§1.2 TENSOR CONCEPTS AND TRANSFORMATIONS

For $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$, $\widehat{\mathbf{e}}_3$ independent orthogonal unit vectors (base vectors), we may write any vector $\vec{A}$ as

$$\vec{A} = A_1\,\widehat{\mathbf{e}}_1 + A_2\,\widehat{\mathbf{e}}_2 + A_3\,\widehat{\mathbf{e}}_3$$

where (A_1, A_2, A_3) are the coordinates of $\vec{A}$ relative to the base vectors chosen. These components are the projection of $\vec{A}$ onto the base vectors and

$$\vec{A} = (\vec{A}\cdot\widehat{\mathbf{e}}_1)\,\widehat{\mathbf{e}}_1 + (\vec{A}\cdot\widehat{\mathbf{e}}_2)\,\widehat{\mathbf{e}}_2 + (\vec{A}\cdot\widehat{\mathbf{e}}_3)\,\widehat{\mathbf{e}}_3.$$

Select any three independent orthogonal vectors, $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$, not necessarily of unit length, we can then write

$$\widehat{\mathbf{e}}_1 = \frac{\vec{E}_1}{|\vec{E}_1|}, \qquad \widehat{\mathbf{e}}_2 = \frac{\vec{E}_2}{|\vec{E}_2|}, \qquad \widehat{\mathbf{e}}_3 = \frac{\vec{E}_3}{|\vec{E}_3|},$$

and consequently, the vector $\vec{A}$ can be expressed as

$$\vec{A} = \left(\frac{\vec{A}\cdot\vec{E}_1}{\vec{E}_1\cdot\vec{E}_1}\right)\vec{E}_1 + \left(\frac{\vec{A}\cdot\vec{E}_2}{\vec{E}_2\cdot\vec{E}_2}\right)\vec{E}_2 + \left(\frac{\vec{A}\cdot\vec{E}_3}{\vec{E}_3\cdot\vec{E}_3}\right)\vec{E}_3.$$

Here we say that

$$\frac{\vec{A}\cdot\vec{E}_{(i)}}{\vec{E}_{(i)}\cdot\vec{E}_{(i)}}, \quad i = 1, 2, 3$$

are the components of $\vec{A}$ relative to the chosen base vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$. Recall that the parenthesis about the subscript i denotes that there is no summation on this subscript. It is then treated as a free subscript which can have any of the values $1, 2$ or 3.

Reciprocal Basis

Consider a set of any three independent vectors $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ which are not necessarily orthogonal, nor of unit length. In order to represent the vector $\vec{A}$ in terms of these vectors we must find components (A^1, A^2, A^3) such that

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3.$$

This can be done by taking appropriate projections and obtaining three equations and three unknowns from which the components are determined. A much easier way to find the components (A^1, A^2, A^3) is to construct a reciprocal basis $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$. Recall that two bases $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ and $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$ are said to be reciprocal if they satisfy the condition

$$\vec{E}_i\cdot\vec{E}^j = \delta_i^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Note that $\vec{E}_2\cdot\vec{E}^1 = \delta_2^1 = 0$ and $\vec{E}_3\cdot\vec{E}^1 = \delta_3^1 = 0$ so that the vector $\vec{E}^1$ is perpendicular to both the vectors $\vec{E}_2$ and $\vec{E}_3$. (i.e. A vector from one basis is orthogonal to two of the vectors from the other basis.) We can therefore write $\vec{E}^1 = V^{-1}\vec{E}_2\times\vec{E}_3$ where V is a constant to be determined. By taking the dot product of both sides of this equation with the vector $\vec{E}_1$ we find that $V = \vec{E}_1\cdot(\vec{E}_2\times\vec{E}_3)$ is the volume of the parallelepiped formed by the three vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$ when their origins are made to coincide. In a similar manner it can be demonstrated that for $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ a given set of basis vectors, then the reciprocal basis vectors are determined from the relations

$$\vec{E}^1 = \frac{1}{V}\vec{E}_2 \times \vec{E}_3, \qquad \vec{E}^2 = \frac{1}{V}\vec{E}_3 \times \vec{E}_1, \qquad \vec{E}^3 = \frac{1}{V}\vec{E}_1 \times \vec{E}_2,$$

where $V = \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3) \neq 0$ is a triple scalar product and represents the volume of the parallelepiped having the basis vectors for its sides.

Let $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ and $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$ denote a system of reciprocal bases. We can represent any vector $\vec{A}$ with respect to either of these bases. If we select the basis $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ and represent $\vec{A}$ in the form

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3, \tag{1.2.1}$$

then the components (A^1, A^2, A^3) of $\vec{A}$ relative to the basis vectors $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ are called the contravariant components of $\vec{A}$. These components can be determined from the equations

$$\vec{A} \cdot \vec{E}^1 = A^1, \qquad \vec{A} \cdot \vec{E}^2 = A^2, \qquad \vec{A} \cdot \vec{E}^3 = A^3.$$

Similarly, if we choose the reciprocal basis $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$ and represent $\vec{A}$ in the form

$$\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3, \tag{1.2.2}$$

then the components (A_1, A_2, A_3) relative to the basis $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$ are called the covariant components of $\vec{A}$. These components can be determined from the relations

$$\vec{A} \cdot \vec{E}_1 = A_1, \qquad \vec{A} \cdot \vec{E}_2 = A_2, \qquad \vec{A} \cdot \vec{E}_3 = A_3.$$

The contravariant and covariant components are different ways of representing the same vector with respect to a set of reciprocal basis vectors. There is a simple relationship between these components which we now develop. We introduce the notation

$$\vec{E}_i \cdot \vec{E}_j = g_{ij} = g_{ji}, \qquad \text{and} \qquad \vec{E}^i \cdot \vec{E}^j = g^{ij} = g^{ji} \tag{1.2.3}$$

where g_{ij} are called the metric components of the space and g^{ij} are called the conjugate metric components of the space. We can then write

$$\vec{A} \cdot \vec{E}_1 = A_1(\vec{E}^1 \cdot \vec{E}_1) + A_2(\vec{E}^2 \cdot \vec{E}_1) + A_3(\vec{E}^3 \cdot \vec{E}_1) = A_1$$
$$\vec{A} \cdot \vec{E}_1 = A^1(\vec{E}_1 \cdot \vec{E}_1) + A^2(\vec{E}_2 \cdot \vec{E}_1) + A^3(\vec{E}_3 \cdot \vec{E}_1) = A_1$$

or

$$A_1 = A^1 g_{11} + A^2 g_{12} + A^3 g_{13}. \tag{1.2.4}$$

In a similar manner, by considering the dot products $\vec{A} \cdot \vec{E}_2$ and $\vec{A} \cdot \vec{E}_3$ one can establish the results

$$A_2 = A^1 g_{21} + A^2 g_{22} + A^3 g_{23} \qquad A_3 = A^1 g_{31} + A^2 g_{32} + A^3 g_{33}.$$

These results can be expressed with the index notation as

$$A_i = g_{ik}A^k. \tag{1.2.6}$$

Forming the dot products $\vec{A} \cdot \vec{E}^1, \quad \vec{A} \cdot \vec{E}^2, \quad \vec{A} \cdot \vec{E}^3$ it can be verified that

$$A^i = g^{ik}A_k. \tag{1.2.7}$$

The equations (1.2.6) and (1.2.7) are relations which exist between the contravariant and covariant components of the vector $\vec{A}$. Similarly, if for some value j we have $\vec{E}^j = \alpha\,\vec{E}_1 + \beta\,\vec{E}_2 + \gamma\,\vec{E}_3$, then one can show that $\vec{E}^j = g^{ij}\vec{E}_i$. This is left as an exercise.

Coordinate Transformations

Consider a coordinate transformation from a set of coordinates (x, y, z) to (u, v, w) defined by a set of transformation equations

$$\begin{aligned} x &= x(u, v, w) \\ y &= y(u, v, w) \\ z &= z(u, v, w) \end{aligned} \tag{1.2.8}$$

It is assumed that these transformations are single valued, continuous and possess the inverse transformation

$$\begin{aligned} u &= u(x, y, z) \\ v &= v(x, y, z) \\ w &= w(x, y, z). \end{aligned} \tag{1.2.9}$$

These transformation equations define a set of coordinate surfaces and coordinate curves. The coordinate surfaces are defined by the equations

$$\begin{aligned} u(x, y, z) &= c_1 \\ v(x, y, z) &= c_2 \\ w(x, y, z) &= c_3 \end{aligned} \tag{1.2.10}$$

where c_1, c_2, c_3 are constants. These surfaces intersect in the coordinate curves

$$\vec{r}(u, c_2, c_3), \qquad \vec{r}(c_1, v, c_3), \qquad \vec{r}(c_1, c_2, w), \tag{1.2.11}$$

where

$$\vec{r}(u, v, w) = x(u, v, w)\,\widehat{\mathbf{e}}_1 + y(u, v, w)\,\widehat{\mathbf{e}}_2 + z(u, v, w)\,\widehat{\mathbf{e}}_3.$$

The general situation is illustrated in the figure 1.2-1.

Consider the vectors

$$\vec{E}^1 = \operatorname{grad} u = \nabla u, \qquad \vec{E}^2 = \operatorname{grad} v = \nabla v, \qquad \vec{E}^3 = \operatorname{grad} w = \nabla w \tag{1.2.12}$$

evaluated at the common point of intersection (c_1, c_2, c_3) of the coordinate surfaces. The system of vectors $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$ can be selected as a system of basis vectors which are normal to the coordinate surfaces. Similarly, the vectors

$$\vec{E}_1 = \frac{\partial \vec{r}}{\partial u}, \qquad \vec{E}_2 = \frac{\partial \vec{r}}{\partial v}, \qquad \vec{E}_3 = \frac{\partial \vec{r}}{\partial w} \tag{1.2.13}$$

when evaluated at the common point of intersection (c_1, c_2, c_3) forms a system of vectors $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ which we can select as a basis. This basis is a set of tangent vectors to the coordinate curves. It is now demonstrated that the normal basis $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$ and the tangential basis $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ are a set of reciprocal bases.

Recall that $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3$ denotes the position vector of a variable point. By substitution for x, y, z from (1.2.8) there results

$$\vec{r} = \vec{r}(u, v, w) = x(u, v, w)\,\widehat{\mathbf{e}}_1 + y(u, v, w)\,\widehat{\mathbf{e}}_2 + z(u, v, w)\,\widehat{\mathbf{e}}_3. \tag{1.2.14}$$

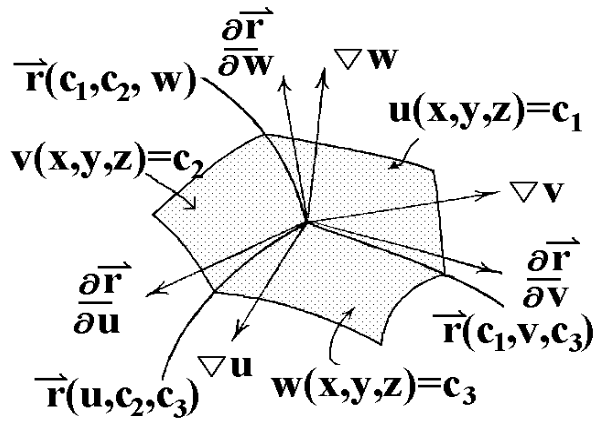

Figure 1.2-1. Coordinate curves and coordinate surfaces.

A small change in $\vec{r}$ is denoted

$$d\vec{r} = dx\,\widehat{\mathbf{e}}_1 + dy\,\widehat{\mathbf{e}}_2 + dz\,\widehat{\mathbf{e}}_3 = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv + \frac{\partial \vec{r}}{\partial w}\,dw \tag{1.2.15}$$

where

$$\begin{aligned}\frac{\partial \vec{r}}{\partial u} &= \frac{\partial x}{\partial u}\,\widehat{\mathbf{e}}_1 + \frac{\partial y}{\partial u}\,\widehat{\mathbf{e}}_2 + \frac{\partial z}{\partial u}\,\widehat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial v} &= \frac{\partial x}{\partial v}\,\widehat{\mathbf{e}}_1 + \frac{\partial y}{\partial v}\,\widehat{\mathbf{e}}_2 + \frac{\partial z}{\partial v}\,\widehat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial w} &= \frac{\partial x}{\partial w}\,\widehat{\mathbf{e}}_1 + \frac{\partial y}{\partial w}\,\widehat{\mathbf{e}}_2 + \frac{\partial z}{\partial w}\,\widehat{\mathbf{e}}_3.\end{aligned} \tag{1.2.16}$$

In terms of the u,v,w coordinates, this change can be thought of as moving along the diagonal of a parallelepiped having the vector sides $\frac{\partial \vec{r}}{\partial u}\,du$, $\frac{\partial \vec{r}}{\partial v}\,dv$, and $\frac{\partial \vec{r}}{\partial w}\,dw$.

Assume $u = u(x,y,z)$ is defined by equation (1.2.9) and differentiate this relation to obtain

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz. \tag{1.2.17}$$

The equation (1.2.15) enables us to represent this differential in the form:

$$\begin{aligned} du &= \operatorname{grad} u \cdot d\vec{r}\\ du &= \operatorname{grad} u \cdot \left(\frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv + \frac{\partial \vec{r}}{\partial w}\,dw\right)\\ du &= \left(\operatorname{grad} u \cdot \frac{\partial \vec{r}}{\partial u}\right)du + \left(\operatorname{grad} u \cdot \frac{\partial \vec{r}}{\partial v}\right)dv + \left(\operatorname{grad} u \cdot \frac{\partial \vec{r}}{\partial w}\right)dw.\end{aligned} \tag{1.2.18}$$

By comparing like terms in this last equation we find that

$$\vec{E}^1 \cdot \vec{E}_1 = 1, \qquad \vec{E}^1 \cdot \vec{E}_2 = 0, \qquad \vec{E}^1 \cdot \vec{E}_3 = 0. \tag{1.2.19}$$

Similarly, from the other equations in equation (1.2.9) which define $v = v(x,y,z)$, and $w = w(x,y,z)$ it can be demonstrated that

$$dv = \left(\operatorname{grad} v \cdot \frac{\partial \vec{r}}{\partial u}\right)du + \left(\operatorname{grad} v \cdot \frac{\partial \vec{r}}{\partial v}\right)dv + \left(\operatorname{grad} v \cdot \frac{\partial \vec{r}}{\partial w}\right)dw \tag{1.2.20}$$

and

$$dw = \left(\operatorname{grad} w \cdot \frac{\partial \vec{r}}{\partial u}\right) du + \left(\operatorname{grad} w \cdot \frac{\partial \vec{r}}{\partial v}\right) dv + \left(\operatorname{grad} w \cdot \frac{\partial \vec{r}}{\partial w}\right) dw. \tag{1.2.21}$$

By comparing like terms in equations (1.2.20) and (1.2.21) we find

$$\begin{aligned} &\vec{E}^2 \cdot \vec{E}_1 = 0, \qquad \vec{E}^2 \cdot \vec{E}_2 = 1, \qquad \vec{E}^2 \cdot \vec{E}_3 = 0 \\ &\vec{E}^3 \cdot \vec{E}_1 = 0, \qquad \vec{E}^3 \cdot \vec{E}_2 = 0, \qquad \vec{E}^3 \cdot \vec{E}_3 = 1. \end{aligned} \tag{1.2.22}$$

The equations (1.2.22) and (1.2.19) show us that the basis vectors defined by equations (1.2.12) and (1.2.13) are reciprocal.

Introducing the notation

$$(x^1, x^2, x^3) = (u, v, w) \qquad (y^1, y^2, y^3) = (x, y, z) \tag{1.2.23}$$

where the $x's$ denote the generalized coordinates and the $y's$ denote the rectangular Cartesian coordinates, the above equations can be expressed in a more concise form with the index notation. For example, if

$$x^i = x^i(x, y, z) = x^i(y^1, y^2, y^3), \quad \text{and} \quad y^i = y^i(u, v, w) = y^i(x^1, x^2, x^3), \quad i = 1, 2, 3 \tag{1.2.24}$$

then the reciprocal basis vectors can be represented

$$\vec{E}^i = \operatorname{grad} x^i, \qquad i = 1, 2, 3 \tag{1.2.25}$$

and

$$\vec{E}_i = \frac{\partial \vec{r}}{\partial x^i}, \qquad i = 1, 2, 3. \tag{1.2.26}$$

We now show that these basis vectors are reciprocal. Observe that $\vec{r} = \vec{r}(x^1, x^2, x^3)$ with

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x^m}\, dx^m \tag{1.2.27}$$

and consequently

$$dx^i = \operatorname{grad} x^i \cdot d\vec{r} = \operatorname{grad} x^i \cdot \frac{\partial \vec{r}}{\partial x^m}\, dx^m = \left(\vec{E}^i \cdot \vec{E}_m\right) dx^m = \delta^i_m\, dx^m, \qquad i = 1, 2, 3 \tag{1.2.28}$$

Comparing like terms in this last equation establishes the result that

$$\vec{E}^i \cdot \vec{E}_m = \delta^i_m, \quad i, m = 1, 2, 3 \tag{1.2.29}$$

which demonstrates that the basis vectors are reciprocal.

Scalars, Vectors and Tensors

Tensors are quantities which obey certain transformation laws. That is, scalars, vectors, matrices and higher order arrays can be thought of as components of a tensor quantity. We shall be interested in finding how these components are represented in various coordinate systems. We desire knowledge of these transformation laws in order that we can represent various physical laws in a form which is independent of the coordinate system chosen. Before defining different types of tensors let us examine what we mean by a coordinate transformation.

Coordinate transformations of the type found in equations (1.2.8) and (1.2.9) can be generalized to higher dimensions. Let x^i, $i = 1, 2, \ldots, N$ denote N variables. These quantities can be thought of as representing a variable point $(x^1, x^2, \ldots, x^N)$ in an N dimensional space V_N. Another set of N quantities, call them barred quantities, $\overline{x}^i$, $i = 1, 2, \ldots, N$, can be used to represent a variable point $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N)$ in an N dimensional space $\overline{V}_N$. When the $x's$ are related to the $\overline{x}'s$ by equations of the form

$$x^i = x^i(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N), \quad i = 1, 2, \ldots, N \tag{1.2.30}$$

then a transformation is said to exist between the coordinates x^i and $\overline{x}^i$, $i = 1, 2, \ldots, N$. Whenever the relations (1.2.30) are functionally independent, single valued and possess partial derivatives such that the Jacobian of the transformation

$$J\left(\frac{x}{\overline{x}}\right) = J\left(\frac{x^1, x^2, \ldots, x^N}{\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N}\right) = \begin{vmatrix} \frac{\partial x^1}{\partial \overline{x}^1} & \frac{\partial x^1}{\partial \overline{x}^2} & \cdots & \frac{\partial x^1}{\partial \overline{x}^N} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x^N}{\partial \overline{x}^1} & \frac{\partial x^N}{\partial \overline{x}^2} & \cdots & \frac{\partial x^N}{\partial \overline{x}^N} \end{vmatrix} \tag{1.2.31}$$

is different from zero, then there exists an inverse transformation

$$\overline{x}^i = \overline{x}^i(x^1, x^2, \ldots, x^N), \quad i = 1, 2, \ldots, N. \tag{1.2.32}$$

For brevity the transformation equations (1.2.30) and (1.2.32) are sometimes expressed by the notation

$$x^i = x^i(\overline{x}),\ i = 1, \ldots, N \qquad \text{and} \qquad \overline{x}^i = \overline{x}^i(x),\ i = 1, \ldots, N. \tag{1.2.33}$$

Consider a sequence of transformations from x to $\bar{x}$ and then from $\bar{x}$ to $\bar{\bar{x}}$ coordinates. For simplicity let $\bar{x} = y$ and $\bar{\bar{x}} = z$. If we denote by T_1, T_2 and T_3 the transformations

$$\begin{aligned} T_1: &\qquad y^i = y^i(x^1, \ldots, x^N) \quad i = 1, \ldots, N \quad \text{or} \quad T_1 x = y \\ T_2: &\qquad z^i = z^i(y^1, \ldots, y^N) \quad i = 1, \ldots, N \quad \text{or} \quad T_2 y = z \end{aligned}$$

Then the transformation T_3 obtained by substituting T_1 into T_2 is called the product of two successive transformations and is written

$$T_3: \qquad z^i = z^i(y^1(x^1, \ldots, x^N), \ldots, y^N(x^1, \ldots, x^N)) \quad i = 1, \ldots, N \quad \text{or} \quad T_3 x = T_2 T_1 x = z.$$

This product transformation is denoted symbolically by $T_3 = T_2 T_1$.

The Jacobian of the product transformation is equal to the product of Jacobians associated with the product transformation and $J_3 = J_2 J_1$.

Transformations Form a Group

A group G is a nonempty set of elements together with a law, for combining the elements. The combined elements are denoted by a product. Thus, if a and b are elements in G then no matter how you define the law for combining elements, the product combination is denoted ab. The set G and combining law forms a group if the following properties are satisfied:

(i) For all $a, b \in G$, then $ab \in G$. This is called the closure property.

(ii) There exists an identity element I such that for all $a \in G$ we have $Ia = aI = a$.

(iii) There exists an inverse element. That is, for all $a \in G$ there exists an inverse element a^{-1} such that $a\,a^{-1} = a^{-1}a = I$.

(iv) The associative law holds under the combining law and $a(bc) = (ab)c$ for all $a, b, c \in G$.

For example, the set of elements $G = \{1, -1, i, -i\}$, where $i^2 = -1$ together with the combining law of ordinary multiplication, forms a group. This can be seen from the multiplication table.

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
-i	-i	i	1	-1
i	i	-i	-1	1

The set of all coordinate transformations of the form found in equation (1.2.30), with Jacobian different from zero, forms a group because:

(i) The product transformation, which consists of two successive transformations, belongs to the set of transformations. (closure)

(ii) The identity transformation exists in the special case that $\overline{x}$ and x are the same coordinates.

(iii) The inverse transformation exists because the Jacobian of each individual transformation is different from zero.

(iv) The associative law is satisfied in that the transformations satisfy the property $T_3(T_2T_1) = (T_3T_2)T_1$.

When the given transformation equations contain a parameter the combining law is often times represented as a product of symbolic operators. For example, we denote by T_α a transformation of coordinates having a parameter α. The inverse transformation can be denoted by T_α^{-1} and one can write $T_\alpha x = \overline{x}$ or $x = T_\alpha^{-1}\overline{x}$. We let T_β denote the same transformation, but with a parameter β, then the transitive property is expressed symbolically by $T_\alpha T_\beta = T_\gamma$ where the product $T_\alpha T_\beta$ represents the result of performing two successive transformations. The first coordinate transformation uses the given transformation equations and uses the parameter α in these equations. This transformation is then followed by another coordinate transformation using the same set of transformation equations, but this time the parameter value is β. The above symbolic product is used to demonstrate that the result of applying two successive transformations produces a result which is equivalent to performing a single transformation of coordinates having the parameter value γ. Usually some relationship can then be established between the parameter values α, β and γ.

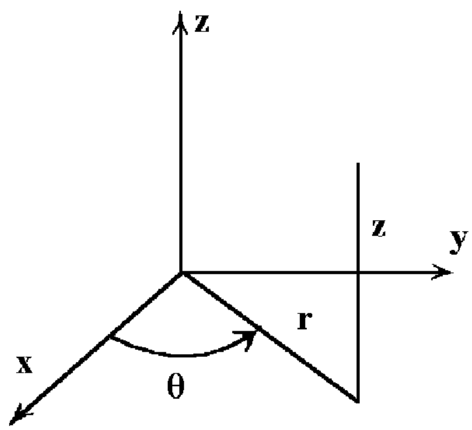

Figure 1.2-2. Cylindrical coordinates.

In this symbolic notation, we let T_θ denote the identity transformation. That is, using the parameter value of θ in the given set of transformation equations produces the identity transformation. The inverse transformation can then be expressed in the form of finding the parameter value β such that $T_\alpha T_\beta = T_\theta$.

Cartesian Coordinates

At times it is convenient to introduce an orthogonal Cartesian coordinate system having coordinates $y^i, \quad i = 1, 2, \ldots, N$. This space is denoted E_N and represents an N-dimensional Euclidean space. Whenever the generalized independent coordinates x^i, $i = 1, \ldots, N$ are functions of the $y's$, and these equations are functionally independent, then there exists independent transformation equations

$$y^i = y^i(x^1, x^2, \ldots, x^N), \quad i = 1, 2, \ldots, N, \tag{1.2.34}$$

with Jacobian different from zero. Similarly, if there is some other set of generalized coordinates, say a barred system $\overline{x}^i$, $i = 1, \ldots, N$ where the $\overline{x}'s$ are independent functions of the $y's$, then there will exist another set of independent transformation equations

$$y^i = y^i(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N), \quad i = 1, 2, \ldots, N, \tag{1.2.35}$$

with Jacobian different from zero. The transformations found in the equations (1.2.34) and (1.2.35) imply that there exists relations between the $x's$ and $\overline{x}'s$ of the form (1.2.30) with inverse transformations of the form (1.2.32). It should be remembered that the concepts and ideas developed in this section can be applied to a space V_N of any finite dimension. Two dimensional surfaces ($N = 2$) and three dimensional spaces ($N = 3$) will occupy most of our applications. In relativity, one must consider spaces where $N = 4$.

EXAMPLE 1.2-1. (cylindrical coordinates (r, θ, z)**)** Consider the transformation

$$x = x(r, \theta, z) = r\cos\theta \qquad y = y(r, \theta, z) = r\sin\theta \qquad z = z(r, \theta, z) = z$$

from rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z), illustrated in the figure 1.2-2. By letting

$$y^1 = x, \quad y^2 = y, \quad y^3 = z \qquad x^1 = r, \quad x^2 = \theta, \quad x^3 = z$$

the above set of equations are examples of the transformation equations (1.2.8) with $u = r$, $v = \theta$, $w = z$ as the generalized coordinates.

■

EXAMPLE 1.2.2. (Spherical Coordinates) (ρ, θ, ϕ)

Consider the transformation

$$x = x(\rho, \theta, \phi) = \rho \sin\theta \cos\phi \qquad y = y(\rho, \theta, \phi) = \rho \sin\theta \sin\phi \qquad z = z(\rho, \theta, \phi) = \rho \cos\theta$$

from rectangular coordinates (x, y, z) to spherical coordinates (ρ, θ, ϕ). By letting

$$y^1 = x,\ y^2 = y,\ y^3 = z \qquad x^1 = \rho,\ x^2 = \theta\,, x^3 = \phi$$

the above set of equations has the form found in equation (1.2.8) with $u = \rho$, $v = \theta$, $w = \phi$ the generalized coordinates. One could place bars over the $x's$ in this example in order to distinguish these coordinates from the $x's$ of the previous example. The spherical coordinates (ρ, θ, ϕ) are illustrated in the figure 1.2-3.

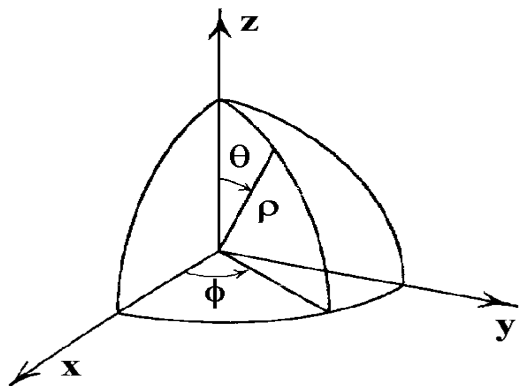

Figure 1.2-3. Spherical coordinates.

■

<u>**Scalar Functions and Invariance**</u>

We are now at a point where we can begin to define what tensor quantities are. The first definition is for a scalar invariant or tensor of order zero.

Definition: (Absolute scalar field) Assume there exists a coordinate transformation of the type (1.2.30) with Jacobian J different from zero. Let the scalar function

$$f = f(x^1, x^2, \dots, x^N) \tag{1.2.36}$$

be a function of the coordinates x^i, $i = 1, \dots, N$ in a space V_N. Whenever there exists a function

$$\overline{f} = \overline{f}(\overline{x}^1, \overline{x}^2, \dots, \overline{x}^N) \tag{1.2.37}$$

which is a function of the coordinates $\overline{x}^i$, $i = 1, \dots, N$ such that $\overline{f} = J^W f$, then f is called a tensor of rank or order zero of weight W in the space V_N. Whenever $W = 0$, the scalar f is called the component of an absolute scalar field and is referred to as an absolute tensor of rank or order zero.

That is, an absolute scalar field is an invariant object in the space V_N with respect to the group of coordinate transformations. It has a single component in each coordinate system. For any scalar function of the type defined by equation (1.2.36), we can substitute the transformation equations (1.2.30) and obtain

$$f = f(x^1, \dots, x^N) = f(x^1(\overline{x}), \dots, x^N(\overline{x})) = \overline{f}(\overline{x}^1, \dots, \overline{x}^N). \tag{1.2.38}$$

Vector Transformation, Contravariant Components

In V_N consider a curve C defined by the set of parametric equations

$$C: \quad x^i = x^i(t), \quad i = 1, \dots, N$$

where t is a parameter. The tangent vector to the curve C is the vector

$$\vec{T} = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \dots, \frac{dx^N}{dt} \right).$$

In index notation, which focuses attention on the components, this tangent vector is denoted

$$T^i = \frac{dx^i}{dt}, \qquad i = 1, \dots, N.$$

For a coordinate transformation of the type defined by equation (1.2.30) with its inverse transformation defined by equation (1.2.32), the curve C is represented in the barred space by

$$\overline{x}^i = \overline{x}^i(x^1(t), x^2(t), \dots, x^N(t)) = \overline{x}^i(t), \quad i = 1, \dots, N,$$

with t unchanged. The tangent to the curve in the barred system of coordinates is represented by

$$\frac{d\overline{x}^i}{dt} = \frac{\partial \overline{x}^i}{\partial x^j} \frac{dx^j}{dt}, \quad i = 1, \dots, N. \tag{1.2.39}$$

Letting $\overline{T}^i$, $i = 1, \ldots, N$ denote the components of this tangent vector in the barred system of coordinates, the equation (1.2.39) can then be expressed in the form

$$\overline{T}^i = \frac{\partial \overline{x}^i}{\partial x^j} T^j, \quad i, j = 1, \ldots, N. \tag{1.2.40}$$

This equation is said to define the transformation law associated with an absolute contravariant tensor of rank or order one. In the case $N = 3$ the matrix form of this transformation is represented

$$\begin{pmatrix} \overline{T}^1 \\ \overline{T}^2 \\ \overline{T}^3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \overline{x}^1}{\partial x^1} & \frac{\partial \overline{x}^1}{\partial x^2} & \frac{\partial \overline{x}^1}{\partial x^3} \\ \frac{\partial \overline{x}^2}{\partial x^1} & \frac{\partial \overline{x}^2}{\partial x^2} & \frac{\partial \overline{x}^2}{\partial x^3} \\ \frac{\partial \overline{x}^3}{\partial x^1} & \frac{\partial \overline{x}^3}{\partial x^2} & \frac{\partial \overline{x}^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} T^1 \\ T^2 \\ T^3 \end{pmatrix} \tag{1.2.41}$$

A more general definition is

Definition: (Contravariant tensor) Whenever N quantities A^i in a coordinate system $(x^1, \ldots, x^N)$ are related to N quantities $\overline{A}^i$ in a coordinate system $(\overline{x}^1, \ldots, \overline{x}^N)$ such that the Jacobian J is different from zero, then if the transformation law

$$\overline{A}^i = J^W \frac{\partial \overline{x}^i}{\partial x^j} A^j$$

is satisfied, these quantities are called the components of a relative tensor of rank or order one with weight W. Whenever $W = 0$ these quantities are called the components of an absolute tensor of rank or order one.

We see that the above transformation law satisfies the group properties.

EXAMPLE 1.2-3. (Transitive Property of Contravariant Transformation)

Show that successive contravariant transformations is also a contravariant transformation.

Solution: Consider the transformation of a vector from an unbarred to a barred system of coordinates. A vector or absolute tensor of rank one $A^i = A^i(x)$, $i = 1, \ldots, N$ will transform like the equation (1.2.40) and

$$\overline{A}^i(\overline{x}) = \frac{\partial \overline{x}^i}{\partial x^j} A^j(x). \tag{1.2.42}$$

Another transformation from $\overline{x} \to \overline{\overline{x}}$ coordinates will produce the components

$$\overline{\overline{A}}^i(\overline{\overline{x}}) = \frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^j} \overline{A}^j(\overline{x}) \tag{1.2.43}$$

Here we have used the notation $A^j(x)$ to emphasize the dependence of the components A^j upon the x coordinates. Changing indices and substituting equation (1.2.42) into (1.2.43) we find

$$\overline{\overline{A}}^i(\overline{\overline{x}}) = \frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^j} \frac{\partial \overline{x}^j}{\partial x^m} A^m(x). \tag{1.2.44}$$

From the fact that

$$\frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial x^m} = \frac{\partial \overline{\overline{x}}^i}{\partial x^m},$$

the equation (1.2.44) simplifies to

$$\overline{\overline{A}}^i(\overline{\overline{x}}) = \frac{\partial \overline{\overline{x}}^i}{\partial x^m} A^m(x) \tag{1.2.45}$$

and hence this transformation is also contravariant. We express this by saying that the above are transitive with respect to the group of coordinate transformations.

Note that from the chain rule one can write

$$\frac{\partial x^m}{\partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial x^n} = \frac{\partial x^m}{\partial \overline{x}^1}\frac{\partial \overline{x}^1}{\partial x^n} + \frac{\partial x^m}{\partial \overline{x}^2}\frac{\partial \overline{x}^2}{\partial x^n} + \frac{\partial x^m}{\partial \overline{x}^3}\frac{\partial \overline{x}^3}{\partial x^n} = \frac{\partial x^m}{\partial x^n} = \delta^m_n.$$

Do not make the mistake of writing

$$\frac{\partial x^m}{\partial \overline{x}^2}\frac{\partial \overline{x}^2}{\partial x^n} = \frac{\partial x^m}{\partial x^n} \qquad \text{or} \qquad \frac{\partial x^m}{\partial \overline{x}^3}\frac{\partial \overline{x}^3}{\partial x^n} = \frac{\partial x^m}{\partial x^n}$$

as these expressions are incorrect. Note that there are no summations in these terms, whereas there is a summation index in the representation of the chain rule.

■

Vector Transformation, Covariant Components

Consider a scalar invariant $A(x) = \overline{A}(\overline{x})$ which is a shorthand notation for the equation

$$A(x^1, x^2, \dots, x^n) = \overline{A}(\overline{x}^1, \overline{x}^2, \dots, \overline{x}^n)$$

involving the coordinate transformation of equation (1.2.30). By the chain rule we differentiate this invariant and find that the components of the gradient must satisfy

$$\frac{\partial \overline{A}}{\partial \overline{x}^i} = \frac{\partial A}{\partial x^j}\frac{\partial x^j}{\partial \overline{x}^i}. \tag{1.2.46}$$

Let

$$A_j = \frac{\partial A}{\partial x^j} \qquad \text{and} \qquad \overline{A}_i = \frac{\partial \overline{A}}{\partial \overline{x}^i},$$

then equation (1.2.46) can be expressed as the transformation law

$$\overline{A}_i = A_j \frac{\partial x^j}{\partial \overline{x}^i}. \tag{1.2.47}$$

This is the transformation law for an absolute covariant tensor of rank or order one. A more general definition is

Definition: (Covariant tensor) Whenever N quantities A_i in a coordinate system $(x^1, \ldots, x^N)$ are related to N quantities $\overline{A}_i$ in a coordinate system $(\overline{x}^1, \ldots, \overline{x}^N)$, with Jacobian J different from zero, such that the transformation law

$$\overline{A}_i = J^W \frac{\partial x^j}{\partial \overline{x}^i} A_j \tag{1.2.48}$$

is satisfied, then these quantities are called the components of a relative covariant tensor of rank or order one having a weight of W. Whenever $W = 0$, these quantities are called the components of an absolute covariant tensor of rank or order one.

Again we note that the above transformation satisfies the group properties. Absolute tensors of rank or order one are referred to as vectors while absolute tensors of rank or order zero are referred to as scalars.

EXAMPLE 1.2-4. (Transitive Property of Covariant Transformation)

Consider a sequence of transformation laws of the type defined by the equation (1.2.47)

$$\begin{array}{ll} x \to \overline{x} & \qquad \overline{A}_i(\overline{x}) = A_j(x) \dfrac{\partial x^j}{\partial \overline{x}^i} \\ \overline{x} \to \overline{\overline{x}} & \qquad \overline{\overline{A}}_k(\overline{\overline{x}}) = \overline{A}_m(\overline{x}) \dfrac{\partial \overline{x}^m}{\partial \overline{\overline{x}}^k} \end{array}$$

We can therefore express the transformation of the components associated with the coordinate transformation $x \to \overline{\overline{x}}$ and

$$\overline{\overline{A}}_k(\overline{\overline{x}}) = \left(A_j(x) \frac{\partial x^j}{\partial \overline{x}^m} \right) \frac{\partial \overline{x}^m}{\partial \overline{\overline{x}}^k} = A_j(x) \frac{\partial x^j}{\partial \overline{\overline{x}}^k},$$

which demonstrates the transitive property of a covariant transformation.

■

Higher Order Tensors

We have shown that first order tensors are quantities which obey certain transformation laws. Higher order tensors are defined in a similar manner and also satisfy the group properties. We assume that we are given transformations of the type illustrated in equations (1.2.30) and (1.2.32) which are single valued and continuous with Jacobian J different from zero. Further, the quantities x^i and $\overline{x}^i$, $i = 1, \ldots, n$ represent the coordinates in any two coordinate systems. The following transformation laws define second order and third order tensors.

Definition: (Second order contravariant tensor) Whenever N-squared quantities A^{ij} in a coordinate system $(x^1, \ldots, x^N)$ are related to N-squared quantities $\overline{A}^{mn}$ in a coordinate system $(\overline{x}^1, \ldots, \overline{x}^N)$ such that the transformation law

$$\overline{A}^{mn}(\overline{x}) = A^{ij}(x) J^W \frac{\partial \overline{x}^m}{\partial x^i} \frac{\partial \overline{x}^n}{\partial x^j} \tag{1.2.49}$$

is satisfied, then these quantities are called components of a relative contravariant tensor of rank or order two with weight W. Whenever $W = 0$ these quantities are called the components of an absolute contravariant tensor of rank or order two.

Definition: (Second order covariant tensor) Whenever N-squared quantities A_{ij} in a coordinate system $(x^1, \ldots, x^N)$ are related to N-squared quantities $\overline{A}_{mn}$ in a coordinate system $(\overline{x}^1, \ldots, \overline{x}^N)$ such that the transformation law

$$\overline{A}_{mn}(\overline{x}) = A_{ij}(x) J^W \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n} \tag{1.2.50}$$

is satisfied, then these quantities are called components of a relative covariant tensor of rank or order two with weight W. Whenever $W = 0$ these quantities are called the components of an absolute covariant tensor of rank or order two.

Definition: (Second order mixed tensor) Whenever N-squared quantities A^i_j in a coordinate system $(x^1, \ldots, x^N)$ are related to N-squared quantities $\overline{A}^m_n$ in a coordinate system $(\overline{x}^1, \ldots, \overline{x}^N)$ such that the transformation law

$$\overline{A}^m_n(\overline{x}) = A^i_j(x) J^W \frac{\partial \overline{x}^m}{\partial x^i} \frac{\partial x^j}{\partial \overline{x}^n} \tag{1.2.51}$$

is satisfied, then these quantities are called components of a relative mixed tensor of rank or order two with weight W. Whenever $W = 0$ these quantities are called the components of an absolute mixed tensor of rank or order two. It is contravariant of order one and covariant of order one.

Higher order tensors are defined in a similar manner. For example, if we can find N-cubed quantities A^m_{np} such that

$$\overline{A}^i_{jk}(\overline{x}) = A^\gamma_{\alpha\beta}(x) J^W \frac{\partial \overline{x}^i}{\partial x^\gamma} \frac{\partial x^\alpha}{\partial \overline{x}^j} \frac{\partial x^\beta}{\partial \overline{x}^k} \tag{1.2.52}$$

then this is a relative mixed tensor of order three with weight W. It is contravariant of order one and covariant of order two.

General Definition

In general a mixed tensor of rank or order $(m+n)$

$$T^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n} \tag{1.2.53}$$

is contravariant of order m and covariant of order n if it obeys the transformation law

$$\overline{T}^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n} = \left[J\left(\frac{x}{\overline{x}}\right)\right]^W T^{a_1 a_2 \ldots a_m}_{b_1 b_2 \ldots b_n} \frac{\partial \overline{x}^{i_1}}{\partial x^{a_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{a_2}} \cdots \frac{\partial \overline{x}^{i_m}}{\partial x^{a_m}} \cdot \frac{\partial x^{b_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{b_2}}{\partial \overline{x}^{j_2}} \cdots \frac{\partial x^{b_n}}{\partial \overline{x}^{j_n}} \tag{1.2.54}$$

where

$$J\left(\frac{x}{\overline{x}}\right) = \left|\frac{\partial x}{\partial \overline{x}}\right| = \frac{\partial(x^1, x^2, \ldots, x^N)}{\partial(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N)}$$

is the Jacobian of the transformation. When $W = 0$ the tensor is called an absolute tensor, otherwise it is called a relative tensor of weight W.

Here superscripts are used to denote contravariant components and subscripts are used to denote covariant components. Thus, if we are given the tensor components in one coordinate system, then the components in any other coordinate system are determined by the transformation law of equation (1.2.54). Throughout the remainder of this text one should treat all tensors as absolute tensors unless specified otherwise.

Dyads and Polyads

Note that vectors can be represented in bold face type with the notation

$$\mathbf{A} = A_i \mathbf{E}^i$$

This notation can also be generalized to tensor quantities. Higher order tensors can also be denoted by bold face type. For example the tensor components T_{ij} and B_{ijk} can be represented in terms of the basis vectors $\mathbf{E}^i, i = 1, \ldots, N$ by using a notation which is similar to that for the representation of vectors. For example,

$$\mathbf{T} = T_{ij} \mathbf{E}^i \mathbf{E}^j$$
$$\mathbf{B} = B_{ijk} \mathbf{E}^i \mathbf{E}^j \mathbf{E}^k.$$

Here $\mathbf{T}$ denotes a tensor with components T_{ij} and $\mathbf{B}$ denotes a tensor with components B_{ijk}. The quantities $\mathbf{E}^i\mathbf{E}^j$ are called unit dyads and $\mathbf{E}^i\mathbf{E}^j\mathbf{E}^k$ are called unit triads. There is no multiplication sign between the basis vectors. This notation is called a polyad notation. A further generalization of this notation is the representation of an arbitrary tensor using the basis and reciprocal basis vectors in bold type. For example, a mixed tensor would have the polyadic representation

$$\mathbf{T} = T^{ij\ldots k}_{lm\ldots n} \mathbf{E}_i \mathbf{E}_j \ldots \mathbf{E}_k \mathbf{E}^l \mathbf{E}^m \ldots \mathbf{E}^n.$$

A dyadic is formed by the outer or direct product of two vectors. For example, the outer product of the vectors

$$\mathbf{a} = a_1 \mathbf{E}^1 + a_2 \mathbf{E}^2 + a_3 \mathbf{E}^3 \quad \text{and} \quad \mathbf{b} = b_1 \mathbf{E}^1 + b_2 \mathbf{E}^2 + b_3 \mathbf{E}^3$$

gives the dyad

$$\begin{aligned}\mathbf{ab} =&a_1b_1\mathbf{E}^1\mathbf{E}^1 + a_1b_2\mathbf{E}^1\mathbf{E}^2 + a_1b_3\mathbf{E}^1\mathbf{E}^3\\ &a_2b_1\mathbf{E}^2\mathbf{E}^1 + a_2b_2\mathbf{E}^2\mathbf{E}^2 + a_2b_3\mathbf{E}^2\mathbf{E}^3\\ &a_3b_1\mathbf{E}^3\mathbf{E}^1 + a_3b_2\mathbf{E}^3\mathbf{E}^2 + a_3b_3\mathbf{E}^3\mathbf{E}^3.\end{aligned}$$

In general, a dyad can be represented

$$\mathbf{A} = A_{ij}\mathbf{E}^i\mathbf{E}^j \qquad i,j = 1,\dots,N$$

where the summation convention is in effect for the repeated indices. The coefficients A_{ij} are called the coefficients of the dyad. When the coefficients are written as an $N \times N$ array it is called a matrix. Every second order tensor can be written as a linear combination of dyads. The dyads form a basis for the second order tensors. As the example above illustrates, the nine dyads $\{\mathbf{E}^1\mathbf{E}^1\,, \mathbf{E}^1\mathbf{E}^2, \dots, \mathbf{E}^3\mathbf{E}^3\,\}$, associated with the outer products of three dimensional base vectors, constitute a basis for the second order tensor $\mathbf{A} = \mathbf{ab}$ having the components $A_{ij} = a_ib_j$ with $i,j = 1,2,3$. Similarly, a triad has the form

$$\mathbf{T} = T_{ijk}\mathbf{E}^i\mathbf{E}^j\mathbf{E}^k \quad \text{Sum on repeated indices}$$

where i,j,k have the range $1,2,\dots,N$. The set of outer or direct products $\{\,\mathbf{E}^i\mathbf{E}^j\mathbf{E}^k\,\}$, with $i,j,k = 1,\dots,N$ constitutes a basis for all third order tensors. Tensor components with mixed suffixes like C^i_{jk} are associated with triad basis of the form

$$\mathbf{C} = C^i_{jk}\mathbf{E}_i\mathbf{E}^j\mathbf{E}^k$$

where i,j,k have the range $1,2,\dots N$. Dyads are associated with the outer product of two vectors, while triads, tetrads,... are associated with higher-order outer products. These higher-order outer or direct products are referred to as polyads.

The polyad notation is a generalization of the vector notation. The subject of how polyad components transform between coordinate systems is the subject of tensor calculus.

In Cartesian coordinates we have $\mathbf{E}^i = \mathbf{E}_i = \widehat{\mathbf{e}}_i$ and a dyadic with components called dyads is written $\mathbf{A} = A_{ij}\,\widehat{\mathbf{e}}_i\,\widehat{\mathbf{e}}_j$ or

$$\begin{aligned}\mathbf{A} =&A_{11}\,\widehat{\mathbf{e}}_1\,\widehat{\mathbf{e}}_1 + A_{12}\,\widehat{\mathbf{e}}_1\,\widehat{\mathbf{e}}_2 + A_{13}\,\widehat{\mathbf{e}}_1\,\widehat{\mathbf{e}}_3\\ &A_{21}\,\widehat{\mathbf{e}}_2\,\widehat{\mathbf{e}}_1 + A_{22}\,\widehat{\mathbf{e}}_2\,\widehat{\mathbf{e}}_2 + A_{23}\,\widehat{\mathbf{e}}_2\,\widehat{\mathbf{e}}_3\\ &A_{31}\,\widehat{\mathbf{e}}_3\,\widehat{\mathbf{e}}_1 + A_{32}\,\widehat{\mathbf{e}}_3\,\widehat{\mathbf{e}}_2 + A_{33}\,\widehat{\mathbf{e}}_3\,\widehat{\mathbf{e}}_3\end{aligned}$$

where the terms $\widehat{\mathbf{e}}_i\,\widehat{\mathbf{e}}_j$ are called unit dyads. Note that a dyadic has nine components as compared with a vector which has only three components. The conjugate dyadic $\mathbf{A}_c$ is defined by a transposition of the unit vectors in $\mathbf{A}$, to obtain

$$\begin{aligned}\mathbf{A}_c =&A_{11}\,\widehat{\mathbf{e}}_1\,\widehat{\mathbf{e}}_1 + A_{12}\,\widehat{\mathbf{e}}_2\,\widehat{\mathbf{e}}_1 + A_{13}\,\widehat{\mathbf{e}}_3\,\widehat{\mathbf{e}}_1\\ &A_{21}\,\widehat{\mathbf{e}}_1\,\widehat{\mathbf{e}}_2 + A_{22}\,\widehat{\mathbf{e}}_2\,\widehat{\mathbf{e}}_2 + A_{23}\,\widehat{\mathbf{e}}_3\,\widehat{\mathbf{e}}_2\\ &A_{31}\,\widehat{\mathbf{e}}_1\,\widehat{\mathbf{e}}_3 + A_{32}\,\widehat{\mathbf{e}}_2\,\widehat{\mathbf{e}}_3 + A_{33}\,\widehat{\mathbf{e}}_3\,\widehat{\mathbf{e}}_3\end{aligned}$$

If a dyadic equals its conjugate $\mathbf{A} = \mathbf{A}_c$, then $A_{ij} = A_{ji}$ and the dyadic is called symmetric. If a dyadic equals the negative of its conjugate $\mathbf{A} = -\mathbf{A}_c$, then $A_{ij} = -A_{ji}$ and the dyadic is called skew-symmetric. A special dyadic called the identical dyadic or idemfactor is defined by

$$J = \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_1 + \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_2 + \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_3.$$

This dyadic has the property that pre or post dot product multiplication of J with a vector $\vec{V}$ produces the same vector $\vec{V}$. For example,

$$\begin{aligned} \vec{V} \cdot J &= (V_1 \widehat{\mathbf{e}}_1 + V_2 \widehat{\mathbf{e}}_2 + V_3 \widehat{\mathbf{e}}_3) \cdot J \\ &= V_1 \widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_1 + V_2 \widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_2 + V_3 \widehat{\mathbf{e}}_3 \cdot \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_3 = \vec{V} \\ \text{and} \quad J \cdot \vec{V} &= J \cdot (V_1 \widehat{\mathbf{e}}_1 + V_2 \widehat{\mathbf{e}}_2 + V_3 \widehat{\mathbf{e}}_3) \\ &= V_1 \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_1 + V_2 \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_2 + V_3 \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_3 \cdot \widehat{\mathbf{e}}_3 = \vec{V} \end{aligned}$$

A dyadic operation often used in physics and chemistry is the double dot product $\mathbf{A} : \mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are both dyadics. Here both dyadics are expanded using the distributive law of multiplication, and then each unit dyad pair $\widehat{\mathbf{e}}_i \widehat{\mathbf{e}}_j : \widehat{\mathbf{e}}_m \widehat{\mathbf{e}}_n$ are combined according to the rule

$$\widehat{\mathbf{e}}_i \widehat{\mathbf{e}}_j : \widehat{\mathbf{e}}_m \widehat{\mathbf{e}}_n = (\widehat{\mathbf{e}}_i \cdot \widehat{\mathbf{e}}_m)(\widehat{\mathbf{e}}_j \cdot \widehat{\mathbf{e}}_n).$$

For example, if $\mathbf{A} = A_{ij} \widehat{\mathbf{e}}_i \widehat{\mathbf{e}}_j$ and $\mathbf{B} = B_{ij} \widehat{\mathbf{e}}_i \widehat{\mathbf{e}}_j$, then the double dot product $\mathbf{A} : \mathbf{B}$ is calculated as follows.

$$\begin{aligned} \mathbf{A} : \mathbf{B} = (A_{ij} \widehat{\mathbf{e}}_i \widehat{\mathbf{e}}_j) : (B_{mn} \widehat{\mathbf{e}}_m \widehat{\mathbf{e}}_n) &= A_{ij} B_{mn} (\widehat{\mathbf{e}}_i \widehat{\mathbf{e}}_j : \widehat{\mathbf{e}}_m \widehat{\mathbf{e}}_n) = A_{ij} B_{mn} (\widehat{\mathbf{e}}_i \cdot \widehat{\mathbf{e}}_m)(\widehat{\mathbf{e}}_j \cdot \widehat{\mathbf{e}}_n) \\ &= A_{ij} B_{mn} \delta_{im} \delta_{jn} = A_{mj} B_{mj} \\ &= A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} \\ &+ A_{21} B_{21} + A_{22} B_{22} + A_{23} B_{23} \\ &+ A_{31} B_{31} + A_{32} B_{32} + A_{33} B_{33} \end{aligned}$$

When operating with dyads, triads and polyads, there is a definite order to the way vectors and polyad components are represented. For example, for $\vec{A} = A_i \widehat{\mathbf{e}}_i$ and $\vec{B} = B_i \widehat{\mathbf{e}}_i$ vectors with outer product

$$\vec{A}\vec{B} = A_m B_n \widehat{\mathbf{e}}_m \widehat{\mathbf{e}}_n = \phi$$

there is produced the dyadic ϕ with components $A_m B_n$. In comparison, the outer product

$$\vec{B}\vec{A} = B_m A_n \widehat{\mathbf{e}}_m \widehat{\mathbf{e}}_n = \psi$$

produces the dyadic ψ with components $B_m A_n$. That is

$$\begin{aligned} \phi = \vec{A}\vec{B} = & A_1 B_1 \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_1 + A_1 B_2 \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_2 + A_1 B_3 \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_3 \\ & A_2 B_1 \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_1 + A_2 B_2 \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_2 + A_2 B_3 \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_3 \\ & A_3 B_1 \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_1 + A_3 B_2 \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_2 + A_3 B_3 \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_3 \\ \text{and} \quad \psi = \vec{B}\vec{A} = & B_1 A_1 \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_1 + B_1 A_2 \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_2 + B_1 A_3 \widehat{\mathbf{e}}_1 \widehat{\mathbf{e}}_3 \\ & B_2 A_1 \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_1 + B_2 A_2 \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_2 + B_2 A_3 \widehat{\mathbf{e}}_2 \widehat{\mathbf{e}}_3 \\ & B_3 A_1 \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_1 + B_3 A_2 \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_2 + B_3 A_3 \widehat{\mathbf{e}}_3 \widehat{\mathbf{e}}_3 \end{aligned}$$

are different dyadics.

The scalar dot product of a dyad with a vector $\vec{C}$ is defined for both pre and post multiplication as

$$\begin{aligned} \phi \cdot \vec{C} = \vec{A}\vec{B} \cdot \vec{C} = & \vec{A}(\vec{B} \cdot \vec{C}) \\ \vec{C} \cdot \phi = \vec{C} \cdot \vec{A}\vec{B} = & (\vec{C} \cdot \vec{A})\vec{B} \end{aligned}$$

These products are, in general, not equal.

Operations Using Tensors

The following are some important tensor operations which are used to derive special equations and to prove various identities.

Addition and Subtraction

Tensors of the same type and weight can be added or subtracted. For example, two third order mixed tensors, when added, produce another third order mixed tensor. Let A^i_{jk} and B^i_{jk} denote two third order mixed tensors. Their sum is denoted

$$C^i_{jk} = A^i_{jk} + B^i_{jk}.$$

That is, like components are added. The sum is also a mixed tensor as we now verify. By hypothesis A^i_{jk} and B^i_{jk} are third order mixed tensors and hence must obey the transformation laws

$$\begin{aligned}\overline{A}^i_{jk} &= A^m_{np}\frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial x^n}{\partial \overline{x}^j}\frac{\partial x^p}{\partial \overline{x}^k}\\ \overline{B}^i_{jk} &= B^m_{np}\frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial x^n}{\partial \overline{x}^j}\frac{\partial x^p}{\partial \overline{x}^k}.\end{aligned}$$

We let $\overline{C}^i_{jk} = \overline{A}^i_{jk} + \overline{B}^i_{jk}$ denote the sum in the transformed coordinates. Then the addition of the above transformation equations produces

$$\overline{C}^i_{jk} = \left(\overline{A}^i_{jk} + \overline{B}^i_{jk}\right) = \left(A^m_{np} + B^m_{np}\right)\frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial x^n}{\partial \overline{x}^j}\frac{\partial x^p}{\partial \overline{x}^k} = C^m_{np}\frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial x^n}{\partial \overline{x}^j}\frac{\partial x^p}{\partial \overline{x}^k}.$$

Consequently, the sum transforms as a mixed third order tensor.

Multiplication (Outer Product)

The product of two tensors is also a tensor. The rank or order of the resulting tensor is the sum of the ranks of the tensors occurring in the multiplication. As an example, let A^i_{jk} denote a mixed third order tensor and let B^l_m denote a mixed second order tensor. The outer product of these two tensors is the fifth order tensor

$$C^{il}_{jkm} = A^i_{jk}B^l_m,\ i,j,k,l,m = 1,2,\ldots,N.$$

Here all indices are free indices as i,j,k,l,m take on any of the integer values $1,2,\ldots,N$. Let $\overline{A}^i_{jk}$ and $\overline{B}^l_m$ denote the components of the given tensors in the barred system of coordinates. We define $\overline{C}^{il}_{jkm}$ as the outer product of these components. Observe that C^{il}_{jkm} is a tensor for by hypothesis A^i_{jk} and B^l_m are tensors and hence obey the transformation laws

$$\begin{aligned}\overline{A}^\alpha_{\beta\gamma} &= A^i_{jk}\frac{\partial \overline{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^\beta}\frac{\partial x^k}{\partial \overline{x}^\gamma}\\ \overline{B}^\delta_\epsilon &= B^l_m\frac{\partial \overline{x}^\delta}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^\epsilon}.\end{aligned}\tag{1.2.55}$$

The outer product of these components produces

$$\begin{aligned}\overline{C}^{\alpha\delta}_{\beta\gamma\epsilon} = \overline{A}^\alpha_{\beta\gamma}\overline{B}^\delta_\epsilon &= A^i_{jk}B^l_m\frac{\partial \overline{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^\beta}\frac{\partial x^k}{\partial \overline{x}^\gamma}\frac{\partial \overline{x}^\delta}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^\epsilon}\\ &= C^{il}_{jkm}\frac{\partial \overline{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^\beta}\frac{\partial x^k}{\partial \overline{x}^\gamma}\frac{\partial \overline{x}^\delta}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^\epsilon}\end{aligned}\tag{1.2.56}$$

which demonstrates that C^{il}_{jkm} transforms as a mixed fifth order absolute tensor. Other outer products are analyzed in a similar way.

Contraction

The operation of contraction on any mixed tensor of rank m is performed when an upper index is set equal to a lower index and the summation convention is invoked. When the summation is performed over the repeated indices the resulting quantity is also a tensor of rank or order $(m-2)$. For example, let A^i_{jk}, $i,j,k = 1,2,\ldots,N$ denote a mixed tensor and perform a contraction by setting j equal to i. We obtain

$$A^i_{ik} = A^1_{1k} + A^2_{2k} + \cdots + A^N_{Nk} = A_k \tag{1.2.57}$$

where k is a free index. To show that A_k is a tensor, we let $\overline{A}^i_{ik} = \overline{A}_k$ denote the contraction on the transformed components of A^i_{jk}. By hypothesis A^i_{jk} is a mixed tensor and hence the components must satisfy the transformation law

$$\overline{A}^i_{jk} = A^m_{np} \frac{\partial \overline{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \overline{x}^j} \frac{\partial x^p}{\partial \overline{x}^k}.$$

Now execute a contraction by setting j equal to i and perform a summation over the repeated index. We find

$$\begin{aligned} \overline{A}^i_{ik} = \overline{A}_k = A^m_{np} \frac{\partial \overline{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \overline{x}^i} \frac{\partial x^p}{\partial \overline{x}^k} &= A^m_{np} \frac{\partial x^n}{\partial x^m} \frac{\partial x^p}{\partial \overline{x}^k} \\ = A^m_{np} \delta^n_m \frac{\partial x^p}{\partial \overline{x}^k} = A^n_{np} \frac{\partial x^p}{\partial \overline{x}^k} &= A_p \frac{\partial x^p}{\partial \overline{x}^k}. \end{aligned} \tag{1.2.58}$$

Hence, the contraction produces a tensor of rank two less than the original tensor. Contractions on other mixed tensors can be analyzed in a similar manner.

New tensors can be constructed from old tensors by performing a contraction on an upper and lower index. This process can be repeated as long as there is an upper and lower index upon which to perform the contraction. Each time a contraction is performed the rank of the resulting tensor is two less than the rank of the original tensor.

Multiplication (Inner Product)

The inner product of two tensors is obtained by:
(i) first taking the outer product of the given tensors and
(ii) performing a contraction on two of the indices.

EXAMPLE 1.2-5. (Inner product)

Let A^i and B_j denote the components of two first order tensors (vectors). The outer product of these tensors is

$$C^i_j = A^i B_j, \; i,j = 1,2,\ldots,N.$$

The inner product of these tensors is the scalar

$$C = A^i B_i = A^1 B_1 + A^2 B_2 + \cdots + A^N B_N.$$

Note that in some situations the inner product is performed by employing only subscript indices. For example, the above inner product is sometimes expressed as

$$C = A_i B_i = A_1 B_1 + A_2 B_2 + \cdots A_N B_N.$$

This notation is discussed later when Cartesian tensors are considered.

■

Quotient Law

Assume B_r^{qs} and C_p^s are arbitrary absolute tensors. Further assume we have a quantity $A(ijk)$ which we think might be a third order mixed tensor A^i_{jk}. By showing that the equation

$$A^r_{qp}B_r^{qs} = C_p^s$$

is satisfied, then it follows that A^r_{qp} must be a tensor. This is an example of the quotient law. Obviously, this result can be generalized to apply to tensors of any order or rank. To prove the above assertion we shall show from the above equation that A^i_{jk} is a tensor. Let x^i and $\overline{x}^i$ denote a barred and unbarred system of coordinates which are related by transformations of the form defined by equation (1.2.30). In the barred system, we assume that

$$\overline{A}^r_{qp}\overline{B}_r^{qs} = \overline{C}_p^s \tag{1.2.59}$$

where by hypothesis B_k^{ij} and C_m^l are arbitrary absolute tensors and therefore must satisfy the transformation equations

$$\begin{aligned}\overline{B}_r^{qs} &= B_k^{ij}\frac{\partial \overline{x}^q}{\partial x^i}\frac{\partial \overline{x}^s}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^r}\\ \overline{C}_p^s &= C_m^l\frac{\partial \overline{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^p}.\end{aligned}$$

We substitute for $\overline{B}_r^{qs}$ and $\overline{C}_p^s$ in the equation (1.2.59) and obtain the equation

$$\begin{aligned}\overline{A}^r_{qp}\left(B_k^{ij}\frac{\partial \overline{x}^q}{\partial x^i}\frac{\partial \overline{x}^s}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^r}\right) &= \left(C_m^l\frac{\partial \overline{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^p}\right)\\ &= A^r_{qm}B_r^{ql}\frac{\partial \overline{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^p}.\end{aligned}$$

Since the summation indices are dummy indices they can be replaced by other symbols. We change l to j, q to i and r to k and write the above equation as

$$\frac{\partial \overline{x}^s}{\partial x^j}\left(\overline{A}^r_{qp}\frac{\partial \overline{x}^q}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^r} - A^k_{im}\frac{\partial x^m}{\partial \overline{x}^p}\right)B_k^{ij} = 0.$$

Use inner multiplication by $\frac{\partial x^n}{\partial \overline{x}^s}$ and simplify this equation to the form

$$\begin{aligned}\delta_j^n\left[\overline{A}^r_{qp}\frac{\partial \overline{x}^q}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^r} - A^k_{im}\frac{\partial x^m}{\partial \overline{x}^p}\right]B_k^{ij} &= 0 \qquad \text{or}\\ \left[\overline{A}^r_{qp}\frac{\partial \overline{x}^q}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^r} - A^k_{im}\frac{\partial x^m}{\partial \overline{x}^p}\right]B_k^{in} &= 0.\end{aligned}$$

Because B_k^{in} is an arbitrary tensor, the quantity inside the brackets is zero and therefore

$$\overline{A}^r_{qp}\frac{\partial \overline{x}^q}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^r} - A^k_{im}\frac{\partial x^m}{\partial \overline{x}^p} = 0.$$

This equation is simplified by inner multiplication by $\frac{\partial x^i}{\partial \overline{x}^j}\frac{\partial \overline{x}^l}{\partial x^k}$ to obtain

$$\begin{aligned}\delta_j^q\delta_r^l\overline{A}^r_{qp} - A^k_{im}\frac{\partial x^m}{\partial \overline{x}^p}\frac{\partial x^i}{\partial \overline{x}^j}\frac{\partial \overline{x}^l}{\partial x^k} &= 0 \qquad \text{or}\\ \overline{A}^l_{jp} = A^k_{im}\frac{\partial x^m}{\partial \overline{x}^p}\frac{\partial x^i}{\partial \overline{x}^j}\frac{\partial \overline{x}^l}{\partial x^k}\end{aligned}$$

which is the transformation law for a third order mixed tensor.

EXERCISE 1.2

▶ **1.** Consider the transformation equations representing a rotation of axes through an angle α.

$$T_\alpha: \quad \begin{cases} x^1 &= \overline{x}^1 \cos\alpha - \overline{x}^2 \sin\alpha \\ x^2 &= \overline{x}^1 \sin\alpha + \overline{x}^2 \cos\alpha \end{cases}$$

Treat α as a parameter and show this set of transformations constitutes a group by finding the value of α which:

(i) gives the identity transformation.

(ii) gives the inverse transformation.

(iii) show the transformation is transitive in that a transformation with $\alpha = \theta_1$ followed by a transformation with $\alpha = \theta_2$ is equivalent to the transformation using $\alpha = \theta_1 + \theta_2$.

▶ **2.** Show the transformation

$$T_\alpha: \quad \begin{cases} \overline{x}^1 &= \alpha x^1 \\ \overline{x}^2 &= \frac{1}{\alpha} x^2 \end{cases}$$

forms a group with α as a parameter. Find the value of α such that:

(i) the identity transformation exists.

(ii) the inverse transformation exists.

(iii) the transitive property is satisfied.

▶ **3.** Show the given transformation forms a group with parameter α.

$$T_\alpha: \quad \begin{cases} \overline{x}^1 &= \frac{x^1}{1-\alpha x^1} \\ \overline{x}^2 &= \frac{x^2}{1-\alpha x^1} \end{cases}$$

▶ **4.** Consider the Lorentz transformation from relativity theory having the velocity parameter V, c is the speed of light and $x_4 = t$ is time.

$$T_V: \quad \begin{cases} \overline{x}^1 &= \frac{x^1 - V x^4}{\sqrt{1-\frac{V^2}{c^2}}} \\ \overline{x}^2 &= x^2 \\ \overline{x}^3 &= x^3 \\ \overline{x}^4 &= \frac{x^4 - \frac{V x^1}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \end{cases}$$

Show this set of transformations constitutes a group, by establishing:

(i) $V = 0$ gives the identity transformation T_0.

(ii) $T_{V_2} \cdot T_{V_1} = T_0$ requires that $V_2 = -V_1$.

(iii) $T_{V_2} \cdot T_{V_1} = T_{V_3}$ requires that

$$V_3 = \frac{V_1 + V_2}{1 + \frac{V_1 V_2}{c^2}}.$$

▶ **5.** For $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ an arbitrary independent basis, (a) Verify that

$$\vec{E}^1 = \frac{1}{V}\vec{E}_2 \times \vec{E}_3, \qquad \vec{E}^2 = \frac{1}{V}\vec{E}_3 \times \vec{E}_1, \qquad \vec{E}^3 = \frac{1}{V}\vec{E}_1 \times \vec{E}_2$$

is a reciprocal basis, where $V = \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$ (b) Show that $\vec{E}^j = g^{ij}\vec{E}_i$.

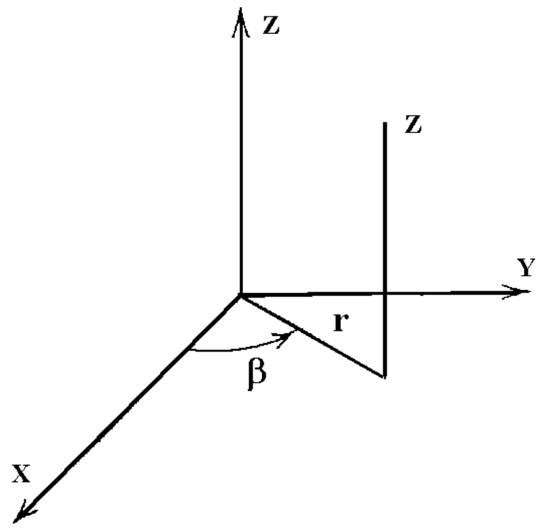

Figure 1.2-4. Cylindrical coordinates (r, β, z).

► **6.** For the cylindrical coordinates (r, β, z) illustrated in the figure 1.2-4.

(a) Write out the transformation equations from rectangular (x, y, z) coordinates to cylindrical (r, β, z) coordinates. Also write out the inverse transformation.

(b) Determine the following basis vectors in cylindrical coordinates and represent your results in terms of cylindrical coordinates.

(i) The tangential basis $\vec{E}_1, \vec{E}_2, \vec{E}_3$. (ii)The normal basis $\vec{E}^1, \vec{E}^2, \vec{E}^3$. (iii) $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\beta, \hat{\mathbf{e}}_z$

where $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\beta, \hat{\mathbf{e}}_z$ are normalized vectors in the directions of the tangential basis.

(c) A vector $\vec{A} = A_x \,\widehat{\mathbf{e}}_1 + A_y \,\widehat{\mathbf{e}}_2 + A_z \,\widehat{\mathbf{e}}_3$ can be represented in any of the forms:

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3$$
$$\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3$$
$$\vec{A} = A_r\hat{\mathbf{e}}_r + A_\beta\hat{\mathbf{e}}_\beta + A_z\hat{\mathbf{e}}_z$$

depending upon the basis vectors selected . In terms of the components A_x, A_y, A_z

(i) Solve for the contravariant components A^1, A^2, A^3.

(ii) Solve for the covariant components A_1, A_2, A_3.

(iii) Solve for the components A_r, A_β, A_z. Express all results in cylindrical coordinates. (Note the components A_r, A_β, A_z are referred to as physical components. Physical components are considered in more detail in a later section.)

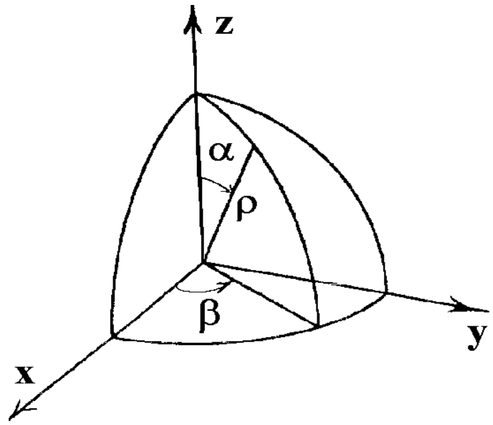

Figure 1.2-5. Spherical coordinates (ρ, α, β).

► **7.** For the spherical coordinates (ρ, α, β) illustrated in the figure 1.2-5.

(a) Write out the transformation equations from rectangular (x, y, z) coordinates to spherical (ρ, α, β) coordinates. Also write out the equations which describe the inverse transformation.

(b) Determine the following basis vectors in spherical coordinates

(i) The tangential basis $\vec{E}_1, \vec{E}_2, \vec{E}_3$.

(ii) The normal basis $\vec{E}^1, \vec{E}^2, \vec{E}^3$.

(iii) $\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\alpha, \hat{\mathbf{e}}_\beta$ which are normalized vectors in the directions of the tangential basis. Express all results in terms of spherical coordinates.

(c) A vector $\vec{A} = A_x \,\widehat{\mathbf{e}}_1 + A_y \,\widehat{\mathbf{e}}_2 + A_z \,\widehat{\mathbf{e}}_3$ can be represented in any of the forms:

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3$$
$$\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3$$
$$\vec{A} = A_\rho\hat{\mathbf{e}}_\rho + A_\alpha\hat{\mathbf{e}}_\alpha + A_\beta\hat{\mathbf{e}}_\beta$$

depending upon the basis vectors selected . Calculate, in terms of the coordinates (ρ, α, β) and the components A_x, A_y, A_z

(i) The contravariant components A^1, A^2, A^3.

(ii) The covariant components A_1, A_2, A_3.

(iii) The components $A_\rho, A_\alpha, A_\beta$ which are called physical components.

► **8.** Work the problems 6,7 and then let $(x^1, x^2, x^3) = (r, \beta, z)$ denote the coordinates in the cylindrical system and let $(\overline{x}^1, \overline{x}^2, \overline{x}^3) = (\rho, \alpha, \beta)$ denote the coordinates in the spherical system.

(a) Write the transformation equations $x \to \overline{x}$ from cylindrical to spherical coordinates. Also find the inverse transformations. (Hint: See the figures 1.2-4 and 1.2-5.)

(b) Use the results from part (a) and the results from problems 6,7 to verify that

$$\overline{A}_i = A_j \frac{\partial x^j}{\partial \overline{x}^i} \quad \text{for} \quad i = 1, 2, 3.$$

(i.e. Substitute A_j from problem 6 to get $\bar{A}_i$ given in problem 7.)

(c) Use the results from part (a) and the results from problems 6,7 to verify that

$$\overline{A}^i = A^j \frac{\partial \overline{x}^i}{\partial x^j} \quad \text{for} \quad i = 1, 2, 3.$$

(i.e. Substitute A^j from problem 6 to get $\bar{A}^i$ given by problem 7.)

► **9.** Pick two arbitrary noncolinear vectors in the x, y plane, say

$$\vec{V}_1 = 5\,\widehat{\mathbf{e}}_1 + \widehat{\mathbf{e}}_2 \quad \text{and} \quad \vec{V}_2 = \widehat{\mathbf{e}}_1 + 5\,\widehat{\mathbf{e}}_2$$

and let $\vec{V}_3 = \widehat{\mathbf{e}}_3$ be a unit vector perpendicular to both $\vec{V}_1$ and $\vec{V}_2$. The vectors $\vec{V}_1$ and $\vec{V}_2$ can be thought of as defining an oblique coordinate system, as illustrated in the figure 1.2-6.

(a) Find the reciprocal basis $(\vec{V}^1, \vec{V}^2, \vec{V}^3)$.

(b) Let

$$\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3 = \alpha\vec{V}_1 + \beta\vec{V}_2 + \gamma\vec{V}_3$$

and show that

$$\begin{aligned} \alpha &= \frac{5x}{24} - \frac{y}{24} \\ \beta &= -\frac{x}{24} + \frac{5y}{24} \\ \gamma &= z \end{aligned}$$

(c) Show

$$\begin{aligned} x &= 5\alpha + \beta \\ y &= \alpha + 5\beta \\ z &= \gamma \end{aligned}$$

(d) For $\gamma = \gamma_0$ constant, show the coordinate lines are described by $\alpha =$ constant and $\beta =$ constant, and sketch some of these coordinate lines. (See figure 1.2-6.)

(e) Find the metrics g_{ij} and conjugate metrices g^{ij} associated with the (α, β, γ) space.

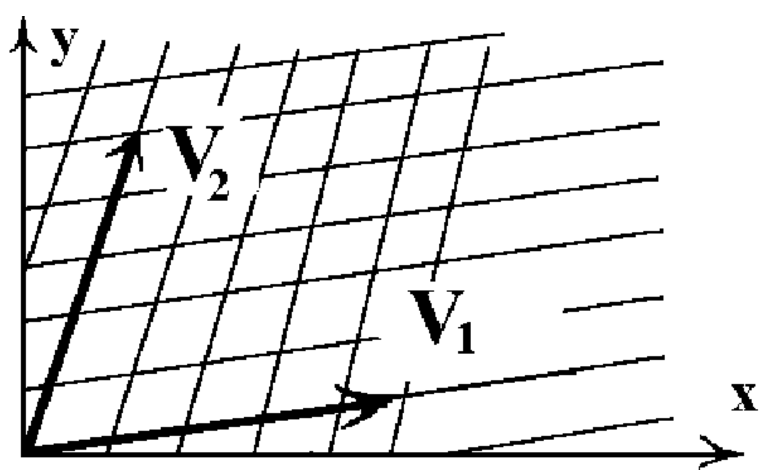

Figure 1.2-6. Oblique coordinates.

► **10.** Consider the transformation equations

$$x = x(u, v, w)$$
$$y = y(u, v, w)$$
$$z = z(u, v, w)$$

substituted into the position vector

$$\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3.$$

Define the basis vectors

$$(\vec{E}_1, \vec{E}_2, \vec{E}_3) = \left(\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}\right)$$

with the reciprocal basis

$$\vec{E}^1 = \frac{1}{V}\vec{E}_2 \times \vec{E}_3, \qquad \vec{E}^2 = \frac{1}{V}\vec{E}_3 \times \vec{E}_1, \qquad \vec{E}^3 = \frac{1}{V}\vec{E}_1 \times \vec{E}_2.$$

where

$$V = \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3).$$

Let $v = \vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3)$ and show that $v \cdot V = 1$.

► **11.** Given the coordinate transformation

$$x = -u - 2v \qquad y = -u - v \qquad z = z$$

(a) Find and illustrate graphically some of the coordinate curves.

(b) For $\vec{r} = \vec{r}(u, v, z)$ a position vector, define the basis vectors

$$\vec{E}_1 = \frac{\partial \vec{r}}{\partial u}, \qquad \vec{E}_2 = \frac{\partial \vec{r}}{\partial v}, \qquad \vec{E}_3 = \frac{\partial \vec{r}}{\partial z}.$$

Calculate these vectors and then calculate the reciprocal basis $\vec{E}^1, \vec{E}^2, \vec{E}^3$.

(c) With respect to the basis vectors in (b) find the contravariant components A^i associated with the vector

$$\vec{A} = \alpha_1\,\widehat{\mathbf{e}}_1 + \alpha_2\,\widehat{\mathbf{e}}_2 + \alpha_3\,\widehat{\mathbf{e}}_3$$

where $(\alpha_1, \alpha_2, \alpha_3)$ are constants.

(d) Find the covariant components A_i associated with the vector $\vec{A}$ given in part (c).

(e) Calculate the metric tensor g_{ij} and conjugate metric tensor g^{ij}.

(f) From the results (e), verify that $g_{ij}g^{jk} = \delta_i^k$

(g) Use the results from (c)(d) and (e) to verify that $A_i = g_{ik}A^k$

(h) Use the results from (c)(d) and (e) to verify that $A^i = g^{ik}A_k$

(i) Find the projection of the vector $\vec{A}$ on unit vectors in the directions $\vec{E}_1, \vec{E}_2, \vec{E}_3$.

(j) Find the projection of the vector $\vec{A}$ on unit vectors the directions $\vec{E}^1, \vec{E}^2, \vec{E}^3$.

► **12.** For $\vec{r} = y^i\,\widehat{\mathbf{e}}_i$ where $y^i = y^i(x^1,x^2,x^3)$, $i=1,2,3$ we have by definition

$$\vec{E}_j = \frac{\partial \vec{r}}{\partial x^j} = \frac{\partial y^i}{\partial x^j}\,\widehat{\mathbf{e}}_i. \quad \text{From this relation show that} \quad \vec{E}^m = \frac{\partial x^m}{\partial y^j}\,\widehat{\mathbf{e}}_j$$

and consequently

$$g_{ij} = \vec{E}_i \cdot \vec{E}_j = \frac{\partial y^m}{\partial x^i}\frac{\partial y^m}{\partial x^j}, \quad \text{and} \quad g^{ij} = \vec{E}^i \cdot \vec{E}^j = \frac{\partial x^i}{\partial y^m}\frac{\partial x^j}{\partial y^m}, \qquad i,j,m = 1,\ldots,3$$

► **13.** Consider the set of all coordinate transformations of the form

$$y^i = a^i_j x^j + b^i$$

where a^i_j and b^i are constants and the determinant of a^i_j is different from zero. Show this set of transformations forms a group.

► **14.** For α_i , β_i constants and t a parameter, $x^i = \alpha_i + t\,\beta_i$,$i=1,2,3$ is the parametric representation of a straight line. Find the parametric equation of the line which passes through the two points $(1,2,3)$ and $(14,7,-3)$. What does the vector $\frac{d\vec{r}}{dt}$ represent?

► **15.** A surface can be represented using two parameters u,v by introducing the parametric equations

$$x^i = x^i(u,v), \quad i=1,2,3, \quad a<u<b \quad \text{and} \quad c<v<d.$$

The parameters u,v are called the curvilinear coordinates of a point on the surface. A point on the surface can be represented by the position vector $\vec{r} = \vec{r}(u,v) = x^1(u,v)\,\widehat{\mathbf{e}}_1 + x^2(u,v)\,\widehat{\mathbf{e}}_2 + x^3(u,v)\,\widehat{\mathbf{e}}_3$. The vectors $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are tangent vectors to the coordinate surface curves $\vec{r}(u,c_2)$ and $\vec{r}(c_1,v)$ respectively. An element of surface area dS on the surface is defined as the area of the elemental parallelogram having the vector sides $\frac{\partial \vec{r}}{\partial u}du$ and $\frac{\partial \vec{r}}{\partial v}dv$. Show that

$$dS = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du dv = \sqrt{g_{11}g_{22} - (g_{12})^2}\, du dv$$

where

$$g_{11} = \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial u} \qquad g_{12} = \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} \qquad g_{22} = \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial v}.$$

Hint: $(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = |\vec{A}\times\vec{B}|^2$ See Exercise 1.1, problem 9(c).

► **16.**

(a) Use the results from problem 15 and find the element of surface area of the circular cone

$$x = u\sin\alpha\cos v \qquad y = u\sin\alpha\sin v \qquad z = u\cos\alpha$$
$$\alpha \text{ a constant} \qquad 0 \le u \le b \qquad 0 \le v \le 2\pi$$

(b) Find the surface area of the above cone.

► **17.** The equation of a plane is defined in terms of two parameters u and v and has the form

$$x^i = \alpha_i\, u + \beta_i\, v + \gamma_i \qquad i = 1,2,3,$$

where α_i β_i and γ_i are constants. Find the equation of the plane which passes through the points $(1,2,3)$, $(14,7,-3)$ and $(5,5,5)$. What does this problem have to do with the position vector $\vec{r}(u,v)$, the vectors $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}$ and $\vec{r}(0,0)$? Hint: See problem 15.

► **18.** Determine the points of intersection of the curve $x^1 = t,\ x^2 = (t)^2,\ x^3 = (t)^3$ with the plane

$$8\,x^1 - 5\,x^2 + x^3 - 4 = 0.$$

► **19.** Verify the relations $V\, e_{ijk}\vec{E}^k = \vec{E}_i \times \vec{E}_j$ and $v^{-1}\, e^{ijk}\vec{E}_k = \vec{E}^i \times \vec{E}^j$ where $v = \vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3)$ and $V = \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$..

► **20.** Let $\bar{x}^i$ and x^i, $i = 1,2,3$ be related by the linear transformation $\bar{x}^i = c^i_j x^j$, where c^i_j are constants such that the determinant $c = det(c^i_j)$ is different from zero. Let γ^n_m denote the cofactor of c^m_n divided by the determinant c.

(a) Show that $c^i_j \gamma^j_k = \gamma^i_j c^j_k = \delta^i_k$.

(b) Show the inverse transformation can be expressed $x^i = \gamma^i_j \bar{x}^j$.

(c) Show that if A^i is a contravariant vector, then its transformed components are $\bar{A}^p = c^p_q A^q$.

(d) Show that if A_i is a covariant vector, then its transformed components are $\bar{A}_i = \gamma^p_i A_p$.

► **21.** Show that the outer product of two contravariant vectors A^i and B^i, $i = 1,2,3$ results in a second order contravariant tensor.

► **22.** Show that for the position vector $\vec{r} = y^i(x^1, x^2, x^3)\,\widehat{\mathbf{e}}_i$ the element of arc length squared is $ds^2 = d\vec{r} \cdot d\vec{r} = g_{ij} dx^i dx^j$ where $g_{ij} = \vec{E}_i \cdot \vec{E}_j = \dfrac{\partial y^m}{\partial x^i}\dfrac{\partial y^m}{\partial x^j}$.

► **23.** For A^i_{jk}, B^m_n and C^p_{tq} absolute tensors, show that if $A^i_{jk} B^k_n = C^i_{jn}$ then $\overline{A}^i_{jk}\overline{B}^k_n = \overline{C}^i_{jn}$.

► **24.** Let A_{ij} denote an absolute covariant tensor of order 2. Show that the determinant $A = det(A_{ij})$ is an invariant of weight 2 and $\sqrt{(A)}$ is an invariant of weight 1.

► **25.** Let B^{ij} denote an absolute contravariant tensor of order 2. Show that the determinant $B = det(B^{ij})$ is an invariant of weight -2 and $\sqrt{B}$ is an invariant of weight -1.

► **26.**

(a) Write out the contravariant components of the following vectors

$$(i)\quad \vec{E}_1 \qquad (ii)\quad \vec{E}_2 \qquad (iii)\quad \vec{E}_3 \quad \text{where} \quad \vec{E}_i = \frac{\partial \vec{r}}{\partial x^i} \quad \text{for} \quad i = 1,2,3.$$

(b) Write out the covariant components of the following vectors

$$(i)\quad \vec{E}^1 \qquad (ii)\quad \vec{E}^2 \qquad (ii)\quad \vec{E}^3 \quad \text{where} \quad \vec{E}^i = \text{grad}\, x^i, \quad \text{for} \quad i = 1,2,3.$$

▶ **27.** Let A_{ij} and A^{ij} denote absolute second order tensors. Show that $\lambda = A_{ij}A^{ij}$ is a scalar invariant.

▶ **28.** Assume that a_{ij}, $i,j = 1,2,3,4$ is a skew-symmetric second order absolute tensor. (a) Show that

$$b_{ijk} = \frac{\partial a_{jk}}{\partial x^i} + \frac{\partial a_{ki}}{\partial x^j} + \frac{\partial a_{ij}}{\partial x^k}$$

is a third order tensor. (b) Show b_{ijk} is skew-symmetric in all pairs of indices and (c) determine the number of independent components this tensor has.

▶ **29.** Show the linear forms $A_1x + B_1y + C_1$ and $A_2x + B_2y + C_2$, with respect to the group of rotations and translations $x = \overline{x}\cos\theta - \overline{y}\sin\theta + h$ and $y = \overline{x}\sin\theta + \overline{y}\cos\theta + k$, have the forms $\overline{A}_1\overline{x} + \overline{B}_1\overline{y} + \overline{C}_1$ and $\overline{A}_2\overline{x} + \overline{B}_2\overline{y} + \overline{C}_2$. Also show that the quantities $A_1B_2 - A_2B_1$ and $A_1A_2 + B_1B_2$ are invariants.

▶ **30.** Show that the curvature of a curve $y = f(x)$ is $\kappa = \pm\, y''(1 + y'^2)^{-3/2}$ and that this curvature remains invariant under the group of rotations given in the problem 1. Hint: Calculate $\frac{dy}{dx} = \frac{dy}{d\overline{x}}\frac{d\overline{x}}{dx}$.

▶ **31.** Show that when the equation of a curve is given in the parametric form $x = x(t)$, $y = y(t)$, then the curvature is $\kappa = \pm\dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$ and remains invariant under the change of parameter $t = t(\overline{t})$, where $\dot{x} = \frac{dx}{dt}$, etc.

▶ **32.** Let A^{ij}_k denote a third order mixed tensor. (a) Show that the contraction A^{ij}_i is a first order contravariant tensor. (b) Show that contraction of i and j produces A^{ii}_k which is not a tensor. This shows that in general, the process of contraction does not always apply to indices at the same level.

▶ **33.** Let $\phi = \phi(x^1, x^2, \ldots, x^N)$ denote an absolute scalar invariant. (a) Is the quantity $\frac{\partial \phi}{\partial x^i}$ a tensor? (b) Is the quantity $\frac{\partial^2 \phi}{\partial x^i \partial x^j}$ a tensor?

▶ **34.** Consider the second order absolute tensor a_{ij}, $i,j = 1,2$ where $a_{11} = 1, a_{12} = 2, a_{21} = 3$ and $a_{22} = 4$. Find the components of $\overline{a}_{ij}$ under the transformation of coordinates $\overline{x}^1 = x^1 + x^2$ and $\overline{x}^2 = x^1 - x^2$.

▶ **35.** Let A_i, B_i denote the components of two covariant absolute tensors of order one. Show that $C_{ij} = A_iB_j$ is an absolute second order covariant tensor.

▶ **36.** Let A^i denote the components of an absolute contravariant tensor of order one and let B_i denote the components of an absolute covariant tensor of order one, show that $C^i_j = A^iB_j$ transforms as an absolute mixed tensor of order two.

▶ **37.** (a) Show the sum and difference of two tensors of the same kind is also a tensor of this kind. (b) Show that the outer product of two tensors is a tensor. Do parts (a) (b) in the special case where one tensor A^i is a relative tensor of weight 4 and the other tensor B^j_k is a relative tensor of weight 3. What is the weight of the outer product tensor $T^{ij}_k = A^iB^j_k$ in this special case?

▶ **38.** Let A^{ij}_{km} denote the components of a mixed tensor of weight M. Form the contraction $B^j_m = A^{ij}_{im}$ and determine how B^j_m transforms. What is its weight?

▶ **39.** Let A^i_j denote the components of an absolute mixed tensor of order two. Show that the scalar contraction $S = A^i_i$ is an invariant.

▶ **40.** Let $A^i = A^i(x^1, x^2, \ldots, x^N)$ denote the components of an absolute contravariant tensor. Form the quantity $B^i_j = \frac{\partial A^i}{\partial x^j}$ and determine if B^i_j transforms like a tensor.

▶ **41.** Let A_i denote the components of a covariant vector. (a) Show that $a_{ij} = \dfrac{\partial A_i}{\partial x^j} - \dfrac{\partial A_j}{\partial x^i}$ are the components of a second order tensor. (b) Show that $\dfrac{\partial a_{ij}}{\partial x^k} + \dfrac{\partial a_{jk}}{\partial x^i} + \dfrac{\partial a_{ki}}{\partial x^j} = 0.$

▶ **42.** Show that $x^i = K\, e^{ijk} A_j B_k$, with $K \neq 0$ and arbitrary, is a general solution of the system of equations $A_i x^i = 0$, $B_i x^i = 0$, $i = 1, 2, 3$. Give a geometric interpretation of this result in terms of vectors.

▶ **43.** Given the vector $\vec{A} = y\,\widehat{\mathbf{e}}_1 + z\,\widehat{\mathbf{e}}_2 + x\,\widehat{\mathbf{e}}_3$ where $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3$ denote a set of unit basis vectors which define a set of orthogonal x, y, z axes. Let $\vec{E}_1 = 3\,\widehat{\mathbf{e}}_1 + 4\,\widehat{\mathbf{e}}_2$, $\vec{E}_2 = 4\,\widehat{\mathbf{e}}_1 + 7\,\widehat{\mathbf{e}}_2$ and $\vec{E}_3 = \widehat{\mathbf{e}}_3$ denote a set of basis vectors which define a set of u, v, w axes. (a) Find the coordinate transformation between these two sets of axes. (b) Find a set of reciprocal vectors $\vec{E}^1, \vec{E}^3, \vec{E}^3$. (c) Calculate the covariant components of $\vec{A}$. (d) Calculate the contravariant components of $\vec{A}$.

▶ **44.** Let $\mathbf{A} = A_{ij}\,\widehat{\mathbf{e}}_i\,\widehat{\mathbf{e}}_j$ denote a dyadic. Show that

$$\mathbf{A} : \mathbf{A}_c = A_{11}A_{11} + A_{12}A_{21} + A_{13}A_{31} + A_{21}A_{12} + A_{22}A_{22} + A_{23}A_{32} + A_{31}A_{13} + A_{32}A_{23} + A_{23}A_{33}$$

▶ **45.** Let $\vec{A} = A_i\,\widehat{\mathbf{e}}_i$, $\vec{B} = B_i\,\widehat{\mathbf{e}}_i$, $\vec{C} = C_i\,\widehat{\mathbf{e}}_i$, $\vec{D} = D_i\,\widehat{\mathbf{e}}_i$ denote vectors and let $\phi = \vec{A}\vec{B}$, $\psi = \vec{C}\vec{D}$ denote dyadics which are the outer products involving the above vectors. Show that the double dot product satisfies

$$\phi : \psi = \vec{A}\vec{B} : \vec{C}\vec{D} = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D})$$

▶ **46.** Show that if a_{ij} is a symmetric tensor in one coordinate system, then it is symmetric in all coordinate systems.

▶ **47.** Write the transformation laws for the given tensors. $(a)\quad A^k_{ij} \qquad (b)\quad A^{ij}_k \qquad (c)\quad A^{ijk}_m$

▶ **48.** Show that if $\overline{A}_i = A_j \dfrac{\partial x^j}{\partial \overline{x}^i}$, then $A_i = \overline{A}_j \frac{\partial \overline{x}^j}{\partial x^i}$. Note that this is equivalent to interchanging the bar and unbarred systems.

▶ **49.**

(a) Show that under the linear homogeneous transformation

$$\begin{aligned} x_1 &= a^1_1 \overline{x}_1 + a^2_1 \overline{x}_2 \\ x_2 &= a^1_2 \overline{x}_1 + a^2_2 \overline{x}_2 \end{aligned}$$

the quadratic form

$$Q(x_1, x_2) = g_{11}(x_1)^2 + 2g_{12}x_1x_2 + g_{22}(x_2)^2 \quad \text{becomes} \quad \overline{Q}(\overline{x}_1, \overline{x}_2) = \overline{g}_{11}(\overline{x}_1)^2 + 2\overline{g}_{12}\overline{x}_1\overline{x}_2 + \overline{g}_{22}(\overline{x}_2)^2$$

where $\overline{g}_{ij} = g_{11}a^j_1 a^i_1 + g_{12}(a^i_1 a^j_2 + a^j_1 a^i_2) + g_{22}a^i_2 a^j_2$.

(b) Show $F = g_{11}g_{22} - (g_{12})^2$ is a relative invariant of weight 2 of the quadratic form $Q(x_1, x_2)$ with respect to the group of linear homogeneous transformations. i.e. Show that $\overline{F} = \Delta^2 F$ where $\overline{F} = \overline{g}_{11}\overline{g}_{22} - (\overline{g}_{12})^2$ and $\Delta = (a^1_1 a^2_2 - a^2_1 a^1_2)$.

▶ **50.** Let $\mathbf{a}_i$ and $\mathbf{b}_i$ for $i = 1, \ldots, n$ denote arbitrary vectors and form the dyadic

$$\Phi = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_n\mathbf{b}_n.$$

By definition the first scalar invariant of Φ is

$$\phi_1 = \mathbf{a}_1 \cdot \mathbf{b}_1 + \mathbf{a}_2 \cdot \mathbf{b}_2 + \cdots + \mathbf{a}_n \cdot \mathbf{b}_n$$

where a dot product operator has been placed between the vectors. The first vector invariant of Φ is defined

$$\vec{\phi} = \mathbf{a}_1 \times \mathbf{b}_1 + \mathbf{a}_2 \times \mathbf{b}_2 + \cdots + \mathbf{a}_n \times \mathbf{b}_n$$

where a vector cross product operator has been placed between the vectors.

(a) Show that the first scalar and vector invariant of

$$\Phi = \widehat{\mathbf{e}}_1\,\widehat{\mathbf{e}}_2 + \widehat{\mathbf{e}}_2\,\widehat{\mathbf{e}}_3 + \widehat{\mathbf{e}}_3\,\widehat{\mathbf{e}}_3$$

are respectively 1 and $\widehat{\mathbf{e}}_1 + \widehat{\mathbf{e}}_3$.

(b) From the vector $\mathbf{f} = f_1\,\widehat{\mathbf{e}}_1 + f_2\,\widehat{\mathbf{e}}_2 + f_3\,\widehat{\mathbf{e}}_3$ one can form the dyadic $\nabla\mathbf{f}$ having the matrix components

$$\nabla\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} \\ \frac{\partial f_1}{\partial z} & \frac{\partial f_2}{\partial z} & \frac{\partial f_3}{\partial z} \end{pmatrix}.$$

Show that this dyadic has the first scalar and vector invariants given by

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$
$$\nabla \times \mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\widehat{\mathbf{e}}_1 + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\widehat{\mathbf{e}}_2 + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\widehat{\mathbf{e}}_3$$

▶ **51.** Let Φ denote the dyadic given in problem 50. The dyadic Φ_2 defined by

$$\Phi_2 = \frac{1}{2}\sum_{i,j} \mathbf{a}_i \times \mathbf{a}_j \mathbf{b}_i \times \mathbf{b}_j$$

is called the Gibbs second dyadic of Φ, where the summation is taken over all permutations of i and j. When $i = j$ the dyad vanishes. Note that the permutations i, j and j, i give the same dyad and so occurs twice in the final sum. The factor 1/2 removes this doubling. Associated with the Gibbs dyad Φ_2 are the scalar invariants

$$\phi_2 = \frac{1}{2}\sum_{i,j} (\mathbf{a}_i \times \mathbf{a}_j) \cdot (\mathbf{b}_i \times \mathbf{b}_j)$$
$$\phi_3 = \frac{1}{6}\sum_{i,j,k} (\mathbf{a}_i \times \mathbf{a}_j \cdot \mathbf{a}_k)(\mathbf{b}_i \times \mathbf{b}_j \cdot \mathbf{b}_k)$$

Show that the dyad

$$\Phi = \mathbf{a}\,\mathbf{s} + \mathbf{t}\,\mathbf{q} + \mathbf{c}\,\mathbf{u}$$

has

the first scalar invariant	$\phi_1 = \mathbf{a} \cdot \mathbf{s} + \mathbf{b} \cdot \mathbf{t} + \mathbf{c} \cdot \mathbf{u}$
the first vector invariant	$\vec{\phi} = \mathbf{a} \times \mathbf{s} + \mathbf{b} \times \mathbf{t} + \mathbf{c} \times \mathbf{u}$
Gibbs second dyad	$\Phi_2 = \mathbf{b} \times \mathbf{c}\mathbf{t} \times \mathbf{u} + \mathbf{c} \times \mathbf{a}\mathbf{u} \times \mathbf{s} + \mathbf{a} \times \mathbf{b}\mathbf{s} \times \mathbf{t}$
second scalar of Φ	$\phi_2 = (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{t} \cdot \mathbf{u}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{u} \times \mathbf{s}) + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{s} \times \mathbf{t})$
third scalar of Φ	$\phi_3 = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})(\mathbf{s} \times \mathbf{t} \cdot \mathbf{u})$

▶ **52.** (**Spherical Trigonometry**) Construct a spherical triangle ABC on the surface of a unit sphere with sides and angles less than 180 degrees. Denote by $\mathbf{a}$,$\mathbf{b}$ $\mathbf{c}$ the unit vectors from the origin of the sphere to the vertices A,B and C. Make the construction such that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is positive with $\mathbf{a},\mathbf{b},\mathbf{c}$ forming a right-handed system. Let α,β,γ denote the angles between these unit vectors such that

$$\mathbf{a}\cdot\mathbf{b}=\cos\gamma \qquad \mathbf{c}\cdot\mathbf{a}=\cos\beta \qquad \mathbf{b}\cdot\mathbf{c}=\cos\alpha. \tag{1}$$

The great circles through the vertices A,B,C then make up the sides of the spherical triangle where side α is opposite vertex A, side β is opposite vertex B and side γ is opposite the vertex C. The angles A,B and C between the various planes formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are called the interior dihedral angles of the spherical triangle. Note that the cross products

$$\mathbf{a}\times\mathbf{b}=\sin\gamma\,\overline{\mathbf{c}} \qquad \mathbf{b}\times\mathbf{c}=\sin\alpha\,\overline{\mathbf{a}} \qquad \mathbf{c}\times\mathbf{a}=\sin\beta\,\overline{\mathbf{b}} \tag{2}$$

define unit vectors $\overline{\mathbf{a}},\overline{\mathbf{b}}$ and $\overline{\mathbf{c}}$ perpendicular to the planes determined by the unit vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$. The dot products

$$\overline{\mathbf{a}}\cdot\overline{\mathbf{b}}=\cos\overline{\gamma} \qquad \overline{\mathbf{b}}\cdot\overline{\mathbf{c}}=\cos\overline{\alpha} \qquad \overline{\mathbf{c}}\cdot\overline{\mathbf{a}}=\cos\overline{\beta} \tag{3}$$

define the angles $\overline{\alpha}$,$\overline{\beta}$ and $\overline{\gamma}$ which are called the exterior dihedral angles at the vertices A,B and C and are such that

$$\overline{\alpha}=\pi-A \qquad \overline{\beta}=\pi-B \qquad \overline{\gamma}=\pi-C. \tag{4}$$

(a) Using appropriate scaling, show that the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\overline{\mathbf{a}},\overline{\mathbf{b}},\overline{\mathbf{c}}$ form a reciprocal set.

(b) Show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\sin\alpha\,\mathbf{a}\cdot\overline{\mathbf{a}}=\sin\beta\,\mathbf{b}\cdot\overline{\mathbf{b}}=\sin\gamma\,\mathbf{c}\cdot\overline{\mathbf{c}}$

(c) Show that $\overline{\mathbf{a}}\cdot(\overline{\mathbf{b}}\times\overline{\mathbf{c}})=\sin\overline{\alpha}\,\mathbf{a}\cdot\overline{\mathbf{a}}=\sin\overline{\beta}\,\mathbf{b}\cdot\overline{\mathbf{b}}=\sin\overline{\gamma}\,\mathbf{c}\cdot\overline{\mathbf{c}}$

(d) Using parts (b) and (c) show that

$$\frac{\sin\alpha}{\sin\overline{\alpha}}=\frac{\sin\beta}{\sin\overline{\beta}}=\frac{\sin\gamma}{\sin\overline{\gamma}}$$

(e) Use the results from equation (4) to derive the law of sines for spherical triangles

$$\frac{\sin\alpha}{\sin A}=\frac{\sin\beta}{\sin B}=\frac{\sin\gamma}{\sin C}$$

(f) Using the equations (2) show that

$$\sin\beta\sin\gamma\mathbf{b}\cdot\mathbf{c}=(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{b})-\mathbf{b}\cdot\mathbf{c}$$

and hence show that

$$\cos\alpha=\cos\beta\cos\gamma-\sin\beta\sin\gamma\cos\overline{\alpha}.$$

In a similar manner show also that

$$\cos\overline{\alpha}=\cos\overline{\beta}\cos\overline{\gamma}-\sin\overline{\beta}\sin\overline{\gamma}\cos\alpha.$$

(g) Using part (f) derive the law of cosines for spherical triangles

$$\cos\alpha=\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos A$$
$$\cos A=-\cos B\cos C+\sin B\sin C\cos\alpha$$

A cyclic permutation of the symbols produces similar results involving the other angles and sides of the spherical triangle.