction of the current flow and the fingers of the right hand curl around in the direction of the magnetic field $\vec{B}$. The force on a test charge Q moving with velocity $\vec{V}$ in a magnetic field is

$$\vec{F}_m = Q(\vec{V} \times \vec{B}). \tag{2.6.50}$$

The total electromagnetic force acting on Q is the electric force plus the magnetic force and is

$$\vec{F} = Q\left[\vec{E} + (\vec{V} \times \vec{B})\right] \tag{2.6.51}$$

which is known as the Lorentz force law. The magnetic force due to a line charge density λ^* moving along a curve C is the line integral

$$\vec{F}_{mag} = \int_C \lambda^* \, ds(\vec{V} \times \vec{B}) = \int_C \vec{I} \times \vec{B} ds. \tag{2.6.52}$$

Similarly, for a moving surface charge density moving on a surface

$$\vec{F}_{mag} = \iint_S \mu^* d\sigma(\vec{V} \times \vec{B}) = \iint_S \vec{K} \times \vec{B} \, d\sigma \tag{2.6.53}$$

and for a moving volume charge density

$$\vec{F}_{mag} = \iiint_V \rho^* d\tau(\vec{V} \times \vec{B}) = \iint_V \vec{J} \times \vec{B} \, d\tau \tag{2.6.54}$$

where the quantities $\vec{I} = \lambda^* \vec{V}$, $\vec{K} = \mu^* \vec{V}$ and $\vec{J} = \rho^* \vec{V}$ are respectively the current, the current per unit length, and current per unit area.

A conductor is any material where the charge is free to move. The flow of charge is governed by Ohm's law. Ohm's law states that the current density vector J_i is a linear function of the electric intensity or $J_i = \sigma_{im} E_m$, where σ_{im} is the conductivity tensor of the material. For homogeneous, isotropic conductors $\sigma_{im} = \sigma \delta_{im}$ so that $J_i = \sigma E_i$ where σ is the conductivity and $1/\sigma$ is called the resistivity.

Surround a charge density ρ^* with an arbitrary simple closed surface S having volume V and calculate the flux of the current density across the surface. We find by the divergence theorem

$$\iint_S \vec{J} \cdot \hat{\mathbf{n}} \, d\sigma = \iiint_V \nabla \cdot \vec{J} \, d\tau. \tag{2.6.55}$$

If charge is to be conserved, the current flow out of the volume through the surface must equal the loss due to the time rate of change of charge within the surface which implies

$$\iint_S \vec{J} \cdot \hat{\mathbf{n}} \, d\sigma = \iiint_V \nabla \cdot \vec{J} \, d\tau = -\frac{d}{dt} \iiint_V \rho^* \, d\tau = -\iiint_V \frac{\partial \rho^*}{\partial t} \, d\tau \tag{2.6.56}$$

or

$$\iiint_V \left[\nabla \cdot \vec{J} + \frac{\partial \rho^*}{\partial t} \right] d\tau = 0. \tag{2.6.57}$$

This implies that for an arbitrary volume we must have

$$\nabla \cdot \vec{J} = -\frac{\partial \rho^*}{\partial t}. \tag{2.6.58}$$

Note that equation (2.6.58) has the same form as the continuity equation (2.3.73) for mass conservation and so it is also called a continuity equation for charge conservation. For magnetostatics there exists steady line currents or stationary current so $\frac{\partial \rho^*}{\partial t} = 0$. This requires that $\nabla \cdot \vec{J} = 0$.

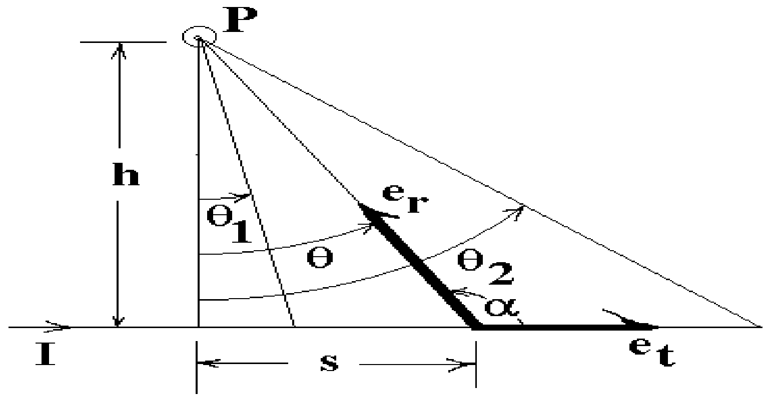

Figure 2.6-6. Magnetic field around wire.

Biot-Savart Law

The Biot-Savart law for magnetostatics describes the magnetic field at a point P due to a steady line current moving along a curve C and is

$$\vec{B}(P) = \frac{\mu_0}{4\pi}\int_C \frac{\vec{I}\times\widehat{\mathbf{e}}_r}{r^2}\,ds \tag{2.6.59}$$

with units [N/amp · m] and where the integration is in the direction of the current flow. In the Biot-Savart law we have the constant $\mu_0 = 4\pi\times 10^{-7}\,\mathrm{N/amp}^2$ which is called the permeability of free space, $\vec{I} = I\,\widehat{\mathbf{e}}_t$ is the current flowing in the direction of the unit tangent vector $\widehat{\mathbf{e}}_t$ to the curve C, $\widehat{\mathbf{e}}_r$ is a unit vector directed from a point on the curve C toward the point P and r is the distance from a point on the curve to the general point P. Note that for a steady current to exist along the curve the magnitude of $\vec{I}$ must be the same everywhere along the curve. Hence, this term can be brought out in front of the integral. For surface currents $\vec{K}$ and volume currents $\vec{J}$ the Biot-Savart law is written

$$\begin{aligned} \vec{B}(P) =& \frac{\mu_0}{4\pi}\iint_S \frac{\vec{K}\times\widehat{\mathbf{e}}_r}{r^2}\,d\sigma \\ \text{and}\qquad \vec{B}(P) =& \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J}\times\widehat{\mathbf{e}}_r}{r^2}\,d\tau. \end{aligned}$$

EXAMPLE 2.6-5.

Calculate the magnetic field $\vec{B}$ a distance h perpendicular to a wire carrying a constant current $\vec{I}$.

Solution: The magnetic field circles around the wire. For the geometry of the figure 2.6-6, the magnetic field points out of the page. We can write

$$\vec{I}\times\widehat{\mathbf{e}}_r = I\,\widehat{\mathbf{e}}_t\times\widehat{\mathbf{e}}_r = I\hat{\mathbf{e}}\sin\alpha$$

where $\hat{\mathbf{e}}$ is a unit vector tangent to the circle of radius h which encircles the wire and cuts the wire perpendicularly.

For this problem the Biot-Savart law is

$$\vec{B}(P) = \frac{\mu_0 I}{4\pi} \int \frac{\hat{\mathbf{e}}}{r^2}\, ds.$$

In terms of θ we find from the geometry of figure 2.6-6

$$\tan\theta = \frac{s}{h} \quad \text{with} \quad ds = h\sec^2\theta\, d\theta \quad \text{and} \quad \cos\theta = \frac{h}{r}.$$

Therefore,

$$\vec{B}(P) = \frac{\mu_0}{\pi} \int_{\theta_1}^{\theta_2} \frac{I\hat{\mathbf{e}}\sin\alpha\, h\sec^2\theta}{h^2/\cos^2\theta}\, d\theta.$$

But, $\alpha = \pi/2 + \theta$ so that $\sin\alpha = \cos\theta$ and consequently

$$\vec{B}(P) = \frac{\mu_0 I\hat{\mathbf{e}}}{4\pi h} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{\mu_0 I\hat{\mathbf{e}}}{4\pi h}(\sin\theta_2 - \sin\theta_1).$$

For a long straight wire $\theta_1 \to -\pi/2$ and $\theta_2 \to \pi/2$ to give the magnetic field $\vec{B}(P) = \dfrac{\mu_0 I\hat{\mathbf{e}}}{2\pi h}$. ■

For volume currents the Biot-Savart law is

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{J} \times \widehat{\mathbf{e}}_r}{r^2}\, d\tau \tag{2.6.60}$$

and consequently (see exercises)

$$\nabla \cdot \vec{B} = 0. \tag{2.6.61}$$

Recall the divergence of an electric field is $\nabla \cdot \vec{E} = \frac{\rho^*}{\epsilon_0}$ is known as the Gauss's law for electric fields and so in analogy the divergence $\nabla \cdot \vec{B} = 0$ is sometimes referred to as Gauss's law for magnetic fields. If $\nabla \cdot \vec{B} = 0$, then there exists a vector field $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$. The vector field $\vec{A}$ is called the vector potential of $\vec{B}$. Note that $\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$. Also the vector potential $\vec{A}$ is not unique since $\vec{B}$ is also derivable from the vector potential $\vec{A} + \nabla\phi$ where ϕ is an arbitrary continuous and differentiable scalar.

Ampere's Law

Ampere's law is associated with the work done in moving around a simple closed path. For example, consider the previous example 2.6-5. In this example the integral of $\vec{B}$ around a circular path of radius h which is centered at some point on the wire can be associated with the work done in moving around this path. The summation of force times distance is

$$\oint_C \vec{B} \cdot d\vec{r} = \oint_C \vec{B} \cdot \hat{\mathbf{e}}\, ds = \frac{\mu_0 I}{2\pi h} \oint_C ds = \mu_0 I \tag{2.6.62}$$

where now $d\vec{r} = \hat{\mathbf{e}}\, ds$ is a tangent vector to the circle encircling the wire and $\oint_C ds = 2\pi h$ is the distance around this circle. The equation (2.6.62) holds not only for circles, but for any simple closed curve around the wire. Using the Stoke's theorem we have

$$\oint_C \vec{B} \cdot d\vec{r} = \iint_S (\nabla \times \vec{B}) \cdot \widehat{\mathbf{e}}_n\, d\sigma = \mu_0 I = \iint_S \mu_0 \vec{J} \cdot \widehat{\mathbf{e}}_n\, d\sigma \tag{2.6.63}$$

where $\iint_S \vec{J} \cdot \widehat{\mathbf{e}}_n \, d\sigma$ is the total flux (current) passing through the surface which is created by encircling some curve about the wire. Equating like terms in equation (2.6.63) gives the differential form of Ampere's law

$$\nabla \times \vec{B} = \mu_0 \vec{J}. \tag{2.6.64}$$

Magnetostatics in Materials

Similar to what happens when charges are introduced into materials we have magnetic fields whenever there are moving charges within materials. For example, when electrons move around an atom tiny current loops are formed. These current loops create what are called magnetic dipole moments $\vec{m}$ throughout the material. When a magnetic field $\vec{B}$ is applied to a material medium there is a net alignment of the magnetic dipoles. The quantity $\vec{M}$, called the magnetization vector is introduced. Here $\vec{M}$ is associated with a dielectric medium and has the units [amp/m] and represents an average magnetic dipole moment per unit volume and is analogous to the polarization vector $\vec{P}$ used in electrostatics. The magnetization vector $\vec{M}$ acts a lot like the previous polarization vector in that it produces bound volume currents $\vec{J}_b$ and surface currents $\vec{K}_b$ where $\nabla \times \vec{M} = \vec{J}_b$ is a volume current density throughout some volume and $\vec{M} \times \widehat{\mathbf{e}}_n = \vec{K}_b$ is a surface current on the boundary of this volume.

From electrostatics note that the time derivative of $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ has the same units as current density. The total current in a magnetized material is then $\vec{J}_t = \vec{J}_b + \vec{J}_f + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ where $\vec{J}_b$ is the bound current, $\vec{J}_f$ is the free current and $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is the induced current. Ampere's law, equation (2.6.64), in magnetized materials then becomes

$$\nabla \times \vec{B} = \mu_0 \vec{J}_t = \mu_0 (\vec{J}_b + \vec{J}_f + \epsilon_0 \frac{\partial \vec{E}}{\partial t}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2.6.65}$$

where $\vec{J} = \vec{J}_b + \vec{J}_f$. The term $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is referred to as a displacement current or as a Maxwell correction to the field equation. This term implies that a changing electric field induces a magnetic field.

An auxiliary magnet field $\vec{H}$ defined by

$$H_i = \frac{1}{\mu_0} B_i - M_i \tag{2.6.66}$$

is introduced which relates the magnetic force vector $\vec{B}$ and magnetization vector $\vec{M}$. This is another constitutive equation which describes material properties. For an anisotropic material (crystal)

$$B_i = \mu_i^j H_j \qquad \text{and} \qquad M_i = \chi_i^j H_j \tag{2.6.67}$$

where μ_i^j is called the magnetic permeability tensor and χ_i^j is called the magnetic permeability tensor. Both of these quantities are dimensionless. For an isotropic material

$$\mu_i^j = \mu \delta_i^j \qquad \text{where} \qquad \mu = \mu_0 k_m. \tag{2.6.68}$$

Here $\mu_0 = 4\pi \times 10^{-7}\,\text{N/amp}^2$ is the permeability of free space and $k_m = \frac{\mu}{\mu_0}$ is the relative permeability coefficient. Similarly, for an isotropic material we have $\chi_i^j = \chi_m \delta_i^j$ where χ_m is called the magnetic susceptibility coefficient and is dimensionless. The magnetic susceptibility coefficient has positive values for materials called paramagnets and negative values for materials called diamagnets. For a linear medium the quantities $\vec{B}$, $\vec{M}$ and $\vec{H}$ are related by

$$B_i = \mu_0(H_i + M_i) = \mu_0 H_i + \mu_0 \chi_m H_i = \mu_0(1+\chi_m)H_i = \mu_0 k_m H_i = \mu H_i \tag{2.6.69}$$

where $\mu = \mu_0 k_m = \mu_0(1+\chi_m)$ is called the permeability of the material.

Note: The auxiliary magnetic vector $\vec{H}$ for magnetostatics in materials plays a role similar to the displacement vector $\vec{D}$ for electrostatics in materials. Be careful in using electromagnetic equations from different texts as many authors interchange the roles of $\vec{B}$ and $\vec{H}$. Some authors call $\vec{H}$ the magnetic field. However, the quantity $\vec{B}$ should be the fundamental quantity.[1]

Electrodynamics

In the nonstatic case of electrodynamics there is an additional quantity $\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$ called the polarization current which satisfies

$$\nabla \cdot \vec{J}_p = \nabla \cdot \frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t}\nabla \cdot \vec{P} = -\frac{\partial \rho_b}{\partial t} \tag{2.6.70}$$

and the current density has three parts

$$\vec{J} = \vec{J}_b + \vec{J}_f + \vec{J}_p = \nabla \times \vec{M} + \vec{J}_f + \frac{\partial \vec{P}}{\partial t} \tag{2.6.71}$$

consisting of bound, free and polarization currents.

Faraday's law states that a changing magnetic field creates an electric field. In particular, the electromagnetic force induced in a closed loop circuit C is proportional to the rate of change of flux of the magnetic field associated with any surface S connected with C. Faraday's law states

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \widehat{\mathbf{e}}_n \, d\sigma.$$

Using the Stoke's theorem, we find

$$\iint_S (\nabla \times \vec{E}) \cdot \widehat{\mathbf{e}}_n \, d\sigma = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot \widehat{\mathbf{e}}_n \, d\sigma.$$

The above equation must hold for an arbitrary surface and loop. Equating like terms we obtain the differential form of Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{2.6.72}$$

This is the first electromagnetic field equation of Maxwell.

Ampere's law, equation (2.6.65), written in terms of the total current from equation (2.6.71) , becomes

$$\nabla \times \vec{B} = \mu_0(\nabla \times \vec{M} + \vec{J}_f + \frac{\partial \vec{P}}{\partial t}) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2.6.73}$$

which can also be written as

$$\nabla \times (\frac{1}{\mu_0}\vec{B} - \vec{M}) = \vec{J}_f + \frac{\partial}{\partial t}(\vec{P} + \epsilon_0 \vec{E})$$

[1] D.J. Griffiths, Introduction to Electrodynamics, Prentice Hall, 1981. P.232.

or

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}. \tag{2.6.74}$$

This is Maxwell's second electromagnetic field equation.

To the equations (2.6.74) and (2.6.73) we add the Gauss's law for magnetization, equation (2.6.61) and Gauss's law for electrostatics, equation (2.6.48). These four equations produce the Maxwell's equations of electrodynamics and are now summarized. The general form of Maxwell's equations involve the quantities

E_i, Electric force vector, $[E_i]$ = Newton/coulomb

B_i, Magnetic force vector, $[B_i]$ = Weber/m^2

H_i, Auxilary magnetic force vector, $[H_i]$ = ampere/m

D_i, Displacement vector, $[D_i]$ = coulomb/m^2

J_i, Free current density, $[J_i]$ = ampere/m^2

P_i, Polarization vector, $[P_i]$ = coulomb/m^2

M_i, Magnetization vector, $[M_i]$ = ampere/m

for $i = 1, 2, 3$. There are also the quantities

ϱ, representing the free charge density, with units $[\varrho]$ = coulomb/m^3

ϵ_0, Permittivity of free space, $[\epsilon_0]$ = farads/m or coulomb^2/Newton $\cdot$ m^2 .

μ_0, Permeability of free space, $[\mu_0]$ = henrys/m or kg $\cdot$ m/coulomb^2

In addition, there arises the material parameters:

μ^i_j, magnetic permeability tensor, which is dimensionless

ϵ^i_j, dielectric tensor, which is dimensionless

α^i_j, electric susceptibility tensor, which is dimensionless

χ^i_j, magnetic susceptibility tensor, which is dimensionless

These parameters are used to express variations in the electric field E_i and magnetic field B_i when acting in a material medium. In particular, P_i, D_i, M_i and H_i are defined from the equations

$$\begin{aligned} D_i =& \epsilon^j_i E_j = \epsilon_0 E_i + P_i \qquad \epsilon^i_j = \epsilon_0 \delta^i_j + \alpha^j_i \\ B_i =& \mu^j_i H_j = \mu_0 H_i + \mu_0 M_i, \qquad \mu^i_j = \mu_0(\delta^i_j + \chi^i_j) \\ P_i =& \alpha^j_i E_j, \qquad \text{and} \qquad M_i = \chi^j_i H_j \quad \text{for } i = 1, 2, 3. \end{aligned}$$

The above quantities obey the following laws:

Faraday's Law This law states the line integral of the electromagnetic force around a loop is proportional to the rate of flux of magnetic induction through the loop. This gives rise to the first electromagnetic field equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{or} \qquad \epsilon^{ijk} E_{k,j} = -\frac{\partial B^i}{\partial t}. \tag{2.6.75}$$

Ampere's Law This law states the line integral of the magnetic force vector around a closed loop is proportional to the sum of the current through the loop and the rate of flux of the displacement vector through the loop. This produces the second electromagnetic field equation:

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \qquad \text{or} \qquad \epsilon^{ijk} H_{k,j} = J_f^i + \frac{\partial D^i}{\partial t}. \tag{2.6.76}$$

Gauss's Law for Electricity This law states that the flux of the electric force vector through a closed surface is proportional to the total charge enclosed by the surface. This results in the third electromagnetic field equation:

$$\nabla \cdot \vec{D} = \rho_f \qquad \text{or} \qquad D^i_{,i} = \rho_f \quad \text{or} \quad \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} D^i \right) = \rho_f. \tag{2.6.77}$$

Gauss's Law for Magnetism This law states the magnetic flux through any closed volume is zero. This produces the fourth electromagnetic field equation:

$$\nabla \cdot \vec{B} = 0 \qquad \text{or} \qquad B^i_{,i} = 0 \qquad \text{or} \quad \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} B^i \right) = 0. \tag{2.6.78}$$

When no confusion arises it is convenient to drop the subscript f from the above Maxwell equations. Special expanded forms of the above Maxwell equations are given on the pages 176 to 179.

Electromagnetic Stress and Energy

Let V denote the volume of some simple closed surface S. Let us calculate the rate at which electromagnetic energy is lost from this volume. This represents the energy flow per unit volume. Begin with the first two Maxwell's equations in Cartesian form

$$\epsilon_{ijk} E_{k,j} = -\frac{\partial B_i}{\partial t} \tag{2.6.79}$$

$$\epsilon_{ijk} H_{k,j} = J_i + \frac{\partial D_i}{\partial t}. \tag{2.6.80}$$

Now multiply equation (2.6.79) by H_i and equation (2.6.80) by E_i. This gives two terms with dimensions of energy per unit volume per unit of time which we write

$$\epsilon_{ijk} E_{k,j} H_i = -\frac{\partial B_i}{\partial t} H_i \tag{2.6.81}$$

$$\epsilon_{ijk} H_{k,j} E_i = J_i E_i + \frac{\partial D_i}{\partial t} E_i. \tag{2.6.82}$$

Subtracting equation (2.6.82) from equation (2.6.81) we find

$$\begin{aligned} \epsilon_{ijk}(E_{k,j} H_i - H_{k,j} E_i) &= -J_i E_i - \frac{\partial D_i}{\partial t} E_i - \frac{\partial B_i}{\partial t} H_i \\ \epsilon_{ijk} \left[(E_k H_i)_{,j} - E_k H_{i,j} + H_{i,j} E_k \right] &= -J_i E_i - \frac{\partial D_i}{\partial t} E_i - \frac{\partial B_i}{\partial t} H_i \end{aligned}$$

Observe that $\epsilon_{jki}(E_k H_i)_{,j}$ is the same as $\epsilon_{ijk}(E_j H_k)_{,i}$ so that the above simplifies to

$$\epsilon_{ijk}(E_j H_k)_{,i} + J_i E_i = -\frac{\partial D_i}{\partial t} E_i - \frac{\partial B_i}{\partial t} H_i. \tag{2.6.83}$$

Now integrate equation (2.6.83) over a volume and apply Gauss's divergence theorem to obtain

$$\iint_S \epsilon_{ijk} E_j H_k n_i \, d\sigma + \iiint_V J_i E_i \, d\tau = -\iiint_V \left(\frac{\partial D_i}{\partial t} E_i + \frac{\partial B_i}{\partial t} H_i\right) d\tau. \tag{2.6.84}$$

The first term in equation (2.6.84) represents the outward flow of energy across the surface enclosing the volume. The second term in equation (2.6.84) represents the loss by Joule heating and the right-hand side is the rate of decrease of stored electric and magnetic energy. The equation (2.6.84) is known as Poynting's theorem and can be written in the vector form

$$\iint_S (\vec{E} \times \vec{H}) \cdot \hat{\mathbf{n}} \, d\sigma = \iiint_V \left(-\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J}\right) d\tau. \tag{2.6.85}$$

For later use we define the quantity

$$S_i = \epsilon_{ijk} E_j H_k \quad \text{or} \quad \vec{S} = \vec{E} \times \vec{H} \quad [\text{Watts/m}^2] \tag{2.6.86}$$

as Poynting's energy flux vector and note that S_i is perpendicular to both E_i and H_i and represents units of energy density per unit time which crosses a unit surface area within the electromagnetic field.

Electromagnetic Stress Tensor

Instead of calculating energy flow per unit volume, let us calculate force per unit volume. Consider a region containing charges and currents but is free from dielectrics and magnetic materials. To obtain terms with units of force per unit volume we take the cross product of equation (2.6.79) with D_i and the cross product of equation (2.6.80) with B_i and subtract to obtain

$$-\epsilon_{irs}\epsilon_{ijk}(E_{k,j} D_s + H_{k,j} B_s) = \epsilon_{ris} J_i B_s + \epsilon_{ris}\left(\frac{\partial D_i}{\partial t} B_s + \frac{\partial B_s}{\partial t} D_i\right)$$

which simplifies using the $e - \delta$ identity to

$$-(\delta_{rj}\delta_{sk} - \delta_{rk}\delta_{sj})(E_{k,j} D_s + H_{k,j} B_s) = \epsilon_{ris} J_i B_s + \epsilon_{ris} \frac{\partial}{\partial t}(D_i B_s)$$

which further simplifies to

$$-E_{s,r} D_s + E_{r,s} D_s - H_{s,r} B_s + H_{r,s} B_s = \epsilon_{ris} J_i B_s + \frac{\partial}{\partial t}(\epsilon_{ris} D_i B_s). \tag{2.6.87}$$

Observe that the first two terms in the equation (2.6.87) can be written

$$\begin{aligned}
E_{r,s} D_s - E_{s,r} D_s =& E_{r,s} D_s - \epsilon_0 E_{s,r} E_s \\
=& (E_r D_s)_{,s} - E_r D_{s,s} - \epsilon_0 (\frac{1}{2} E_s E_s)_{,r} \\
=& (E_r D_s)_{,s} - \rho E_r - \frac{1}{2}(E_j D_j \delta_{sr})_{,s} \\
=& (E_r D_s - \frac{1}{2} E_j D_j \delta_{rs})_{,s} - \rho E_r
\end{aligned}$$

which can be expressed in the form

$$E_{r,s} D_s - E_{s,r} D_s = T^E_{rs,s} - \rho E_r$$

where

$$T^E_{rs} = E_r D_s - \frac{1}{2} E_j D_j \delta_{rs} \tag{2.6.88}$$

is called the electric stress tensor. In matrix form the stress tensor is written

$$T^E_{rs} = \begin{bmatrix} E_1D_1 - \frac{1}{2}E_jD_j & E_1D_2 & E_1D_3 \\ E_2D_1 & E_2D_2 - \frac{1}{2}E_jD_j & E_2D_3 \\ E_3D_1 & E_3D_2 & E_3D_3 - \frac{1}{2}E_jD_j \end{bmatrix}. \tag{2.6.89}$$

By performing similar calculations we can transform the third and fourth terms in the equation (2.6.87) and obtain

$$H_{r,s}B_s - H_{s,r}B_s = T^M_{rs,s} \tag{2.6.90}$$

where

$$T^M_{rs} = H_r B_S - \frac{1}{2} H_j B_j \delta_{rs} \tag{2.6.91}$$

is the magnetic stress tensor. In matrix form the magnetic stress tensor is written

$$T^M_{rs} = \begin{bmatrix} B_1H_1 - \frac{1}{2}B_jH_j & B_1H_2 & B_1H_3 \\ B_2H_1 & B_2H_2 - \frac{1}{2}B_jH_j & B_2H_3 \\ B_3H_1 & B_3H_2 & B_3H_3 - \frac{1}{2}B_jH_j \end{bmatrix}. \tag{2.6.92}$$

The total electromagnetic stress tensor is

$$T_{rs} = T^E_{rs} + T^M_{rs}. \tag{2.6.93}$$

Then the equation (2.6.87) can be written in the form

$$T_{rs,s} - \rho E_r = \epsilon_{ris} J_i B_s + \frac{\partial}{\partial t}(\epsilon_{ris} D_i B_s)$$

or

$$\rho E_r + \epsilon_{ris} J_i B_S = T_{rs,s} - \frac{\partial}{\partial t}(\epsilon_{ris} D_i B_s). \tag{2.6.94}$$

For free space $D_i = \epsilon_0 E_i$ and $B_i = \mu_0 H_i$ so that the last term of equation (2.6.94) can be written in terms of the Poynting vector as

$$\mu_0 \epsilon_0 \frac{\partial S_r}{\partial t} = \frac{\partial}{\partial t}(\epsilon_{ris} D_i B_s). \tag{2.6.95}$$

Now integrate the equation (2.6.94) over the volume to obtain the total electromagnetic force

$$\iiint_V \rho E_r \, d\tau + \iiint_V \epsilon_{ris} J_i B_s \, d\tau = \iiint_V T_{rs,s} \, d\tau - \mu_0 \epsilon_0 \iiint_V \frac{\partial S_r}{\partial t} \, d\tau.$$

Applying the divergence theorem of Gauss gives

$$\iiint_V \rho E_r \, d\tau + \iiint_V \epsilon_{ris} J_i B_s \, d\tau = \iint_S T_{rs} n_s \, d\sigma - \mu_0 \epsilon_0 \iiint_V \frac{\partial S_r}{\partial t} \, d\tau. \tag{2.6.96}$$

The left side of the equation (2.6.96) represents the forces acting on charges and currents contained within the volume element. If the electric and magnetic fields do not vary with time, then the last term on the right is zero. In this case the forces can be expressed as an integral of the electromagnetic stress tensor.

EXERCISE 2.6

► **1.** Find the field lines and equipotential curves associated with a positive charge q located at $(-a,0)$ and a positive charge q located at $(a,0)$. The field lines are illustrated in the figure 2.6-7.

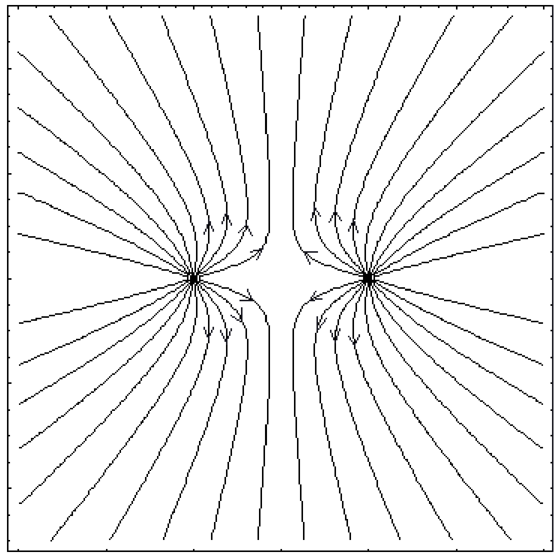

Figure 2.6-7. Lines of electric force between two charges of the same sign.

► **2.** Calculate the lines of force and equipotential curves associated with the electric field $\vec{E}=\vec{E}(x,y)=2y\,\widehat{\mathbf{e}}_1+2x\,\widehat{\mathbf{e}}_2$. Sketch the lines of force and equipotential curves. Put arrows on the lines of force to show direction of the field lines.

► **3.** A right circular cone is defined by

$$x=u\sin\theta_0\cos\phi,\qquad y=u\sin\theta_0\sin\phi,\qquad z=u\cos\theta_0$$

with $0\le\phi\le 2\pi$ and $u\ge 0$. Show the solid angle subtended by this cone is $\Omega=\frac{A}{r^2}=2\pi(1-\cos\theta_0)$.

► **4.** A charge $+q$ is located at the point $(0,a)$ and a charge $-q$ is located at the point $(0,-a)$. Show that the electric force $\vec{E}$ at the position $(x,0)$, where $x>a$ is $\vec{E}=\dfrac{1}{4\pi\epsilon_0}\dfrac{-2aq}{(a^2+x^2)^{3/2}}\,\widehat{\mathbf{e}}_2$.

► **5.** Let the circle $x^2+y^2=a^2$ carry a line charge λ^*. Show the electric field at the point $(0,0,z)$ is $\vec{E}=\dfrac{1}{4\pi\epsilon_0}\dfrac{\lambda^* az(2\pi)\,\widehat{\mathbf{e}}_3}{(a^2+z^2)^{3/2}}$.

► **6.** Use superposition to find the electric field associated with two infinite parallel plane sheets each carrying an equal but opposite sign surface charge density μ^*. Find the field between the planes and outside of each plane. Hint: Fields are of magnitude $\pm\frac{\mu^*}{2\epsilon_0}$ and perpendicular to plates.

► **7.** For a volume current $\vec{J}$ the Biot-Savart law gives $\vec{B}=\dfrac{\mu_0}{4\pi}\displaystyle\iiint_V\dfrac{\vec{J}\times\widehat{\mathbf{e}}_r}{r^2}\,d\tau$. Show that $\nabla\cdot\vec{B}=0$. Hint: Let $\widehat{\mathbf{e}}_r=\dfrac{\vec{r}}{r}$ and consider $\nabla\cdot(\vec{J}\times\dfrac{\vec{r}}{r^3})$. Then use numbers 13 and 10 of the appendix C. Also note that $\nabla\times\vec{J}=0$ because $\vec{J}$ does not depend upon position.

▶ **8.** A homogeneous dielectric is defined by D_i and E_i having parallel vector fields. Show that for a homogeneous dielectric $\epsilon^j_{i,k} = 0$.

▶ **9.** Show that for a homogeneous, isotropic dielectric medium that ϵ is a constant.

▶ **10.** Show that for a homogeneous, isotropic linear dielectric in Cartesian coordinates

$$P_{i,i} = \frac{\alpha_e}{1+\alpha_e}\rho_f.$$

▶ **11.** Verify the Maxwell's equations in Gaussian units for a charge free isotropic homogeneous dielectric.

$$\begin{aligned} \nabla \cdot \vec{E} =& \frac{1}{\epsilon}\nabla \cdot \vec{D} = 0 & \qquad \nabla \times \vec{E} =& -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} \\ \nabla \cdot \vec{B} =& \mu \nabla \vec{H} = 0 & \qquad \nabla \times \vec{H} =& \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J} = \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\sigma \vec{E} \end{aligned}$$

▶ **12.** Verify the Maxwell's equations in Gaussian units for an isotropic homogeneous dielectric with a charge.

$$\begin{aligned} \nabla \cdot \vec{D} =& 4\pi\rho & \qquad \nabla \times \vec{E} =& -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} =& 0 & \qquad \nabla \times \vec{H} =& \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \end{aligned}$$

▶ **13.** For a volume charge ρ in an element of volume $d\tau$ located at a point (ξ, η, ζ) Coulombs law is

$$\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho}{r^2}\,\widehat{\mathbf{e}}_r\,d\tau$$

(a) Show that $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$.

(b) Show that $\widehat{\mathbf{e}}_r = \frac{1}{r}\left((x-\xi)\,\widehat{\mathbf{e}}_1 + (y-\eta)\,\widehat{\mathbf{e}}_2 + (z-\zeta)\,\widehat{\mathbf{e}}_3\right)$.

(c) Show that
$$\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{(x-\xi)\,\widehat{\mathbf{e}}_1 + (y-\eta)\,\widehat{\mathbf{e}}_2 + (z-\zeta)\,\widehat{\mathbf{e}}_3}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}}\rho\,d\xi d\eta d\zeta = \frac{1}{4\pi\epsilon_0}\iiint_V \nabla\left(\frac{\widehat{\mathbf{e}}_r}{r^2}\right)\rho\,d\xi d\eta d\zeta$$

(d) Show that the potential function for $\vec{E}$ is $\mathcal{V} = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\xi,\eta,\zeta)}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}}\,d\xi d\eta d\zeta$

(e) Show that $\vec{E} = -\nabla\mathcal{V}$.

(f) Show that $\nabla^2\mathcal{V} = -\frac{\rho}{\epsilon}$ Hint: Note that the integrand is zero everywhere except at the point where $(\xi,\eta,\zeta) = (x,y,z)$. Consider the integral split into two regions. One region being a small sphere about the point (x,y,z) in the limit as the radius of this sphere approaches zero. Observe the identity $\nabla_{(x,y,z)}\left(\frac{\widehat{\mathbf{e}}_r}{r^2}\right) = -\nabla(\xi,\eta,\zeta)\left(\frac{\widehat{\mathbf{e}}_r}{r^2}\right)$ enables one to employ the Gauss divergence theorem to obtain a surface integral. Use a mean value theorem to show $-\frac{\rho}{4\pi\epsilon_0}\iint_S \frac{\widehat{\mathbf{e}}_r}{r^2}\cdot \hat{n}\,dS = \frac{\rho}{4\pi\epsilon_0}4\pi$ since $\hat{n} = -\,\widehat{\mathbf{e}}_r$.

▶ **14.** Show that for a point charge in space $\rho^* = q\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$, where δ is the Dirac delta function, the equation (2.6.5) can be reduced to the equation (2.6.1).

▶ **15.**

(a) Show the electric field $\vec{E} = \frac{1}{r^2}\,\widehat{\mathbf{e}}_r$ is irrotational. Here $\widehat{\mathbf{e}}_r = \frac{\vec{r}}{r}$ is a unit vector in the direction of r.

(b) Find the potential function $\mathcal{V}$ such that $\vec{E} = -\nabla\mathcal{V}$ which satisfies $\mathcal{V}(r_0) = 0$ for $r_0 > 0$.

▶ **16.**

(a) If $\vec{E}$ is a conservative electric field such that $\vec{E} = -\nabla \mathcal{V}$, then show that $\vec{E}$ is irrotational and satisfies $\nabla \times \vec{E} = \operatorname{curl} \vec{E} = 0$.

(b) If $\nabla \times \vec{E} = \operatorname{curl} \vec{E} = 0$, show that $\vec{E}$ is conservative. (i.e. Show $\vec{E} = -\nabla \mathcal{V}$.)
Hint: The work done on a test charge $Q = 1$ along the straight line segments from (x_0, y_0, z_0) to (x, y_0, z_0) and then from (x, y_0, z_0) to (x, y, z_0) and finally from (x, y, z_0) to (x, y, z) can be written

$$\mathcal{V} = \mathcal{V}(x,y,z) = -\int_{x_0}^{x} E_1(x, y_0, z_0)\, dx - \int_{y_0}^{y} E_2(x, y, z_0)\, dy - \int_{z_0}^{z} E_3(x, y, z)\, dz.$$

Now note that

$$\frac{\partial \mathcal{V}}{\partial y} = -E_2(x, y, z_0) - \int_{z_0}^{z} \frac{\partial E_3(x,y,z)}{\partial y}\, dz$$

and from $\nabla \times \vec{E} = 0$ we find $\dfrac{\partial E_3}{\partial y} = \dfrac{\partial E_2}{\partial z}$, which implies $\dfrac{\partial \mathcal{V}}{\partial y} = -E_2(x, y, z)$. Similar results are obtained for $\dfrac{\partial \mathcal{V}}{\partial x}$ and $\dfrac{\partial \mathcal{V}}{\partial z}$. Hence show $-\nabla \mathcal{V} = \vec{E}$.

▶ **17.**

(a) Show that if $\nabla \cdot \vec{B} = 0$, then there exists some vector field $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$.
The vector field $\vec{A}$ is called the vector potential of $\vec{B}$.
Hint: Let $\vec{A}(x,y,z) = \displaystyle\int_0^1 s\vec{B}(sx, sy, sz) \times \vec{r}\, ds$ where $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3$
and integrate $\displaystyle\int_0^1 \frac{dB_i}{ds} s^2\, ds$ by parts.

(b) Show that $\nabla \cdot (\nabla \times \vec{A}) = 0$.

▶ **18.** Use Faraday's law and Ampere's law to show

$$g^{im}(E^j_{\ ,j})_{,m} - g^{jm}E^i_{\ ,mj} = -\mu_0 \frac{\partial}{\partial t}\left[J^i + \epsilon_0 \frac{\partial E^i}{\partial t}\right]$$

▶ **19.** Assume that $\vec{J} = \sigma \vec{E}$ where σ is the conductivity. Show that for $\rho = 0$ Maxwell's equations produce

$$\begin{aligned} \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} &= \nabla^2 \vec{E} \\ \text{and} \qquad \mu_0 \sigma \frac{\partial \vec{B}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} &= \nabla^2 \vec{B}. \end{aligned}$$

Here both $\vec{E}$ and $\vec{B}$ satisfy the same equation which is known as the telegrapher's equation.

▶ **20.** Show that Maxwell's equations (2.6.75) through (2.6.78) for the electric field under electrostatic conditions reduce to

$$\begin{aligned} \nabla \times \vec{E} &= 0 \\ \nabla \cdot \vec{D} &= \rho_f \end{aligned}$$

Now $\vec{E}$ is irrotational so that $\vec{E} = -\nabla \mathcal{V}$. Show that $\nabla^2 \mathcal{V} = -\dfrac{\rho_f}{\epsilon}$.

▶ **21.** Show that Maxwell's equations (2.6.75) through (2.6.78) for the magnetic field under magnetostatic conditions reduce to $\nabla \times \vec{H} = \vec{J}$ and $\nabla \cdot \vec{B} = 0$. The divergence of $\vec{B}$ being zero implies $\vec{B}$ can be derived from a vector potential function $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$. Here $\vec{A}$ is not unique, see problem 24. If we select $\vec{A}$ such that $\nabla \cdot \vec{A} = 0$ then show for a homogeneous, isotropic material, free of any permanent magnets, that $\nabla^2 \vec{A} = -\mu \vec{J}$.

▶ **22.** Show that under nonsteady state conditions of electrodynamics the Faraday law from Maxwell's equations (2.6.75) through (2.6.78) does not allow one to set $\vec{E} = -\nabla \mathcal{V}$. Why is this? Observe that $\nabla \cdot \vec{B} = 0$ so we can write $\vec{B} = \nabla \times \vec{A}$ for some vector potential $\vec{A}$. Using this vector potential show that Faraday's law can be written $\nabla \times \left(\vec{E} + \dfrac{\partial \vec{A}}{\partial t} \right) = 0$. This shows that the quantity inside the parenthesis is conservative and so we can write $\vec{E} + \dfrac{\partial \vec{A}}{\partial t} = -\nabla \mathcal{V}$ for some scalar potential $\mathcal{V}$. The representation

$$\vec{E} = -\nabla \mathcal{V} - \frac{\partial \vec{A}}{\partial t}$$

is a more general representation of the electric potential. Observe that for steady state conditions $\frac{\partial \vec{A}}{\partial t} = 0$ so that this potential representation reduces to the previous one for electrostatics.

▶ **23.** Using the potential formulation $\vec{E} = -\nabla \mathcal{V} - \dfrac{\partial \vec{A}}{\partial t}$ derived in problem 22, show that in a vacuum

(a) Gauss law can be written $\nabla^2 \mathcal{V} + \dfrac{\partial \nabla \cdot \vec{A}}{\partial t} = -\dfrac{\rho}{\epsilon_0}$

(b) Ampere's law can be written

$$\nabla \times \left(\nabla \times \vec{A} \right) = \mu_0 \vec{J} - \mu_0 \epsilon_0 \nabla \left(\frac{\partial \mathcal{V}}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

(c) Show the result in part (b) can also be expressed in the form

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial \vec{A}}{\partial t} \right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \mathcal{V}}{\partial t} \right) = -\mu_0 \vec{J}$$

▶ **24.** The Maxwell equations in a vacuum have the form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \rho \vec{V} \qquad \nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$

where $\vec{D} = \epsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$ with ϵ_0 and μ_0 constants satisfying $\epsilon_0 \mu_0 = 1/c^2$ where c is the speed of light. Introduce the vector potential $\vec{A}$ and scalar potential $\mathcal{V}$ defined by $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \nabla \mathcal{V}$. Note that the vector potential is not unique. For example, given ψ as a scalar potential we can write $\vec{B} = \nabla \times \vec{A} = \nabla \times (\vec{A} + \nabla \psi)$, since the curl of a gradient is zero. Therefore, it is customary to impose some kind of additional requirement on the potentials. These additional conditions are such that $\vec{E}$ and $\vec{B}$ are not changed. One such condition is that $\vec{A}$ and $\mathcal{V}$ satisfy $\nabla \cdot \vec{A} + \dfrac{1}{c^2} \dfrac{\partial \mathcal{V}}{\partial t} = 0$. This relation is known as the Lorentz relation or Lorentz gauge. Find the Maxwell's equations in a vacuum in terms of $\vec{A}$ and $\mathcal{V}$ and show that

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathcal{V} = -\frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{A} = -\mu_0 \rho \vec{V}.$$

▶ **25.** In a vacuum show that $\vec{E}$ and $\vec{B}$ satisfy

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \qquad \nabla \cdot \vec{E} = 0 \qquad \nabla \vec{B} = 0$$

▶ **26.**

(a) Show that the wave equations in problem 25 have solutions in the form of waves traveling in the x- direction given by

$$\vec{E} = \vec{E}(x,t) = \vec{E}_0 e^{i(kx \pm \omega t)} \qquad \text{and} \qquad \vec{B} = \vec{B}(x,t) = \vec{B}_0 e^{i(kx \pm \omega t)}$$

where $\vec{E}_0$ and $\vec{B}_0$ are constants. Note that wave functions of the form $u = Ae^{i(kx \pm \omega t)}$ are called plane harmonic waves. Sometimes they are called monochromatic waves. Here $i^2 = -1$ is an imaginary unit. Euler's identity shows that the real and imaginary parts of these type wave functions have the form

$$A\cos(kx \pm \omega t) \qquad \text{and} \qquad A\sin(kx \pm \omega t).$$

These represent plane waves. The constant A is the amplitude of the wave , ω is the angular frequency, and $k/2\pi$ is called the wave number. The motion is a simple harmonic motion both in time and space. That is, at a fixed point x the motion is simple harmonic in time and at a fixed time t, the motion is harmonic in space. By examining each term in the sine and cosine terms we find that x has dimensions of length, k has dimension of reciprocal length, t has dimensions of time and ω has dimensions of reciprocal time or angular velocity. The quantity $c = \omega/k$ is the wave velocity. The value $\lambda = 2\pi/k$ has dimension of length and is called the wavelength and $1/\lambda$ is called the wave number. The wave number represents the number of waves per unit of distance along the x-axis. The period of the wave is $T = \lambda/c = 2\pi/\omega$ and the frequency is $f = 1/T$. The frequency represents the number of waves which pass a fixed point in a unit of time.

(b) Show that $\omega = 2\pi f$

(c) Show that $c = f\lambda$

(d) Is the wave motion $u = \sin(kx - \omega t) + \sin(kx + \omega t)$ a traveling wave? Explain.

(e) Show that in general the wave equation $\nabla^2 \phi = \dfrac{1}{c^2}\dfrac{\partial^2 \phi}{\partial t^2}$ have solutions in the form of waves traveling in either the $+x$ or $-x$ direction given by

$$\phi = \phi(x,t) = f(x+ct) + g(x-ct)$$

where f and g are arbitrary twice differentiable functions.

(f) Assume a plane electromagnetic wave is moving in the $+x$ direction. Show that the electric field is in the $xy-$plane and the magnetic field is in the $xz-$plane.
Hint: Assume solutions $E_x = g_1(x-ct)$, $E_y = g_2(x-ct), E_z = g_3(x-ct), B_x = g_4(x-ct)$, $B_y = g_5(x-ct), B_z = g_6(x-ct)$ where $g_i, i = 1, ..., 6$ are arbitrary functions. Then show that E_x does not satisfy $\nabla \cdot \vec{E} = 0$ which implies g_1 must be independent of x and so not a wave function. Do the same for the components of $\vec{B}$. Since both $\nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0$ then $E_x = B_x = 0$. Such waves are called transverse waves because the electric and magnetic fields are perpendicular to the direction of propagation. Faraday's law implies that the $\vec{E}$ and $\vec{B}$ waves must be in phase and be mutually perpendicular to each other.