§1.3 SPECIAL TENSORS

Knowing how tensors are defined and recognizing a tensor when it pops up in front of you are two different things. Some quantities, which are tensors, frequently arise in applied problems and you should learn to recognize these special tensors when they occur. In this section some important tensor quantities are defined. We also consider how these special tensors can in turn be used to define other tensors.

Metric Tensor

Define y^i, $i = 1, \ldots, N$ as independent coordinates in an N dimensional orthogonal Cartesian coordinate system. The distance squared between two points y^i and $y^i + dy^i$, $i = 1, \ldots, N$ is defined by the expression

$$ds^2 = dy^m dy^m = (dy^1)^2 + (dy^2)^2 + \cdots + (dy^N)^2. \tag{1.3.1}$$

Assume that the coordinates y^i are related to a set of independent generalized coordinates x^i, $i = 1, \ldots, N$ by a set of transformation equations

$$y^i = y^i(x^1, x^2, \ldots, x^N), \quad i = 1, \ldots, N. \tag{1.3.2}$$

To emphasize that each y^i depends upon the x coordinates we sometimes use the notation $y^i = y^i(x)$, for $i = 1, \ldots, N$. The differential of each coordinate can be written as

$$dy^m = \frac{\partial y^m}{\partial x^j}\, dx^j, \quad m = 1, \ldots, N, \tag{1.3.3}$$

and consequently in the x-generalized coordinates the distance squared, found from the equation (1.3.1), becomes a quadratic form. Substituting equation (1.3.3) into equation (1.3.1) we find

$$ds^2 = \frac{\partial y^m}{\partial x^i}\frac{\partial y^m}{\partial x^j}\, dx^i dx^j = g_{ij}\, dx^i dx^j \tag{1.3.4}$$

where

$$g_{ij} = \frac{\partial y^m}{\partial x^i}\frac{\partial y^m}{\partial x^j}, \quad i, j = 1, \ldots, N \tag{1.3.5}$$

are called the metrices of the space defined by the coordinates x^i, $i = 1, \ldots, N$. Here the g_{ij} are functions of the x coordinates and is sometimes written as $g_{ij} = g_{ij}(x)$. Further, the metrices g_{ij} are symmetric in the indices i and j so that $g_{ij} = g_{ji}$ for all values of i and j over the range of the indices. If we transform to another coordinate system, say $\overline{x}^i$, $i = 1, \ldots, N$, then the element of arc length squared is expressed in terms of the barred coordinates and $ds^2 = \overline{g}_{ij}\, d\overline{x}^i d\overline{x}^j$, where $\overline{g}_{ij} = \overline{g}_{ij}(\overline{x})$ is a function of the barred coordinates. The following example demonstrates that these metrices are second order covariant tensors.

EXAMPLE 1.3-1. Show the metric components g_{ij} are covariant tensors of the second order.
Solution: In a coordinate system $x^i, i = 1, \ldots, N$ the element of arc length squared is

$$ds^2 = g_{ij} dx^i dx^j \tag{1.3.6}$$

while in a coordinate system $\overline{x}^i, i = 1, \ldots, N$ the element of arc length squared is represented in the form

$$ds^2 = \overline{g}_{mn} d\overline{x}^m d\overline{x}^n. \tag{1.3.7}$$

The element of arc length squared is to be an invariant and so we require that

$$\overline{g}_{mn} d\overline{x}^m d\overline{x}^n = g_{ij} dx^i dx^j \tag{1.3.8}$$

Here it is assumed that there exists a coordinate transformation of the form defined by equation (1.2.30) together with an inverse transformation, as in equation (1.2.32), which relates the barred and unbarred coordinates. In general, if $x^i = x^i(\overline{x})$, then for $i = 1, \ldots, N$ we have

$$dx^i = \frac{\partial x^i}{\partial \overline{x}^m} d\overline{x}^m \quad \text{and} \quad dx^j = \frac{\partial x^j}{\partial \overline{x}^n} d\overline{x}^n \tag{1.3.9}$$

Substituting these differentials in equation (1.3.8) gives us the result

$$\overline{g}_{mn} d\overline{x}^m d\overline{x}^n = g_{ij} \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n} d\overline{x}^m d\overline{x}^n \qquad \text{or} \qquad \left(\overline{g}_{mn} - g_{ij} \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n} \right) d\overline{x}^m d\overline{x}^n = 0$$

For arbitrary changes in $d\overline{x}^m$ this equation implies that $\overline{g}_{mn} = g_{ij} \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n}$ and consequently g_{ij} transforms as a second order absolute covariant tensor.

■

EXAMPLE 1.3-2. (Curvilinear coordinates) Consider a set of general transformation equations from rectangular coordinates (x, y, z) to curvilinear coordinates (u, v, w). These transformation equations and the corresponding inverse transformations are represented

$$\begin{aligned} x &= x(u, v, w) \qquad & u &= u(x, y, z) \\ y &= y(u, v, w) \qquad & v &= v(x, y, z) \\ z &= z(u, v, w). \qquad & w &= w(x, y, z) \end{aligned} \tag{1.3.10}$$

Here $y^1 = x,\ y^2 = y,\ y^3 = z$ and $x^1 = u,\ x^2 = v,\ x^3 = w$ are the Cartesian and generalized coordinates and $N = 3$. The intersection of the coordinate surfaces $u = c_1$,$v = c_2$ and $w = c_3$ define coordinate curves of the curvilinear coordinate system. The substitution of the given transformation equations (1.3.10) into the position vector $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3$ produces the position vector which is a function of the generalized coordinates and

$$\vec{r} = \vec{r}(u, v, w) = x(u, v, w)\,\widehat{\mathbf{e}}_1 + y(u, v, w)\,\widehat{\mathbf{e}}_2 + z(u, v, w)\,\widehat{\mathbf{e}}_3$$

and consequently $d\vec{r} = \frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv + \frac{\partial \vec{r}}{\partial w}\, dw$, where

$$\begin{aligned}
\vec{E}_1 &= \frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}\,\widehat{\mathbf{e}}_1 + \frac{\partial y}{\partial u}\,\widehat{\mathbf{e}}_2 + \frac{\partial z}{\partial u}\,\widehat{\mathbf{e}}_3 \\
\vec{E}_2 &= \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}\,\widehat{\mathbf{e}}_1 + \frac{\partial y}{\partial v}\,\widehat{\mathbf{e}}_2 + \frac{\partial z}{\partial v}\,\widehat{\mathbf{e}}_3 \\
\vec{E}_3 &= \frac{\partial \vec{r}}{\partial w} = \frac{\partial x}{\partial w}\,\widehat{\mathbf{e}}_1 + \frac{\partial y}{\partial w}\,\widehat{\mathbf{e}}_2 + \frac{\partial z}{\partial w}\,\widehat{\mathbf{e}}_3.
\end{aligned} \tag{1.3.11}$$

are tangent vectors to the coordinate curves. The element of arc length in the curvilinear coordinates is

$$\begin{aligned}
ds^2 = d\vec{r}\cdot d\vec{r} = &\frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial u}\,dudu + \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v}\,dudv + \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial w}\,dudw \\
+ &\frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial u}\,dvdu + \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial v}\,dvdv + \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial w}\,dvdw \\
+ &\frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial u}\,dwdu + \frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial v}\,dwdv + \frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial w}\,dwdw.
\end{aligned} \tag{1.3.12}$$

Utilizing the summation convention, the above can be expressed in the index notation. Define the quantities

$$\begin{aligned}
g_{11} &= \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial u} \qquad & g_{12} &= \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} \qquad & g_{13} &= \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial w} \\
g_{21} &= \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial u} & g_{22} &= \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial v} & g_{23} &= \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial w} \\
g_{31} &= \frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial u} & g_{32} &= \frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial v} & g_{33} &= \frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial w}
\end{aligned}$$

and let $x^1 = u, \quad x^2 = v, \quad x^3 = w$. Then the above element of arc length can be expressed as

$$ds^2 = \vec{E}_i\cdot\vec{E}_j\,dx^i dx^j = g_{ij}dx^i dx^j, \qquad\qquad i,j = 1,2,3$$

where

$$g_{ij} = \vec{E}_i\cdot\vec{E}_j = \frac{\partial \vec{r}}{\partial x^i}\cdot\frac{\partial \vec{r}}{\partial x^j} = \frac{\partial y^m}{\partial x^i}\frac{\partial y^m}{\partial x^j}, \qquad i,j \text{ free indices} \tag{1.3.13}$$

are called the metric components of the curvilinear coordinate system. The metric components may be thought of as the elements of a symmetric matrix, since $g_{ij} = g_{ji}$. In the rectangular coordinate system x, y, z, the element of arc length squared is $ds^2 = dx^2 + dy^2 + dz^2$. In this space the metric components are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

■

EXAMPLE 1.3-3. (Cylindrical coordinates (r, θ, z))

The transformation equations from rectangular coordinates to cylindrical coordinates can be expressed as $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$. Here $y^1 = x,\, y^2 = y,\, y^3 = z$ and $x^1 = r,\, x^2 = \theta,\, x^3 = z$, and the position vector can be expressed $\vec{r} = \vec{r}(r, \theta, z) = r\cos\theta\,\widehat{\mathbf{e}}_1 + r\sin\theta\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3$. The derivatives of this position vector are calculated and we find

$$\vec{E}_1 = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\widehat{\mathbf{e}}_1 + \sin\theta\,\widehat{\mathbf{e}}_2, \quad \vec{E}_2 = \frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\widehat{\mathbf{e}}_1 + r\cos\theta\,\widehat{\mathbf{e}}_2, \quad \vec{E}_3 = \frac{\partial \vec{r}}{\partial z} = \widehat{\mathbf{e}}_3.$$

From the results in equation (1.3.13), the metric components of this space are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that since $g_{ij} = 0$ when $i \neq j$, the coordinate system is orthogonal.

■

Given a set of transformations of the form found in equation (1.3.10), one can readily determine the metric components associated with the generalized coordinates. For future reference we list several different coordinate systems together with their metric components. Each of the listed coordinate systems are orthogonal and so $g_{ij} = 0$ for $i \neq j$. The metric components of these orthogonal systems have the form

$$g_{ij} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

and the element of arc length squared is

$$ds^2 = h_1^2 (dx^1)^2 + h_2^2 (dx^2)^2 + h_3^2 (dx^3)^2.$$

1. Cartesian coordinates (x, y, z)

$$\begin{aligned} x &= x \qquad & h_1 &= 1 \\ y &= y \qquad & h_2 &= 1 \\ z &= z \qquad & h_3 &= 1 \end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$x =$Constant, $y =$Constant and $z =$Constant.

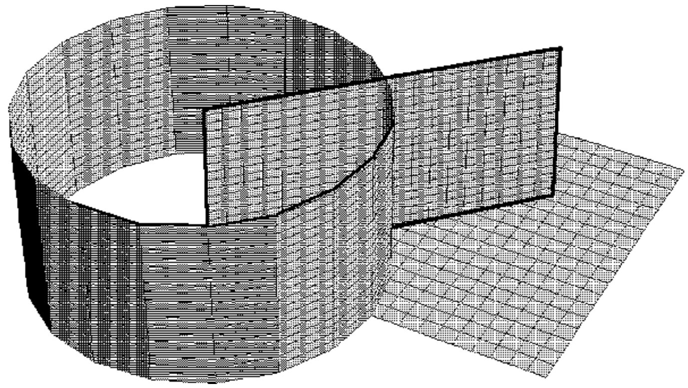

Figure 1.3-1. Cylindrical coordinates.

2. Cylindrical coordinates (r, θ, z)

$$
\begin{array}{lll}
x = r\cos\theta & r \geq 0 & h_1 = 1 \\
y = r\sin\theta & 0 \leq \theta \leq 2\pi & h_2 = r \\
z = z & -\infty < z < \infty & h_3 = 1
\end{array}
$$

The coordinate curves, illustrated in the figure 1.3-1, are formed by the intersection of the coordinate surfaces

$$
\begin{array}{rl}
x^2 + y^2 = r^2, & \text{Cylinders} \\
y/x = \tan\theta & \text{Planes} \\
z = Constant & \text{Planes.}
\end{array}
$$

3. Spherical coordinates (ρ, θ, ϕ)

$$
\begin{array}{lll}
x = \rho\sin\theta\cos\phi & \rho \geq 0 & h_1 = 1 \\
y = \rho\sin\theta\sin\phi & 0 \leq \theta \leq \pi & h_2 = \rho \\
z = \rho\cos\theta & 0 \leq \phi \leq 2\pi & h_3 = \rho\sin\theta
\end{array}
$$

The coordinate curves, illustrated in the figure 1.3-2, are formed by the intersection of the coordinate surfaces

$$
\begin{array}{rl}
x^2 + y^2 + z^2 = \rho^2 & \text{Spheres} \\
x^2 + y^2 = \tan^2\theta\, z^2 & \text{Cones} \\
y = x\,\tan\phi & \text{Planes.}
\end{array}
$$

4. Parabolic cylindrical coordinates (ξ, η, z)

$$
\begin{array}{lll}
x = \xi\eta & -\infty < \xi < \infty & h_1 = \sqrt{\xi^2 + \eta^2} \\
y = \frac{1}{2}(\xi^2 - \eta^2) & -\infty < z < \infty & h_2 = \sqrt{\xi^2 + \eta^2} \\
z = z & \eta \geq 0 & h_3 = 1
\end{array}
$$

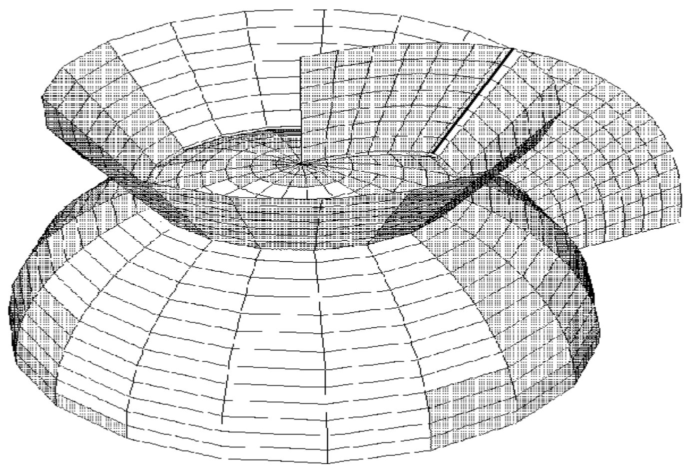

Figure 1.3-2. Spherical coordinates.

The coordinate curves, illustrated in the figure 1.3-3, are formed by the intersection of the coordinate surfaces

$$x^2 = -2\xi^2(y - \frac{\xi^2}{2}) \qquad \text{Parabolic cylinders}$$
$$x^2 = 2\eta^2(y + \frac{\eta^2}{2}) \qquad \text{Parabolic cylinders}$$
$$z = Constant \qquad \text{Planes.}$$

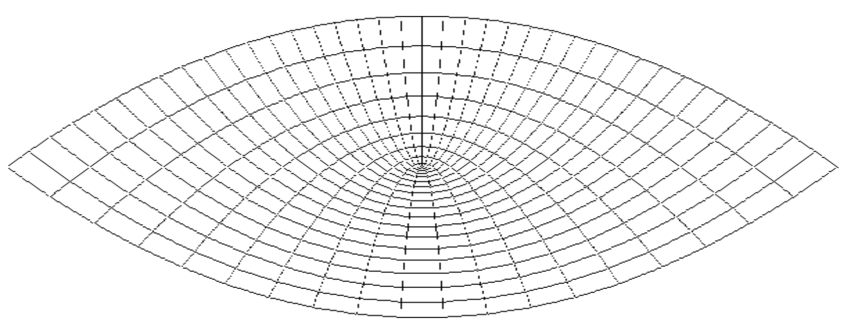

Figure 1.3-3. Parabolic cylindrical coordinates in plane $z = 0$.

5. Parabolic coordinates (ξ, η, ϕ)

$$\begin{array}{lll} x = \xi\eta\cos\phi & \xi \geq 0 & h_1 = \sqrt{\xi^2 + \eta^2} \\ y = \xi\eta\sin\phi & \eta \geq 0 & h_2 = \sqrt{\xi^2 + \eta^2} \\ z = \frac{1}{2}(\xi^2 - \eta^2) & 0 < \phi < 2\pi & h_3 = \xi\eta \end{array}$$

The coordinate curves, illustrated in the figure 1.3-4, are formed by the intersection of the coordinate surfaces

$$x^2 + y^2 = -2\xi^2 (z - \frac{\xi^2}{2}) \qquad \text{Paraboloids}$$

$$x^2 + y^2 = 2\eta^2 (z + \frac{\eta^2}{2}) \qquad \text{Paraboloids}$$

$$y = x\,\tan\phi \qquad \text{Planes.}$$

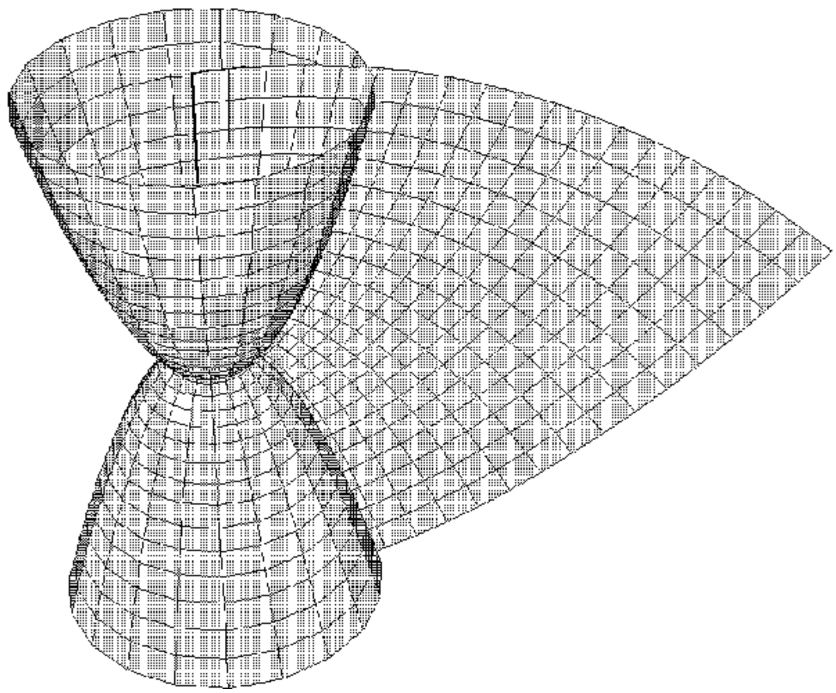

Figure 1.3-4. Parabolic coordinates, $\phi = \pi/4$.

6. Elliptic cylindrical coordinates (ξ, η, z)

$$\begin{array}{lll} x = \cosh\xi\cos\eta & \xi \geq 0 & h_1 = \sqrt{\sinh^2\xi + \sin^2\eta} \\ y = \sinh\xi\sin\eta & 0 \leq \eta \leq 2\pi & h_2 = \sqrt{\sinh^2\xi + \sin^2\eta} \\ z = z & -\infty < z < \infty & h_3 = 1 \end{array}$$

The coordinate curves, illustrated in the figure 1.3-5, are formed by the intersection of the coordinate surfaces

$$\frac{x^2}{\cosh^2\xi} + \frac{y^2}{\sinh^2\xi} = 1 \qquad \text{Elliptic cylinders}$$

$$\frac{x^2}{\cos^2\eta} - \frac{y^2}{\sin^2\eta} = 1 \qquad \text{Hyperbolic cylinders}$$

$$z = Constant \qquad \text{Planes.}$$

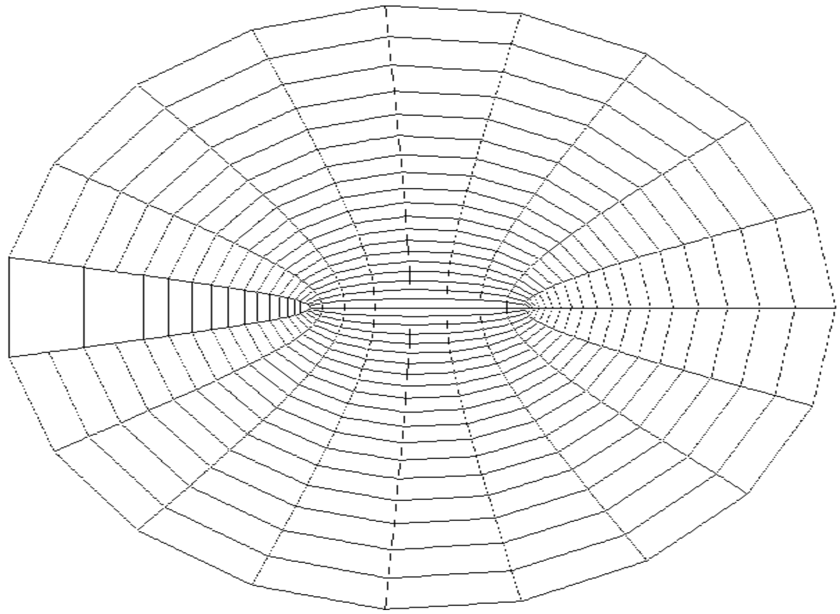

Figure 1.3-5. Elliptic cylindrical coordinates in the plane $z = 0$.

7. Elliptic coordinates (ξ, η, ϕ)

$$
\begin{aligned}
x &= \sqrt{(1-\eta^2)(\xi^2-1)}\cos\phi \qquad & 1 \le \xi < \infty \qquad & h_1 = \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}} \\
y &= \sqrt{(1-\eta^2)(\xi^2-1)}\sin\phi \qquad & -1 \le \eta \le 1 \qquad & h_2 = \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}} \\
z &= \xi\eta & 0 \le \phi < 2\pi \qquad & h_3 = \sqrt{(1-\eta^2)(\xi^2-1)}
\end{aligned}
$$

The coordinate curves, illustrated in the figure 1.3-6, are formed by the intersection of the coordinate surfaces

$$
\begin{aligned}
\frac{x^2}{\xi^2-1} + \frac{y^2}{\xi^2-1} + \frac{z^2}{\xi^2} &= 1 \qquad \text{Prolate ellipsoid} \\
\frac{z^2}{\eta^2} - \frac{x^2}{1-\eta^2} - \frac{y^2}{1-\eta^2} &= 1 \qquad \text{Two-sheeted hyperboloid} \\
y = x\,\tan\phi & \quad \text{Planes}
\end{aligned}
$$

8. Bipolar coordinates (u, v, z)

$$
\begin{aligned}
x &= \frac{a\sinh v}{\cosh v - \cos u}, \qquad & 0 \le u < 2\pi \qquad & h_1^2 = h_2^2 \\
y &= \frac{a\sin u}{\cosh v - \cos u}, \qquad & -\infty < v < \infty \qquad & h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2} \\
z &= z \qquad -\infty < z < \infty & & h_3^2 = 1
\end{aligned}
$$

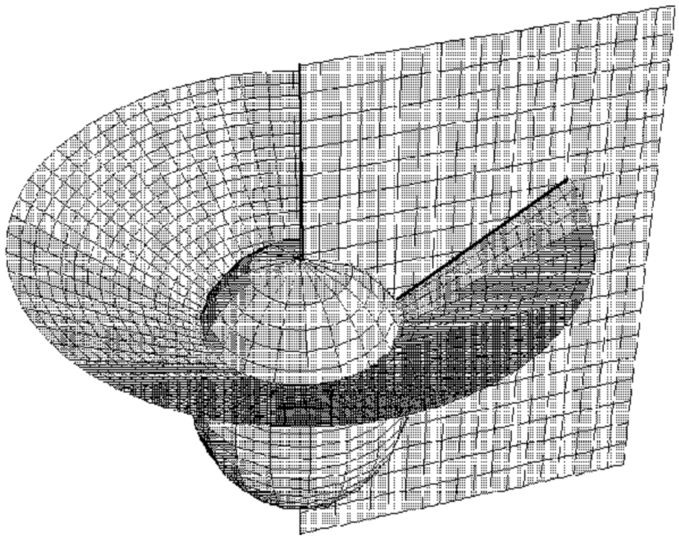

Figure 1.3-6. Elliptic coordinates $\phi = \pi/4$.

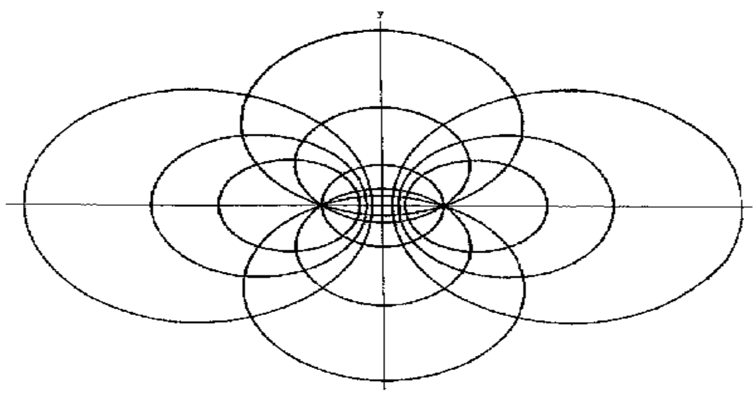

Figure 1.3-7. Bipolar coordinates.

The coordinate curves, illustrated in the figure 1.3-7, are formed by the intersection of the coordinate surfaces

$$(x - a\coth v)^2 + y^2 = \frac{a^2}{\sinh^2 v} \qquad \text{Cylinders}$$

$$x^2 + (y - a\cot u)^2 = \frac{a^2}{\sin^2 u} \qquad \text{Cylinders}$$

$$z = Constant \qquad \text{Planes.}$$

9. Conical coordinates (u, v, w)

$$
\begin{aligned}
x &= \frac{uvw}{ab}, \qquad b^2 > v^2 > a^2 > w^2, \quad u \geq 0 & h_1^2 &= 1 \\
y &= \frac{u}{a}\sqrt{\frac{(v^2-a^2)(w^2-a^2)}{a^2-b^2}} & h_2^2 &= \frac{u^2(v^2-w^2)}{(v^2-a^2)(b^2-v^2)} \\
z &= \frac{u}{b}\sqrt{\frac{(v^2-b^2)(w^2-b^2)}{b^2-a^2}} & h_3^2 &= \frac{u^2(v^2-w^2)}{(w^2-a^2)(w^2-b^2)}
\end{aligned}
$$

The coordinate curves, illustrated in the figure 1.3-8, are formed by the intersection of the coordinate surfaces

$$
\begin{aligned}
x^2 + y^2 + z^2 &= u^2 \qquad \text{Spheres} \\
\frac{x^2}{v^2} + \frac{y^2}{v^2-a^2} + \frac{z^2}{v^2-b^2} &= 0, \qquad \text{Cones} \\
\frac{x^2}{w^2} + \frac{y^2}{w^2-a^2} + \frac{z^2}{w^2-b^2} &= 0, \qquad \text{Cones.}
\end{aligned}
$$

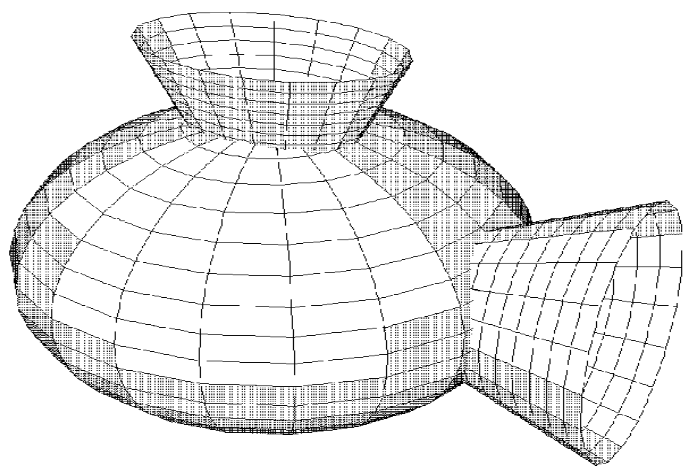

Figure 1.3-8. Conical coordinates.

10. Prolate spheroidal coordinates (u, v, ϕ)

$$
\begin{aligned}
x &= a\sinh u \sin v \cos\phi, \quad u \geq 0 & h_1^2 &= h_2^2 \\
y &= a\sinh u \sin v \sin\phi, \quad 0 \leq v \leq \pi & h_2^2 &= a^2(\sinh^2 u + \sin^2 v) \\
z &= a\cosh u \cos v, \quad 0 \leq \phi < 2\pi & h_3^2 &= a^2 \sinh^2 u \sin^2 v
\end{aligned}
$$

The coordinate curves, illustrated in the figure 1.3-9, are formed by the intersection of the coordinate surfaces

$$
\begin{aligned}
\frac{x^2}{(a\sinh u)^2} + \frac{y^2}{(a\sinh u)^2} + \frac{z^2}{(a\cosh u)^2} &= 1, \qquad \text{Prolate ellipsoids} \\
\frac{z^2}{(a\cos v)^2} - \frac{x^2}{(a\sin v)^2} - \frac{y^2}{(a\sin v)^2} &= 1, \qquad \text{Two-sheeted hyperboloid} \\
y &= x\,\tan\phi, \qquad \text{Planes.}
\end{aligned}
$$

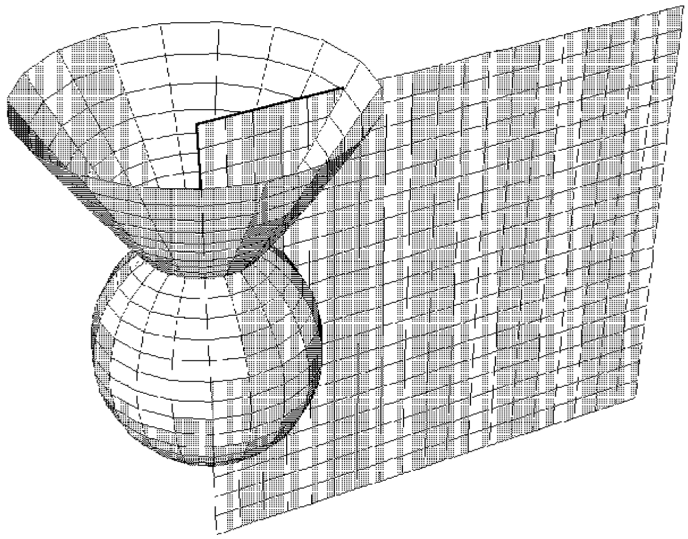

Figure 1.3-9. Prolate spheroidal coordinates

11. Oblate spheroidal coordinates (ξ, η, ϕ)

$$\begin{aligned} x &= a\cosh\xi\cos\eta\cos\phi, \qquad \xi \geq 0 & h_1^2 &= h_2^2 \\ y &= a\cosh\xi\cos\eta\sin\phi, \qquad -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2} & h_2^2 &= a^2(\sinh^2\xi + \sin^2\eta) \\ z &= a\sinh\xi\sin\eta, \quad 0 \leq \phi \leq 2\pi & h_3^2 &= a^2\cosh^2\xi\cos^2\eta \end{aligned}$$

The coordinate curves, illustrated in the figure 1.3-10, are formed by the intersection of the coordinate surfaces

$$\frac{x^2}{(a\cosh\xi)^2} + \frac{y^2}{(a\cosh\xi)^2} + \frac{z^2}{(a\sinh\xi)^2} = 1, \qquad \text{Oblate ellipsoids}$$

$$\frac{x^2}{(a\cos\eta)^2} + \frac{y^2}{(a\cos\eta)^2} - \frac{z^2}{(a\sin\eta)^2} = 1, \qquad \text{One-sheet hyperboloids}$$

$$y = x\,\tan\phi, \qquad \text{Planes.}$$

12. Toroidal coordinates (u, v, ϕ)

$$\begin{aligned} x &= \frac{a\sinh v\cos\phi}{\cosh v - \cos u}, \quad 0 \leq u < 2\pi & h_1^2 &= h_2^2 \\ y &= \frac{a\sinh v\sin\phi}{\cosh v - \cos u}, \quad -\infty < v < \infty & h_2^2 &= \frac{a^2}{(\cosh v - \cos u)^2} \\ z &= \frac{a\sin u}{\cosh v - \cos u}, \quad 0 \leq \phi < 2\pi & h_3^2 &= \frac{a^2\sinh^2 v}{(\cosh v - \cos u)^2} \end{aligned}$$

The coordinate curves, illustrated in the figure 1.3-11, are formed by the intersection of the coordinate surfaces

$$x^2 + y^2 + \left(z - \frac{a\cos u}{\sin u}\right)^2 = \frac{a^2}{\sin^2 u}, \qquad \text{Spheres}$$

$$\left(\sqrt{x^2+y^2} - a\frac{\cosh v}{\sinh v}\right)^2 + z^2 = \frac{a^2}{\sinh^2 v}, \qquad \text{Torus}$$

$$y = x\,\tan\phi, \qquad \text{planes}$$

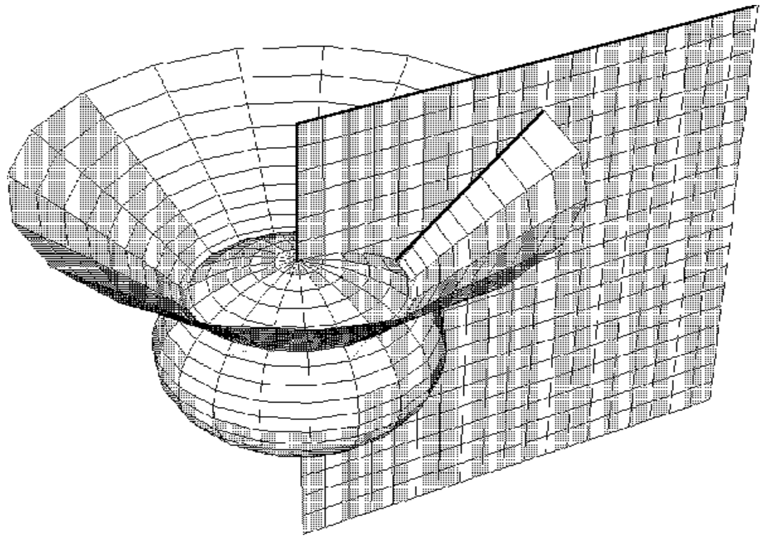

Figure 1.3-10. Oblate spheroidal coordinates

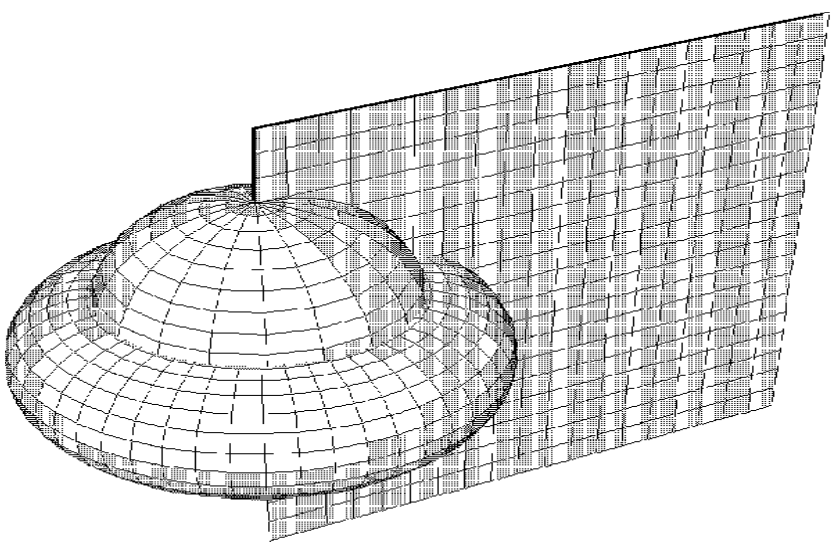

Figure 1.3-11. Toroidal coordinates

EXAMPLE 1.3-4. Show the Kronecker delta δ^i_j is a mixed second order tensor.

Solution: Assume we have a coordinate transformation $x^i = x^i(\overline{x}), i = 1, \ldots, N$ of the form (1.2.30) and possessing an inverse transformation of the form (1.2.32). Let $\overline{\delta}^i_j$ and δ^i_j denote the Kronecker delta in the barred and unbarred system of coordinates. By definition the Kronecker delta is defined

$$\overline{\delta}^i_j = \delta^i_j = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases}.$$

Employing the chain rule we write

$$\frac{\partial \overline{x}^m}{\partial \overline{x}^n} = \frac{\partial \overline{x}^m}{\partial x^i}\frac{\partial x^i}{\partial \overline{x}^n} = \frac{\partial \overline{x}^m}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^n}\delta^i_k \tag{1.3.14}$$

By hypothesis, the $\overline{x}^i, i = 1,\dots,N$ are independent coordinates and therefore we have $\frac{\partial \overline{x}^m}{\partial \overline{x}^n} = \overline{\delta}^m_n$ and (1.3.14) simplifies to

$$\overline{\delta}^m_n = \delta^i_k \frac{\partial \overline{x}^m}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^n}.$$

Therefore, the Kronecker delta transforms as a mixed second order tensor.

■

Conjugate Metric Tensor

Let g denote the determinant of the matrix having the metric tensor $g_{ij}, i,j = 1,\dots,N$ as its elements. In our study of cofactor elements of a matrix we have shown that

$$cof(g_{1j})g_{1k} + cof(g_{2j})g_{2k} + \ldots + cof(g_{Nj})g_{Nk} = g\delta^j_k. \tag{1.3.15}$$

We can use this fact to find the elements in the inverse matrix associated with the matrix having the components g_{ij}. The elements of this inverse matrix are

$$g^{ij} = \frac{1}{g} cof(g_{ij}) \tag{1.3.16}$$

and are called the conjugate metric components. We examine the summation $g^{ij}g_{ik}$ and find:

$$\begin{aligned} g^{ij}g_{ik} &= g^{1j}g_{1k} + g^{2j}g_{2k} + \ldots + g^{Nj}g_{Nk} \\ &= \frac{1}{g}\left[cof(g_{1j})g_{1k} + cof(g_{2j})g_{2k} + \ldots + cof(g_{Nj})g_{Nk}\right] \\ &= \frac{1}{g}\left[g\delta^j_k\right] = \delta^j_k \end{aligned}$$

The equation

$$g^{ij}g_{ik} = \delta^j_k \tag{1.3.17}$$

is an example where we can use the quotient law to show g^{ij} is a second order contravariant tensor. Because of the symmetry of g^{ij} and g_{ij} the equation (1.3.17) can be represented in other forms.

EXAMPLE 1.3-5. Let A_i and A^i denote respectively the covariant and contravariant components of a vector $\vec{A}$. Show these components are related by the equations

$$A_i = g_{ij}A^j \tag{1.3.18}$$

$$A^k = g^{jk}A_j \tag{1.3.19}$$

where g_{ij} and g^{ij} are the metric and conjugate metric components of the space.

Solution: We multiply the equation (1.3.18) by g^{im} (inner product) and use equation (1.3.17) to simplify the results. This produces the equation $g^{im}A_i = g^{im}g_{ij}A^j = \delta^m_j A^j = A^m$. Changing indices produces the result given in equation (1.3.19). Conversely, if we start with equation (1.3.19) and multiply by g_{km} (inner product) we obtain $g_{km}A^k = g_{km}g^{jk}A_j = \delta^j_m A_j = A_m$ which is another form of the equation (1.3.18) with the indices changed.

Notice the consequences of what the equations (1.3.18) and (1.3.19) imply when we are in an orthogonal Cartesian coordinate system where

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this special case, we have

$$\begin{aligned} A_1 &= g_{11}A^1 + g_{12}A^2 + g_{13}A^3 = A^1 \\ A_2 &= g_{21}A^1 + g_{22}A^2 + g_{23}A^3 = A^2 \\ A_3 &= g_{31}A^1 + g_{32}A^2 + g_{33}A^3 = A^3. \end{aligned}$$

These equations tell us that in a Cartesian coordinate system the contravariant and covariant components are identically the same.

■

EXAMPLE 1.3-6. We have previously shown that if A_i is a covariant tensor of rank 1 its components in a barred system of coordinates are

$$\overline{A}_i = A_j \frac{\partial x^j}{\partial \overline{x}^i}. \tag{1.3.20}$$

Solve for the A_j in terms of the $\overline{A}_j$. (i.e. find the inverse transformation).

Solution: Multiply equation (1.3.20) by $\frac{\partial \overline{x}^i}{\partial x^m}$ (inner product) and obtain

$$\overline{A}_i \frac{\partial \overline{x}^i}{\partial x^m} = A_j \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial \overline{x}^i}{\partial x^m}. \tag{1.3.21}$$

In the above product we have $\dfrac{\partial x^j}{\partial \overline{x}^i} \dfrac{\partial \overline{x}^i}{\partial x^m} = \dfrac{\partial x^j}{\partial x^m} = \delta^j_m$ since x^j and x^m are assumed to be independent coordinates. This reduces equation (1.3.21) to the form

$$\overline{A}_i \frac{\partial \overline{x}^i}{\partial x^m} = A_j \delta^j_m = A_m \tag{1.3.22}$$

which is the desired inverse transformation.

This result can be obtained in another way. Examine the transformation equation (1.3.20) and ask the question, "When we have two coordinate systems, say a barred and an unbarred system, does it matter which system we call the barred system?" With some thought it should be obvious that it doesn't matter which system you label as the barred system. Therefore, we can interchange the barred and unbarred symbols in equation (1.3.20) and obtain the result $A_i = \overline{A}_j \dfrac{\partial \overline{x}^j}{\partial x^i}$ which is the same form as equation (1.3.22), but with a different set of indices.

■

Associated Tensors

Associated tensors can be constructed by taking the inner product of known tensors with either the metric or conjugate metric tensor.

Definition: (Associated tensor) Any tensor constructed by multiplying (inner product) a given tensor with the metric or conjugate metric tensor is called an associated tensor.

Associated tensors are different ways of representing a tensor. The multiplication of a tensor by the metric or conjugate metric tensor has the effect of lowering or raising indices. For example the covariant and contravariant components of a vector are different representations of the same vector in different forms. These forms are associated with one another by way of the metric and conjugate metric tensor and

$$g^{ij}A_i = A^j \qquad g_{ij}A^j = A_i.$$

EXAMPLE 1.3-7. The following are some examples of associated tensors.

$$\begin{aligned} A^j &= g^{ij}A_i & A_j &= g_{ij}A^i \\ A^m_{.jk} &= g^{mi}A_{ijk} & A^{i.k}_{m} &= g_{mj}A^{ijk} \\ A^{.nm}_{i..} &= g^{mk}g^{nj}A_{ijk} & A_{mjk} &= g_{im}A^i_{.jk} \end{aligned}$$

Sometimes 'dots'are used as indices in order to represent the location of the index that was raised or lowered. If a tensor is symmetric, the position of the index is immaterial and so a dot is not needed. For example, if A_{mn} is a symmetric tensor, then it is easy to show that $A^n_{.m}$ and $A^{.n}_{m}$ are equal and therefore can be written as A^n_m without confusion.

Higher order tensors are similarly related. For example, if we find a fourth order covariant tensor T_{ijkm} we can then construct the fourth order contravariant tensor T^{pqrs} from the relation

$$T^{pqrs} = g^{pi}g^{qj}g^{rk}g^{sm}T_{ijkm}.$$

This fourth order tensor can also be expressed as a mixed tensor. Some mixed tensors associated with the given fourth order covariant tensor are:

$$T^p_{.jkm} = g^{pi}T_{ijkm}, \qquad T^{pq}_{..km} = g^{qj}T^p_{.jkm}.$$

■

Riemann Space V_N

A Riemannian space V_N is said to exist if the element of arc length squared has the form

$$ds^2 = g_{ij}dx^i dx^j \tag{1.3.23}$$

where the metrices $g_{ij} = g_{ij}(x^1, x^2, \dots, x^N)$ are continuous functions of the coordinates and are different from constants. In the special case $g_{ij} = \delta_{ij}$ the Riemannian space V_N reduces to a Euclidean space E_N. The element of arc length squared defined by equation (1.3.23) is called the Riemannian metric and any geometry which results by using this metric is called a Riemannian geometry. A space V_N is called flat if it is possible to find a coordinate transformation where the element of arclength squared is $ds^2 = \epsilon_i(dx^i)^2$ where each ϵ_i is either $+1$ or -1. A space which is not flat is called curved.

Geometry in V_N

Given two vectors $\vec{A} = A^i\vec{E}_i$ and $\vec{B} = B^j\vec{E}_j$, then their dot product can be represented

$$\vec{A}\cdot\vec{B} = A^iB^j\vec{E}_i\cdot\vec{E}_j = g_{ij}A^iB^j = A_jB^j = A^iB_i = g^{ij}A_jB_i = |\vec{A}||\vec{B}|\cos\theta. \tag{1.3.24}$$

Consequently, in an N dimensional Riemannian space V_N the dot or inner product of two vectors $\vec{A}$ and $\vec{B}$ is defined:

$$g_{ij}A^iB^j = A_jB^j = A^iB_i = g^{ij}A_jB_i = AB\cos\theta. \tag{1.3.25}$$

In this definition A is the magnitude of the vector A^i, the quantity B is the magnitude of the vector B_i and θ is the angle between the vectors when their origins are made to coincide. In the special case that $\theta = 90^\circ$ we have $g_{ij}A^iB^j = 0$ as the condition that must be satisfied in order that the given vectors A^i and B^i are orthogonal to one another. Consider also the special case of equation (1.3.25) when $A^i = B^i$ and $\theta = 0$. In this case the equations (1.3.25) inform us that

$$g^{in}A_nA_i = A^iA_i = g_{in}A^iA^n = (A)^2. \tag{1.3.26}$$

From this equation one can determine the magnitude of the vector A^i. The magnitudes A and B can be written $A = (g_{in}A^iA^n)^{\frac{1}{2}}$ and $B = (g_{pq}B^pB^q)^{\frac{1}{2}}$ and so we can express equation (1.3.24) in the form

$$\cos\theta = \frac{g_{ij}A^iB^j}{(g_{mn}A^mA^n)^{\frac{1}{2}}(g_{pq}B^pB^q)^{\frac{1}{2}}}. \tag{1.3.27}$$

An import application of the above concepts arises in the dynamics of rigid body motion. Note that if a vector A^i has constant magnitude and the magnitude of $\frac{dA^i}{dt}$ is different from zero, then the vectors A^i and $\frac{dA^i}{dt}$ must be orthogonal to one another due to the fact that $g_{ij}A^i\frac{dA^j}{dt} = 0$. As an example, consider the unit vectors $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2$ and $\widehat{\mathbf{e}}_3$ on a rotating system of Cartesian axes. We have for constants c_i, $i = 1, 6$ that

$$\frac{d\,\widehat{\mathbf{e}}_1}{dt} = c_1\,\widehat{\mathbf{e}}_2 + c_2\,\widehat{\mathbf{e}}_3 \qquad \frac{d\,\widehat{\mathbf{e}}_2}{dt} = c_3\,\widehat{\mathbf{e}}_3 + c_4\,\widehat{\mathbf{e}}_1 \qquad \frac{d\,\widehat{\mathbf{e}}_3}{dt} = c_5\,\widehat{\mathbf{e}}_1 + c_6\,\widehat{\mathbf{e}}_2$$

because the derivative of any $\widehat{\mathbf{e}}_i$ (i fixed) constant vector must lie in a plane containing the vectors $\widehat{\mathbf{e}}_j$ and $\widehat{\mathbf{e}}_k$, ($j \neq i$, $k \neq i$ and $j \neq k$), since any vector in this plane must be perpendicular to $\widehat{\mathbf{e}}_i$.

The above definition of a dot product in V_N can be used to define unit vectors in V_N.

Definition: (Unit vector) Whenever the magnitude of a vector A^i is unity, the vector is called a unit vector. In this case we have

$$g_{ij}A^iA^j = 1. \tag{1.3.28}$$

EXAMPLE 1.3-8. (Unit vectors)

In V_N the element of arc length squared is expressed $ds^2 = g_{ij}\,dx^i dx^j$ which can be expressed in the form $1 = g_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}$. This equation states that the vector $\dfrac{dx^i}{ds}$, $i = 1,\dots,N$ is a unit vector. One application of this equation is to consider a particle moving along a curve in V_N which is described by the parametric equations $x^i = x^i(t)$, for $i = 1,\dots,N$. The vector $V^i = \frac{dx^i}{dt}$, $i = 1,\dots,N$ represents a velocity vector of the particle. By chain rule differentiation we have

$$V^i = \frac{dx^i}{dt} = \frac{dx^i}{ds}\frac{ds}{dt} = V\frac{dx^i}{ds}, \tag{1.3.29}$$

where $V = \frac{ds}{dt}$ is the scalar speed of the particle and $\frac{dx^i}{ds}$ is a unit tangent vector to the curve. The equation (1.3.29) shows that the velocity is directed along the tangent to the curve and has a magnitude V. That is

$$\left(\frac{ds}{dt}\right)^2 = (V)^2 = g_{ij}V^iV^j.$$

■

EXAMPLE 1.3-9. (Curvilinear coordinates)

Find an expression for the cosine of the angles between the coordinate curves associated with the transformation equations

$$x = x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w).$$

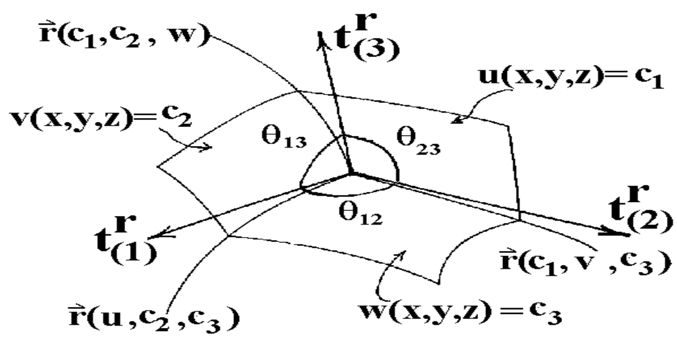

Figure 1.3-12. Angles between curvilinear coordinates.

Solution: Let $y^1 = x$, $y^2 = y$, $y^3 = z$ and $x^1 = u$, $x^2 = v$, $x^3 = w$ denote the Cartesian and curvilinear coordinates respectively. With reference to the figure 1.3-12 we can interpret the intersection of the surfaces $v = c_2$ and $w = c_3$ as the curve $\vec{r} = \vec{r}(u, c_2, c_3)$ which is a function of the parameter u. By moving only along this curve we have $d\vec{r} = \dfrac{\partial \vec{r}}{\partial u}\, du$ and consequently

$$ds^2 = d\vec{r} \cdot d\vec{r} = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u}\, du du = g_{11}(dx^1)^2,$$

or

$$1 = \frac{d\vec{r}}{ds} \cdot \frac{d\vec{r}}{ds} = g_{11} \left(\frac{dx^1}{ds}\right)^2.$$

This equation shows that the vector $\frac{dx^1}{ds} = \frac{1}{\sqrt{g_{11}}}$ is a unit vector along this curve. This tangent vector can be represented by $t^r_{(1)} = \frac{1}{\sqrt{g_{11}}}\delta^r_1$.

The curve which is defined by the intersection of the surfaces $u = c_1$ and $w = c_3$ has the unit tangent vector $t^r_{(2)} = \frac{1}{\sqrt{g_{22}}}\delta^r_2$. Similarly, the curve which is defined as the intersection of the surfaces $u = c_1$ and $v = c_2$ has the unit tangent vector $t^r_{(3)} = \frac{1}{\sqrt{g_{33}}}\delta^r_3$. The cosine of the angle θ_{12}, which is the angle between the unit vectors $t^r_{(1)}$ and $t^r_{(2)}$, is obtained from the result of equation (1.3.25). We find

$$\cos\theta_{12} = g_{pq} t^p_{(1)} t^q_{(2)} = g_{pq} \frac{1}{\sqrt{g_{11}}}\delta^p_1 \frac{1}{\sqrt{g_{22}}}\delta^q_2 = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}.$$

For θ_{13} the angle between the directions $t^i_{(1)}$ and $t^i_{(3)}$ we find

$$\cos\theta_{13} = \frac{g_{13}}{\sqrt{g_{11}}\sqrt{g_{33}}}.$$

Finally, for θ_{23} the angle between the directions $t^i_{(2)}$ and $t^i_{(3)}$ we find

$$\cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}}\sqrt{g_{33}}}.$$

When $\theta_{13} = \theta_{12} = \theta_{23} = 90°$, we have $g_{12} = g_{13} = g_{23} = 0$ and the coordinate curves which make up the curvilinear coordinate system are orthogonal to one another.

In an orthogonal coordinate system we adopt the notation

$$g_{11} = (h_1)^2, \qquad g_{22} = (h_2)^2, \qquad g_{33} = (h_3)^2 \qquad \text{and} \qquad g_{ij} = 0,\ i \neq j.$$

∎

Epsilon Permutation Symbol

Associated with the $e-$permutation symbols there are the epsilon permutation symbols defined by the relations

$$\epsilon_{ijk} = \sqrt{g} e_{ijk} \qquad \text{and} \qquad \epsilon^{ijk} = \frac{1}{\sqrt{g}} e^{ijk} \tag{1.3.30}$$

where g is the determinant of the metrices g_{ij}.

It can be demonstrated that the e_{ijk} permutation symbol is a relative tensor of weight -1 whereas the ϵ_{ijk} permutation symbol is an absolute tensor. Similarly, the e^{ijk} permutation symbol is a relative tensor of weight $+1$ and the corresponding ϵ^{ijk} permutation symbol is an absolute tensor.

EXAMPLE 1.3-10. (ϵ permutation symbol)

Show that e_{ijk} is a relative tensor of weight -1 and the corresponding ϵ_{ijk} permutation symbol is an absolute tensor.

Solution: Examine the Jacobian

$$J\left(\frac{x}{\overline{x}}\right) = \begin{vmatrix} \frac{\partial x^1}{\partial \overline{x}^1} & \frac{\partial x^1}{\partial \overline{x}^2} & \frac{\partial x^1}{\partial \overline{x}^3} \\ \frac{\partial x^2}{\partial \overline{x}^1} & \frac{\partial x^2}{\partial \overline{x}^2} & \frac{\partial x^2}{\partial \overline{x}^3} \\ \frac{\partial x^3}{\partial \overline{x}^1} & \frac{\partial x^3}{\partial \overline{x}^2} & \frac{\partial x^3}{\partial \overline{x}^3} \end{vmatrix}$$

and make the substitution

$$a^i_j = \frac{\partial x^i}{\partial \overline{x}^j}, \; i, j = 1, 2, 3.$$

From the definition of a determinant we may write

$$e_{ijk} a^i_m a^j_n a^k_p = J(\frac{x}{\overline{x}}) e_{mnp}. \tag{1.3.31}$$

By definition, $\overline{e}_{mnp} = e_{mnp}$ in all coordinate systems and hence equation (1.3.31) can be expressed in the form

$$\left[J(\frac{x}{\overline{x}})\right]^{-1} e_{ijk} \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n} \frac{\partial x^k}{\partial \overline{x}^p} = \overline{e}_{mnp} \tag{1.3.32}$$

which demonstrates that e_{ijk} transforms as a relative tensor of weight -1.

We have previously shown the metric tensor g_{ij} is a second order covariant tensor and transforms according to the rule $\overline{g}_{ij} = g_{mn} \dfrac{\partial x^m}{\partial \overline{x}^i} \dfrac{\partial x^n}{\partial \overline{x}^j}$. Taking the determinant of this result we find

$$\overline{g} = |\overline{g}_{ij}| = |g_{mn}| \left| \frac{\partial x^m}{\partial \overline{x}^i} \right|^2 = g \left[J(\frac{x}{\overline{x}})\right]^2 \tag{1.3.33}$$

where g is the determinant of (g_{ij}) and $\overline{g}$ is the determinant of $(\overline{g}_{ij})$. This result demonstrates that g is a scalar invariant of weight $+2$. Taking the square root of this result we find that

$$\sqrt{\overline{g}} = \sqrt{g} J(\frac{x}{\overline{x}}). \tag{1.3.34}$$

Consequently, we call $\sqrt{g}$ a scalar invariant of weight $+1$. Now multiply both sides of equation (1.3.32) by $\sqrt{g}$ and use (1.3.34) to verify the relation

$$\sqrt{g}\, e_{ijk} \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n} \frac{\partial x^k}{\partial \overline{x}^p} = \sqrt{\overline{g}}\, \overline{e}_{mnp}. \tag{1.3.35}$$

This equation demonstrates that the quantity $\epsilon_{ijk} = \sqrt{g}\, e_{ijk}$ transforms like an absolute tensor.

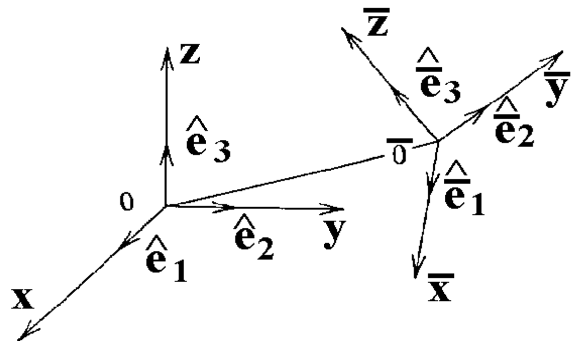

Figure 1.3-14. Translation followed by rotation of axes

In a similar manner one can show e^{ijk} is a relative tensor of weight $+1$ and $\epsilon^{ijk} = \frac{1}{\sqrt{g}} e^{ijk}$ is an absolute tensor. This is left as an exercise.

■

Another exercise found at the end of this section is to show that a generalization of the $e - \delta$ identity is the epsilon identity

$$g^{ij} \epsilon_{ipt} \epsilon_{jrs} = g_{pr} g_{ts} - g_{ps} g_{tr}. \tag{1.3.36}$$

Cartesian Tensors

Consider the motion of a rigid rod in two dimensions. No matter how complicated the movement of the rod is we can describe the motion as a translation followed by a rotation. Consider the rigid rod $\overline{AB}$ illustrated in the figure 1.3-13.

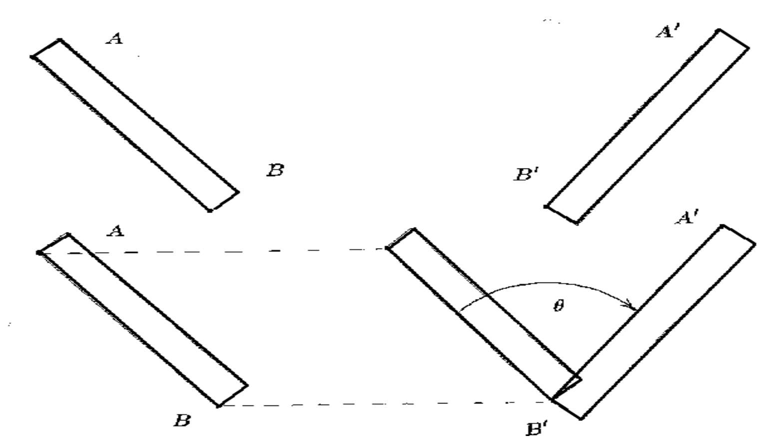

Figure 1.3-13. Motion of rigid rod

In this figure there is a before and after picture of the rod's position. By moving the point B to B' we have a translation. This is then followed by a rotation holding B fixed.

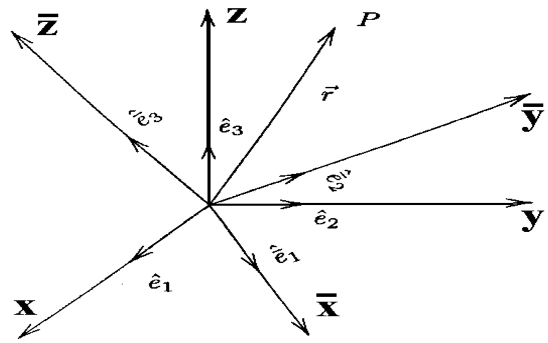

Figure 1.3-15. Rotation of axes

A similar situation exists in three dimensions. Consider two sets of Cartesian axes, say a barred and unbarred system as illustrated in the figure 1.3-14. Let us translate the origin 0 to $\overline{0}$ and then rotate the (x, y, z) axes until they coincide with the $(\overline{x}, \overline{y}, \overline{z})$ axes. We consider first the rotation of axes when the origins 0 and $\overline{0}$ coincide as the translational distance can be represented by a vector b^k, $k = 1, 2, 3$. When the origin 0 is translated to $\overline{0}$ we have the situation illustrated in the figure 1.3-15, where the barred axes can be thought of as a transformation due to rotation.

Let

$$\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3 \tag{1.3.37}$$

denote the position vector of a variable point P with coordinates (x, y, z) with respect to the origin 0 and the unit vectors $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$, $\widehat{\mathbf{e}}_3$. This same point, when referenced with respect to the origin $\overline{0}$ and the unit vectors $\hat{\overline{\mathbf{e}}}_1$, $\hat{\overline{\mathbf{e}}}_2$, $\hat{\overline{\mathbf{e}}}_3$, has the representation

$$\vec{r} = \overline{x}\,\hat{\overline{\mathbf{e}}}_1 + \overline{y}\,\hat{\overline{\mathbf{e}}}_2 + \overline{z}\,\hat{\overline{\mathbf{e}}}_3. \tag{1.3.38}$$

By considering the projections of $\vec{r}$ upon the barred and unbarred axes we can construct the transformation equations relating the barred and unbarred axes. We calculate the projections of $\vec{r}$ onto the x, y and z axes and find:

$$\begin{aligned}
\vec{r}\cdot\widehat{\mathbf{e}}_1 = x &= \overline{x}(\hat{\overline{\mathbf{e}}}_1\cdot\widehat{\mathbf{e}}_1) + \overline{y}(\hat{\overline{\mathbf{e}}}_2\cdot\widehat{\mathbf{e}}_1) + \overline{z}(\hat{\overline{\mathbf{e}}}_3\cdot\widehat{\mathbf{e}}_1)\\
\vec{r}\cdot\widehat{\mathbf{e}}_2 = y &= \overline{x}(\hat{\overline{\mathbf{e}}}_1\cdot\widehat{\mathbf{e}}_2) + \overline{y}(\hat{\overline{\mathbf{e}}}_2\cdot\widehat{\mathbf{e}}_2) + \overline{z}(\hat{\overline{\mathbf{e}}}_3\cdot\widehat{\mathbf{e}}_2)\\
\vec{r}\cdot\widehat{\mathbf{e}}_3 = z &= \overline{x}(\hat{\overline{\mathbf{e}}}_1\cdot\widehat{\mathbf{e}}_3) + \overline{y}(\hat{\overline{\mathbf{e}}}_2\cdot\widehat{\mathbf{e}}_3) + \overline{z}(\hat{\overline{\mathbf{e}}}_3\cdot\widehat{\mathbf{e}}_3).
\end{aligned} \tag{1.3.39}$$

We also calculate the projection of $\vec{r}$ onto the $\overline{x}, \overline{y}, \overline{z}$ axes and find:

$$\begin{aligned}
\vec{r}\cdot\hat{\overline{\mathbf{e}}}_1 = \overline{x} &= x(\widehat{\mathbf{e}}_1\cdot\hat{\overline{\mathbf{e}}}_1) + y(\widehat{\mathbf{e}}_2\cdot\hat{\overline{\mathbf{e}}}_1) + z(\widehat{\mathbf{e}}_3\cdot\hat{\overline{\mathbf{e}}}_1)\\
\vec{r}\cdot\hat{\overline{\mathbf{e}}}_2 = \overline{y} &= x(\widehat{\mathbf{e}}_1\cdot\hat{\overline{\mathbf{e}}}_2) + y(\widehat{\mathbf{e}}_2\cdot\hat{\overline{\mathbf{e}}}_2) + z(\widehat{\mathbf{e}}_3\cdot\hat{\overline{\mathbf{e}}}_2)\\
\vec{r}\cdot\hat{\overline{\mathbf{e}}}_3 = \overline{z} &= x(\widehat{\mathbf{e}}_1\cdot\hat{\overline{\mathbf{e}}}_3) + y(\widehat{\mathbf{e}}_2\cdot\hat{\overline{\mathbf{e}}}_3) + z(\widehat{\mathbf{e}}_3\cdot\hat{\overline{\mathbf{e}}}_3).
\end{aligned} \tag{1.3.40}$$

By introducing the notation $(y_1, y_2, y_3) = (x, y, z)$ $\quad(\overline{y}_1, \overline{y}_2, \overline{y}_3) = (\overline{x}, \overline{y}, \overline{z})$ and defining θ_{ij} as the angle between the unit vectors $\widehat{\mathbf{e}}_i$ and $\hat{\overline{\mathbf{e}}}_j$, we can represent the above transformation equations in a more concise form. We observe that the direction cosines can be written as

$$\begin{array}{lll} \ell_{11} = \widehat{\mathbf{e}}_1 \cdot \hat{\overline{\mathbf{e}}}_1 = \cos\theta_{11} & \ell_{12} = \widehat{\mathbf{e}}_1 \cdot \hat{\overline{\mathbf{e}}}_2 = \cos\theta_{12} & \ell_{13} = \widehat{\mathbf{e}}_1 \cdot \hat{\overline{\mathbf{e}}}_3 = \cos\theta_{13} \\ \ell_{21} = \widehat{\mathbf{e}}_2 \cdot \hat{\overline{\mathbf{e}}}_1 = \cos\theta_{21} & \ell_{22} = \widehat{\mathbf{e}}_2 \cdot \hat{\overline{\mathbf{e}}}_2 = \cos\theta_{22} & \ell_{23} = \widehat{\mathbf{e}}_2 \cdot \hat{\overline{\mathbf{e}}}_3 = \cos\theta_{23} \\ \ell_{31} = \widehat{\mathbf{e}}_3 \cdot \hat{\overline{\mathbf{e}}}_1 = \cos\theta_{31} & \ell_{32} = \widehat{\mathbf{e}}_3 \cdot \hat{\overline{\mathbf{e}}}_2 = \cos\theta_{32} & \ell_{33} = \widehat{\mathbf{e}}_3 \cdot \hat{\overline{\mathbf{e}}}_3 = \cos\theta_{33} \end{array} \tag{1.3.41}$$

which enables us to write the equations (1.3.39) and (1.3.40) in the form

$$y_i = \ell_{ij}\overline{y}_j \qquad \text{and} \qquad \overline{y}_i = \ell_{ji}y_j. \tag{1.3.42}$$

Using the index notation we represent the unit vectors as:

$$\hat{\overline{\mathbf{e}}}_r = \ell_{pr}\,\widehat{\mathbf{e}}_p \qquad \text{or} \qquad \widehat{\mathbf{e}}_p = \ell_{pr}\,\hat{\overline{\mathbf{e}}}_r \tag{1.3.43}$$

where ℓ_{pr} are the direction cosines. In both the barred and unbarred system the unit vectors are orthogonal and consequently we must have the dot products

$$\hat{\overline{\mathbf{e}}}_r \cdot \hat{\overline{\mathbf{e}}}_p = \delta_{rp} \qquad \text{and} \qquad \widehat{\mathbf{e}}_m \cdot \widehat{\mathbf{e}}_n = \delta_{mn} \tag{1.3.44}$$

where δ_{ij} is the Kronecker delta. Substituting equation (1.3.43) into equation (1.3.44) we find the direction cosines ℓ_{ij} must satisfy the relations:

$$\begin{array}{rl} & \hat{\overline{\mathbf{e}}}_r \cdot \hat{\overline{\mathbf{e}}}_s = \ell_{pr}\,\widehat{\mathbf{e}}_p \cdot \ell_{ms}\,\widehat{\mathbf{e}}_m = \ell_{pr}\ell_{ms}\,\widehat{\mathbf{e}}_p \cdot \widehat{\mathbf{e}}_m = \ell_{pr}\ell_{ms}\delta_{pm} = \ell_{mr}\ell_{ms} = \delta_{rs} \\ \text{and} \qquad & \widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_s = \ell_{rm}\,\hat{\overline{\mathbf{e}}}_m \cdot \ell_{sn}\,\hat{\overline{\mathbf{e}}}_n = \ell_{rm}\ell_{sn}\,\hat{\overline{\mathbf{e}}}_m \cdot \hat{\overline{\mathbf{e}}}_n = \ell_{rm}\ell_{sn}\delta_{mn} = \ell_{rm}\ell_{sm} = \delta_{rs}. \end{array}$$

The relations

$$\ell_{mr}\ell_{ms} = \delta_{rs} \qquad \text{and} \qquad \ell_{rm}\ell_{sm} = \delta_{rs}, \tag{1.3.45}$$

with summation index m, are important relations which are satisfied by the direction cosines associated with a rotation of axes.

Combining the rotation and translation equations we find

$$y_i = \underbrace{\ell_{ij}\overline{y}_j}_{rotation} + \underbrace{b_i}_{translation}. \tag{1.3.46}$$

We multiply this equation by ℓ_{ik} and make use of the relations (1.3.45) to find the inverse transformation

$$\overline{y}_k = \ell_{ik}(y_i - b_i). \tag{1.3.47}$$

These transformations are called linear or affine transformations.

Consider the $\overline{x}_i$ axes as fixed, while the x_i axes are rotating with respect to the $\overline{x}_i$ axes where both sets of axes have a common origin. Let $\vec{A} = A^i\,\widehat{\mathbf{e}}_i$ denote a vector fixed in and rotating with the x_i axes. We denote by $\left.\frac{d\vec{A}}{dt}\right|_f$ and $\left.\frac{d\vec{A}}{dt}\right|_r$ the derivatives of $\vec{A}$ with respect to the fixed (f) and rotating (r) axes. We can write, with respect to the fixed axes, that $\left.\dfrac{d\vec{A}}{dt}\right|_f = \dfrac{dA^i}{dt}\,\widehat{\mathbf{e}}_i + A^i\dfrac{d\,\widehat{\mathbf{e}}_i}{dt}$. Note that $\dfrac{d\,\widehat{\mathbf{e}}_i}{dt}$ is the derivative of a vector with constant magnitude. Therefore there exists constants ω_i, $i = 1,\dots,6$ such that

$$\frac{d\,\widehat{\mathbf{e}}_1}{dt} = \omega_3\;\widehat{\mathbf{e}}_2 - \omega_2\,\widehat{\mathbf{e}}_3 \qquad \frac{d\,\widehat{\mathbf{e}}_2}{dt} = \omega_1\;\widehat{\mathbf{e}}_3 - \omega_4\,\widehat{\mathbf{e}}_1 \qquad \frac{d\,\widehat{\mathbf{e}}_3}{dt} = \omega_5\;\widehat{\mathbf{e}}_1 - \omega_6\,\widehat{\mathbf{e}}_2$$

i.e. see page 80. From the dot product $\widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_2 = 0$ we obtain by differentiation $\widehat{\mathbf{e}}_1 \cdot \frac{d\,\widehat{\mathbf{e}}_2}{dt} + \frac{d\,\widehat{\mathbf{e}}_1}{dt} \cdot \widehat{\mathbf{e}}_2 = 0$ which implies $\omega_4 = \omega_3$. Similarly, from the dot products $\widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_3$ and $\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_3$ we obtain by differentiation the additional relations $\omega_5 = \omega_2$ and $\omega_6 = \omega_1$. The derivative of $\vec{A}$ with respect to the fixed axes can now be represented

$$\left.\frac{d\vec{A}}{dt}\right|_f = \frac{dA^i}{dt}\,\widehat{\mathbf{e}}_i + (\omega_2 A_3 - \omega_3 A_2)\,\widehat{\mathbf{e}}_1 + (\omega_3 A_1 - \omega_1 A_3)\,\widehat{\mathbf{e}}_2 + (\omega_1 A_2 - \omega_2 A_1)\,\widehat{\mathbf{e}}_3 = \left.\frac{d\vec{A}}{dt}\right|_r + \vec{\omega}\times\vec{A}$$

where $\vec{\omega} = \omega_i\,\widehat{\mathbf{e}}_i$ is called an angular velocity vector of the rotating system. The term $\vec{\omega}\times\vec{A}$ represents the velocity of the rotating system relative to the fixed system and $\left.\dfrac{d\vec{A}}{dt}\right|_r = \dfrac{dA^i}{dt}\,\widehat{\mathbf{e}}_i$ represents the derivative with respect to the rotating system.

Employing the special transformation equations (1.3.46) let us examine how tensor quantities transform when subjected to a translation and rotation of axes. These are our special transformation laws for Cartesian tensors. We examine only the transformation laws for first and second order Cartesian tensor as higher order transformation laws are easily discerned. We have previously shown that in general the first and second order tensor quantities satisfy the transformation laws:

$$\overline{A}_i = A_j\frac{\partial y_j}{\partial \overline{y}_i} \tag{1.3.48}$$

$$\overline{A}^i = A^j\frac{\partial \overline{y}_i}{\partial y_j} \tag{1.3.49}$$

$$\overline{A}^{mn} = A^{ij}\frac{\partial \overline{y}_m}{\partial y_i}\frac{\partial \overline{y}_n}{\partial y_j} \tag{1.3.50}$$

$$\overline{A}_{mn} = A_{ij}\frac{\partial y_i}{\partial \overline{y}_m}\frac{\partial y_j}{\partial \overline{y}_n} \tag{1.3.51}$$

$$\overline{A}^m_n = A^i_j\frac{\partial \overline{y}_m}{\partial y_i}\frac{\partial y_j}{\partial \overline{y}_n} \tag{1.3.52}$$

For the special case of Cartesian tensors we assume that y_i and $\overline{y}_i$, $i = 1, 2, 3$ are linearly independent. We differentiate the equations (1.3.46) and (1.3.47) and find

$$\frac{\partial y_i}{\partial \overline{y}_k} = \ell_{ij}\frac{\partial \overline{y}_j}{\partial \overline{y}_k} = \ell_{ij}\delta_{jk} = \ell_{ik}, \qquad \text{and} \qquad \frac{\partial \overline{y}_k}{\partial y_m} = \ell_{ik}\frac{\partial y_i}{\partial y_m} = \ell_{ik}\delta_{im} = \ell_{mk}.$$

Substituting these derivatives into the transformation equations (1.3.48) through (1.3.52) we produce the transformation equations

$$\begin{aligned}
\overline{A}_i &= A_j\ell_{ji}\\
\overline{A}^i &= A^j\ell_{ji}\\
\overline{A}^{mn} &= A^{ij}\ell_{im}\ell_{jn}\\
\overline{A}_{mn} &= A_{ij}\ell_{im}\ell_{jn}\\
\overline{A}^m_n &= A^i_j\ell_{im}\ell_{jn}.
\end{aligned}$$

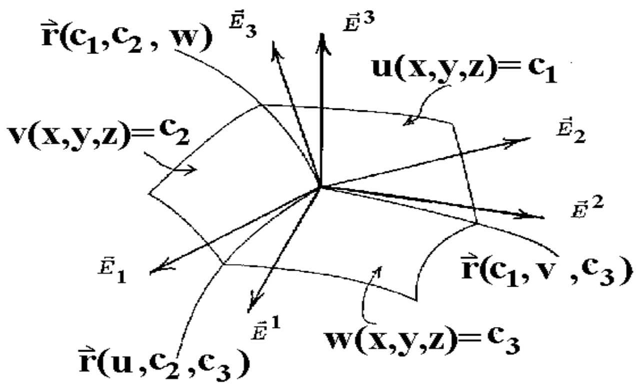

Figure 1.3-16. Transformation to curvilinear coordinates

These are the transformation laws when moving from one orthogonal system to another. In this case the direction cosines ℓ_{im} are constants and satisfy the relations given in equation (1.3.45). The transformation laws for higher ordered tensors are similar in nature to those given above.

In the unbarred system (y_1, y_2, y_3) the metric tensor and conjugate metric tensor are:

$$g_{ij} = \delta_{ij} \qquad \text{and} \qquad g^{ij} = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. In the barred system of coordinates, which is also orthogonal, we have

$$\overline{g}_{ij} = \frac{\partial y_m}{\partial \overline{y}_i}\frac{\partial y_m}{\partial \overline{y}_j}.$$

From the orthogonality relations (1.3.45) we find

$$\overline{g}_{ij} = \ell_{mi}\ell_{mj} = \delta_{ij} \qquad \text{and} \qquad \overline{g}^{ij} = \delta_{ij}.$$

We examine the associated tensors

$$\begin{aligned} A^i &= g^{ij}A_j & A_i &= g_{ij}A^j \\ A^{ij} &= g^{im}g^{jn}A_{mn} & A_{mn} &= g_{mi}g_{nj}A^{ij} \\ A^i_n &= g^{im}A_{mn} & A^i_n &= g_{nj}A^{ij} \end{aligned}$$

and find that the contravariant and covariant components are identical to one another. This holds also in the barred system of coordinates. Also note that these special circumstances allow the representation of contractions using subscript quantities only. This type of a contraction is not allowed for general tensors. It is left as an exercise to try a contraction on a general tensor using only subscripts to see what happens. Note that such a contraction does not produce a tensor. These special situations are considered in the exercises.

Physical Components

We have previously shown an arbitrary vector $\vec{A}$ can be represented in many forms depending upon the coordinate system and basis vectors selected. For example, consider the figure 1.3-16 which illustrates a Cartesian coordinate system and a curvilinear coordinate system.

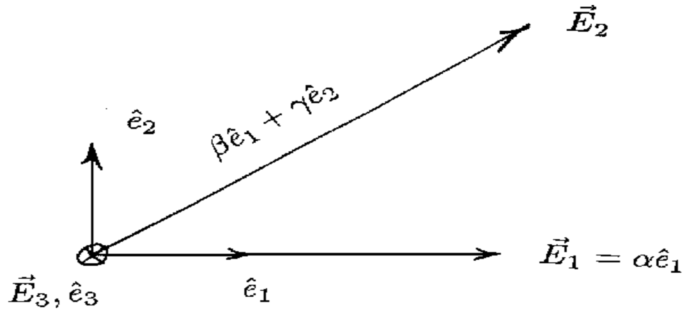

Figure 1.3-17. Physical components

In the Cartesian coordinate system we can represent a vector $\vec{A}$ as

$$\vec{A} = A_x\,\widehat{\mathbf{e}}_1 + A_y\,\widehat{\mathbf{e}}_2 + A_z\,\widehat{\mathbf{e}}_3$$

where $(\,\widehat{\mathbf{e}}_1,\,\widehat{\mathbf{e}}_2,\,\widehat{\mathbf{e}}_3)$ are the basis vectors. Consider a coordinate transformation to a more general coordinate system, say (x^1, x^2, x^3). The vector $\vec{A}$ can be represented with contravariant components as

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3 \tag{1.3.53}$$

with respect to the tangential basis vectors $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$. Alternatively, the same vector $\vec{A}$ can be represented in the form

$$\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3 \tag{1.3.54}$$

having covariant components with respect to the gradient basis vectors $(\vec{E}^1, \vec{E}^2, \vec{E}^3)$. These equations are just different ways of representing the same vector. In the above representations the basis vectors need not be orthogonal and they need not be unit vectors. In general, the physical dimensions of the components A^i and A_j are not the same.

The physical components of the vector $\vec{A}$ in a direction is defined as the projection of $\vec{A}$ upon a unit vector in the desired direction. For example, the physical component of $\vec{A}$ in the direction $\vec{E}_1$ is

$$\vec{A}\cdot\frac{\vec{E}_1}{|\vec{E}_1|} = \frac{A_1}{|\vec{E}_1|} = \text{projection of } \vec{A} \text{ on } \vec{E}_1. \tag{1.3.58}$$

Similarly, the physical component of $\vec{A}$ in the direction $\vec{E}^1$ is

$$\vec{A}\cdot\frac{\vec{E}^1}{|\vec{E}^1|} = \frac{A^1}{|\vec{E}^1|} = \text{projection of } \vec{A} \text{ on } \vec{E}^1. \tag{1.3.59}$$

EXAMPLE 1.3-11. (Physical components) Let α, β, γ denote nonzero positive constants such that the product relation $\alpha\gamma = 1$ is satisfied. Consider the nonorthogonal basis vectors

$$\vec{E}_1 = \alpha\,\widehat{\mathbf{e}}_1, \qquad \vec{E}_2 = \beta\,\widehat{\mathbf{e}}_1 + \gamma\,\widehat{\mathbf{e}}_2, \qquad \vec{E}_3 = \widehat{\mathbf{e}}_3$$

illustrated in the figure 1.3-17.

It is readily verified that the reciprocal basis is

$$\vec{E}^1 = \gamma\,\widehat{\mathbf{e}}_1 - \beta\,\widehat{\mathbf{e}}_2, \qquad \vec{E}^2 = \alpha\,\widehat{\mathbf{e}}_2, \qquad \vec{E}^3 = \widehat{\mathbf{e}}_3.$$

Consider the problem of representing the vector $\vec{A} = A_x\,\widehat{\mathbf{e}}_1 + A_y\,\widehat{\mathbf{e}}_2$ in the contravariant vector form

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 \quad \text{or tensor form} \quad A^i,\ i = 1, 2.$$

This vector has the contravariant components

$$A^1 = \vec{A}\cdot\vec{E}^1 = \gamma A_x - \beta A_y \qquad \text{and} \qquad A^2 = \vec{A}\cdot\vec{E}^2 = \alpha A_y.$$

Alternatively, this same vector can be represented as the covariant vector

$$\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 \quad \text{which has the tensor form} \quad A_i,\ i = 1, 2.$$

The covariant components are found from the relations

$$A_1 = \vec{A}\cdot\vec{E}_1 = \alpha A_x \qquad\qquad A_2 = \vec{A}\cdot\vec{E}_2 = \beta A_x + \gamma A_y.$$

The physical components of $\vec{A}$ in the directions $\vec{E}^1$ and $\vec{E}^2$ are found to be:

$$\begin{aligned}
\vec{A}\cdot\frac{\vec{E}^1}{|\vec{E}^1|} &= \frac{A^1}{|\vec{E}^1|} = \frac{\gamma A_x - \beta A_y}{\sqrt{\gamma^2+\beta^2}} = A(1)\\
\vec{A}\cdot\frac{\vec{E}^2}{|\vec{E}^2|} &= \frac{A^2}{|\vec{E}^2|} = \frac{\alpha A_y}{\alpha} = A_y = A(2).
\end{aligned}$$

Note that these same results are obtained from the dot product relations using either form of the vector $\vec{A}$. For example, we can write

$$\begin{aligned}
&\vec{A}\cdot\frac{\vec{E}^1}{|\vec{E}^1|} = \frac{A_1(\vec{E}^1\cdot\vec{E}^1) + A_2(\vec{E}^2\cdot\vec{E}^1)}{|\vec{E}^1|} = A(1)\\
\text{and}\qquad &\vec{A}\cdot\frac{\vec{E}^2}{|\vec{E}^2|} = \frac{A_1(\vec{E}^1\cdot\vec{E}^2) + A_2(\vec{E}^2\cdot\vec{E}^2)}{|\vec{E}^2|} = A(2).
\end{aligned}$$

In general, the physical components of a vector $\vec{A}$ in a direction of a unit vector λ^i is the generalized dot product in V_N. This dot product is an invariant and can be expressed

$$g_{ij}A^i\lambda^j = A^i\lambda_i = A_i\lambda^i = \text{projection of } \vec{A} \text{ in direction of } \lambda^i$$

■

Physical Components For Orthogonal Coordinates

In orthogonal coordinates observe the element of arc length squared in V_3 is

$$ds^2 = g_{ij}dx^i dx^j = (h_1)^2(dx^1)^2 + (h_2)^2(dx^2)^2 + (h_3)^2(dx^3)^2$$

where

$$g_{ij} = \begin{pmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & (h_3)^2 \end{pmatrix}. \tag{1.3.60}$$

In this case the curvilinear coordinates are orthogonal and

$$h^2_{(i)} = g_{(i)(i)} \quad i \text{ not summed and} \quad g_{ij} = 0,\ i \neq j.$$

At an arbitrary point in this coordinate system we take $\lambda^i, i = 1,2,3$ as a unit vector in the direction of the coordinate x^1. We then obtain

$$\lambda^1 = \frac{dx^1}{ds}, \quad \lambda^2 = 0, \quad \lambda^3 = 0.$$

This is a unit vector since

$$1 = g_{ij}\lambda^i\lambda^j = g_{11}\lambda^1\lambda^1 = h_1^2(\lambda^1)^2$$

or $\lambda^1 = \frac{1}{h_1}$. Here the curvilinear coordinate system is orthogonal and in this case the physical component of a vector A^i, in the direction x^i, is the projection of A^i on λ^i in V_3. The projection in the x^1 direction is determined from

$$A(1) = g_{ij}A^i\lambda^j = g_{11}A^1\lambda^1 = h_1^2 A^1 \frac{1}{h_1} = h_1 A^1.$$

Similarly, we choose unit vectors μ^i and ν^i, $i = 1,2,3$ in the x^2 and x^3 directions. These unit vectors can be represented

$$\begin{aligned} \mu^1 &= 0, & \mu^2 &= \frac{dx^2}{ds} = \frac{1}{h_2}, & \mu^3 &= 0 \\ \nu^1 &= 0, & \nu^2 &= 0, & \nu^3 &= \frac{dx^3}{ds} = \frac{1}{h_3} \end{aligned}$$

and the physical components of the vector A^i in these directions are calculated as

$$A(2) = h_2 A^2 \qquad \text{and} \qquad A(3) = h_3 A^3.$$

In summary, we can say that in an orthogonal coordinate system the physical components of a contravariant tensor of order one can be determined from the equations

$$A(i) = h_{(i)}A^{(i)} = \sqrt{g_{(i)(i)}}A^{(i)}, \quad i = 1, 2\, or\, 3 \quad \text{no summation on i,}$$

which is a short hand notation for the physical components (h_1A^1, h_2A^2, h_3A^3). In an orthogonal coordinate system the nonzero conjugate metric components are

$$g^{(i)(i)} = \frac{1}{g_{(i)(i)}}, \quad i = 1, 2,\ or\, 3 \quad \text{no summation on i.}$$

These components are needed to calculate the physical components associated with a covariant tensor of order one. For example, in the x^1-direction, we have the covariant components

$$\lambda_1 = g_{11}\lambda^1 = h_1^2 \frac{1}{h_1} = h_1, \qquad \lambda_2 = 0, \qquad \lambda_3 = 0$$

and consequently the projection in V_3 can be represented

$$g_{ij}A^i\lambda^j = g_{ij}A^i g^{jm}\lambda_m = A_j g^{jm}\lambda_m = A_1\lambda_1 g^{11} = A_1 h_1 \frac{1}{h_1^2} = \frac{A_1}{h_1} = A(1).$$

In a similar manner we calculate the relations

$$A(2) = \frac{A_2}{h_2} \qquad \text{and} \qquad A(3) = \frac{A_3}{h_3}$$

for the other physical components in the directions x^2 and x^3. These physical components can be represented in the short hand notation

$$A(i) = \frac{A_{(i)}}{h_{(i)}} = \frac{A_{(i)}}{\sqrt{g_{(i)(i)}}}, \quad i = 1, 2\, or\, 3 \qquad \text{no summation on } i.$$

In an orthogonal coordinate system the physical components associated with both the contravariant and covariant components are the same. To show this we note that when $A^i g_{ij} = A_j$ is summed on i we obtain

$$A^1 g_{1j} + A^2 g_{2j} + A^3 g_{3j} = A_j.$$

Since $g_{ij} = 0$ for $i \neq j$ this equation reduces to

$$A^{(i)} g_{(i)(i)} = A_{(i)}, \quad \text{i not summed.}$$

Another form for this equation is

$$A(i) = A^{(i)}\sqrt{g_{(i)(i)}} = \frac{A_{(i)}}{\sqrt{g_{(i)(i)}}} \quad \text{i not summed,}$$

which demonstrates that the physical components associated with the contravariant and covariant components are identical.

NOTATION The physical components are sometimes expressed by symbols with subscripts which represent the coordinate curve along which the projection is taken. For example, let H^i denote the contravariant components of a first order tensor. The following are some examples of the representation of the physical components of H^i in various coordinate systems:

orthogonal coordinates	coordinate system	tensor components	physical components
general	(x^1, x^2, x^3)	H^i	$H(1), H(2), H(3)$
rectangular	(x, y, z)	H^i	H_x, H_y, H_z
cylindrical	(r, θ, z)	H^i	H_r, H_θ, H_z
spherical	(ρ, θ, ϕ)	H^i	H_ρ, H_θ, H_ϕ
general	(u, v, w)	H^i	H_u, H_v, H_w

Higher Order Tensors

The physical components associated with higher ordered tensors are defined by projections in V_N just like the case with first order tensors. For an nth ordered tensor $T_{ij\ldots k}$ we can select n unit vectors $\lambda^i, \mu^i, \ldots, \nu^i$ and form the inner product (projection)

$$T_{ij\ldots k}\lambda^i\mu^j\ldots\nu^k.$$

When projecting the tensor components onto the coordinate curves, there are N choices for each of the unit vectors. This produces N^n physical components.

The above inner product represents the physical component of the tensor $T_{ij\ldots k}$ along the directions of the unit vectors $\lambda^i, \mu^i, \ldots, \nu^i$. The selected unit vectors may or may not be orthogonal. In the cases where the selected unit vectors are all orthogonal to one another, the calculation of the physical components is greatly simplified. By relabeling the unit vectors $\lambda^i_{(m)}, \lambda^i_{(n)}, \ldots, \lambda^i_{(p)}$ where $(m), (n), ..., (p)$ represent one of the N directions, the physical components of a general nth order tensor is represented

$$T(m\,n\ldots p) = T_{ij\ldots k}\lambda^i_{(m)}\lambda^j_{(n)}\ldots\lambda^k_{(p)}$$

EXAMPLE 1.3-12. (Physical components)

In an orthogonal curvilinear coordinate system V_3 with metric g_{ij}, $i,j = 1,2,3$, find the physical components of

(i) the second order tensor A_{ij}. (ii) the second order tensor A^{ij}. (iii) the second order tensor A^i_j.

Solution: The physical components of $A_{mn}, m, n = 1,2,3$ along the directions of two unit vectors λ^i and μ^i is defined as the inner product in V_3. These physical components can be expressed

$$A(ij) = A_{mn}\lambda^m_{(i)}\mu^n_{(j)} \qquad i,j = 1,2,3,$$

where the subscripts (i) and (j) represent one of the coordinate directions. Dropping the subscripts (i) and (j), we make the observation that in an orthogonal curvilinear coordinate system there are three choices for the direction of the unit vector λ^i and also three choices for the direction of the unit vector μ^i. These three choices represent the directions along the x^1, x^2 or x^3 coordinate curves which emanate from a point of the curvilinear coordinate system. This produces a total of nine possible physical components associated with the tensor A_{mn}.

For example, we can obtain the components of the unit vector $\lambda^i, i = 1,2,3$ in the x^1 direction directly from an examination of the element of arc length squared

$$ds^2 = (h_1)^2(dx^1)^2 + (h_2)^2(dx^2)^2 + (h_3)^2(dx^3)^2.$$

By setting $dx^2 = dx^3 = 0$, we find

$$\frac{dx^1}{ds} = \frac{1}{h_1} = \lambda^1, \qquad \lambda^2 = 0, \qquad \lambda^3 = 0.$$

This is the vector $\lambda^i_{(1)}, i = 1,2,3$. Similarly, if we choose to select the unit vector $\lambda^i, i = 1,2,3$ in the x^2 direction, we set $dx^1 = dx^3 = 0$ in the element of arc length squared and find the components

$$\lambda^1 = 0, \qquad \lambda^2 = \frac{dx^2}{ds} = \frac{1}{h_2}, \qquad \lambda^3 = 0.$$

This is the vector $\lambda^i_{(2)}, i = 1,2,3$. Finally, if we select $\lambda^i, i = 1,2,3$ in the x^3 direction, we set $dx^1 = dx^2 = 0$ in the element of arc length squared and determine the unit vector

$$\lambda^1 = 0, \qquad \lambda^2 = 0, \qquad \lambda^3 = \frac{dx^3}{ds} = \frac{1}{h_3}.$$

This is the vector $\lambda^i_{(3)}, i = 1,2,3$. Similarly, the unit vector μ^i can be selected as one of the above three directions. Examining all nine possible combinations for selecting the unit vectors, we calculate the physical components in an orthogonal coordinate system as:

$$\begin{array}{lll} A(11) = \dfrac{A_{11}}{h_1 h_1} & A(12) = \dfrac{A_{12}}{h_1 h_2} & A(13) = \dfrac{A_{13}}{h_1 h_3} \\ A(21) = \dfrac{A_{21}}{h_1 h_2} & A(22) = \dfrac{A_{22}}{h_2 h_2} & A(23) = \dfrac{A_{23}}{h_2 h_3} \\ A(31) = \dfrac{A_{31}}{h_3 h_1} & A(32) = \dfrac{A_{32}}{h_3 h_2} & A(33) = \dfrac{A_{33}}{h_3 h_3} \end{array}$$

These results can be written in the more compact form

$$A(ij) = \frac{A_{(i)(j)}}{h_{(i)} h_{(j)}} \quad \text{no summation on } i \text{ or } j \ . \tag{1.3.61}$$

For mixed tensors we have

$$A^i_j = g^{im} A_{mj} = g^{i1} A_{1j} + g^{i2} A_{2j} + g^{i3} A_{3j}. \tag{1.3.62}$$

From the fact $g^{ij} = 0$ for $i \neq j$, together with the physical components from equation (1.3.61), the equation (1.3.62) reduces to

$$A^{(i)}_{(j)} = g^{(i)(i)} A_{(i)(j)} = \frac{1}{h^2_{(i)}} \cdot h_{(i)} h_{(j)} A(ij) \quad \text{no summation on } i \text{ and } i,j = 1,2 \text{ or } 3.$$

This can also be written in the form

$$A(ij) = A^{(i)}_{(j)} \frac{h_{(i)}}{h_{(j)}} \quad \text{no summation on } i \text{ or } j. \tag{1.3.63}$$

Hence, the physical components associated with the mixed tensor A^i_j in an orthogonal coordinate system can be expressed as

$$\begin{array}{lll} A(11) = A^1_1 & A(12) = A^1_2 \dfrac{h_1}{h_2} & A(13) = A^1_3 \dfrac{h_1}{h_3} \\ A(21) = A^2_1 \dfrac{h_2}{h_1} & A(22) = A^2_2 & A(23) = A^2_3 \dfrac{h_2}{h_3} \\ A(31) = A^3_1 \dfrac{h_3}{h_1} & A(32) = A^3_2 \dfrac{h_3}{h_2} & A(33) = A^3_3. \end{array}$$

For second order contravariant tensors we may write

$$A^{ij} g_{jm} = A^i_m = A^{i1} g_{1m} + A^{i2} g_{2m} + A^{i3} g_{3m}.$$

We use the fact $g_{ij}=0$ for $i\neq j$ together with the physical components from equation (1.3.63) to reduce the above equation to the form $A^{(i)}_{(m)}=A^{(i)(m)}g_{(m)(m)}$ no summation on m . In terms of physical components we have

$$\frac{h_{(m)}}{h_{(i)}}A(im)=A^{(i)(m)}h^2_{(m)} \quad \text{or} \quad A(im)=A^{(i)(m)}h_{(i)}h_{(m)}. \quad \text{no summation} \quad i,m=1,2,3 \tag{1.3.64}$$

Examining the results from equation (1.3.64) we find that the physical components associated with the contravariant tensor A^{ij}, in an orthogonal coordinate system, can be written as:

$$\begin{array}{lll} A(11)=A^{11}h_1h_1 & A(12)=A^{12}h_1h_2 & A(13)=A^{13}h_1h_3 \\ A(21)=A^{21}h_2h_1 & A(22)=A^{22}h_2h_2 & A(23)=A^{23}h_2h_3 \\ A(31)=A^{31}h_3h_1 & A(32)=A^{32}h_3h_2 & A(33)=A^{33}h_3h_3. \end{array}$$

■

Physical Components in General

In an orthogonal curvilinear coordinate system, the physical components associated with the nth order tensor $T_{ij\ldots kl}$ along the curvilinear coordinate directions can be represented:

$$T(ij\ldots kl)=\frac{T_{(i)(j)\ldots(k)(l)}}{h_{(i)}h_{(j)}\ldots h_{(k)}h_{(l)}} \quad \text{no summations.}$$

These physical components can be related to the various tensors associated with $T_{ij\ldots kl}$. For example, in an orthogonal coordinate system, the physical components associated with the mixed tensor $T^{ij\ldots m}_{n\ldots kl}$ can be expressed as:

$$T(ij\ldots m\,n\ldots kl)=T^{(i)(j)\ldots(m)}_{(n)\ldots(k)(l)}\frac{h_{(i)}h_{(j)}\ldots h_{(m)}}{h_{(n)}\ldots h_{(k)}h_{(l)}} \qquad \text{no summations.} \tag{1.3.65}$$

EXAMPLE 1.3-13. (Physical components) Let $x^i=x^i(t), i=1,2,3$ denote the position vector of a particle which moves as a function of time t. Assume there exists a coordinate transformation $\overline{x}^i=\overline{x}^i(x)$, for $i=1,2,3$, of the form given by equations (1.2.33). The position of the particle when referenced with respect to the barred system of coordinates can be found by substitution. The generalized velocity of the particle in the unbarred system is a vector with components

$$v^i=\frac{dx^i}{dt},\ i=1,2,3.$$

The generalized velocity components of the same particle in the barred system is obtained from the chain rule. We find this velocity is represented by

$$\overline{v}^i=\frac{d\overline{x}^i}{dt}=\frac{\partial\overline{x}^i}{\partial x^j}\frac{dx^j}{dt}=\frac{\partial\overline{x}^i}{\partial x^j}v^j.$$

This equation implies that the contravariant quantities

$$(v^1,v^2,v^3)=(\frac{dx^1}{dt},\frac{dx^2}{dt},\frac{dx^3}{dt})$$

are tensor quantities. These quantities are called the components of the generalized velocity. The coordinates x^1, x^2, x^3 are generalized coordinates. This means we can select any set of three independent variables for the representation of the motion. The variables selected might not have the same dimensions. For example, in cylindrical coordinates we let $(x^1 = r, x^2 = \theta, x^3 = z)$. Here x^1 and x^3 have dimensions of distance but x^2 has dimensions of angular displacement. The generalized velocities are

$$v^1 = \frac{dx^1}{dt} = \frac{dr}{dt}, \qquad v^2 = \frac{dx^2}{dt} = \frac{d\theta}{dt}, \qquad v^3 = \frac{dx^3}{dt} = \frac{dz}{dt}.$$

Here v^1 and v^3 have units of length divided by time while v^2 has the units of angular velocity or angular change divided by time. Clearly, these dimensions are not all the same. Let us examine the physical components of the generalized velocities. We find in cylindrical coordinates $h_1 = 1$, $h_2 = r$, $h_3 = 1$ and the physical components of the velocity have the forms:

$$v_r = v(1) = v^1 h_1 = \frac{dr}{dt}, \qquad v_\theta = v(2) = v^2 h_2 = r\frac{d\theta}{dt}, \qquad v_z = v(3) = v^3 h_3 = \frac{dz}{dt}.$$

Now the physical components of the velocity all have the same units of length divided by time.

■

Additional examples of the use of physical components are considered later. For the time being, just remember that when tensor equations are derived, the equations are valid in any generalized coordinate system. In particular, we are interested in the representation of physical laws which are to be invariant and independent of the coordinate system used to represent these laws. Once a tensor equation is derived, we can chose any type of generalized coordinates and expand the tensor equations. Before using any expanded tensor equations we must replace all the tensor components by their corresponding physical components in order that the equations are dimensionally homogeneous. It is these expanded equations, expressed in terms of the physical components, which are used to solve applied problems.

Tensors and Multilinear Forms

Tensors can be thought of as being created by multilinear forms defined on some vector space V. Let us define on a vector space V a linear form, a bilinear form and a general multilinear form. We can then illustrate how tensors are created from these forms.

Definition: (Linear form) Let V denote a vector space which contains vectors $\vec{x}, \vec{x}_1, \vec{x}_2, \ldots$. A linear form in $\vec{x}$ is a scalar function $\varphi(\vec{x})$ having a single vector argument $\vec{x}$ which satisfies the linearity properties:

$$\begin{aligned} &(i) \qquad \varphi(\vec{x}_1 + \vec{x}_2) = \varphi(\vec{x}_1) + \varphi(\vec{x}_2) \\ &(ii) \qquad \varphi(\mu\vec{x}_1) = \mu\varphi(\vec{x}_1) \end{aligned} \tag{1.3.66}$$

for all arbitrary vectors $\vec{x}_1, \vec{x}_2$ in V and all real numbers μ.

An example of a linear form is the dot product relation

$$\varphi(\vec{x}) = \vec{A} \cdot \vec{x} \tag{1.3.67}$$

where $\vec{A}$ is a constant vector and $\vec{x}$ is an arbitrary vector belonging to the vector space V.

Note that a linear form in $\vec{x}$ can be expressed in terms of the components of the vector $\vec{x}$ and the base vectors $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$ used to represent $\vec{x}$. To show this, we write the vector $\vec{x}$ in the component form

$$\vec{x} = x^i \, \widehat{\mathbf{e}}_i = x^1 \, \widehat{\mathbf{e}}_1 + x^2 \, \widehat{\mathbf{e}}_2 + x^3 \, \widehat{\mathbf{e}}_3,$$

where $x^i, i = 1, 2, 3$ are the components of $\vec{x}$ with respect to the basis vectors $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$. By the linearity property of φ we can write

$$\begin{aligned}
\varphi(\vec{x}) = \varphi(x^i \, \widehat{\mathbf{e}}_i) &= \varphi(x^1 \, \widehat{\mathbf{e}}_1 + x^2 \, \widehat{\mathbf{e}}_2 + x^3 \, \widehat{\mathbf{e}}_3) \\
&= \varphi(x^1 \, \widehat{\mathbf{e}}_1) + \varphi(x^2 \, \widehat{\mathbf{e}}_2) + \varphi(x^3 \, \widehat{\mathbf{e}}_3) \\
&= x^1 \varphi(\widehat{\mathbf{e}}_1) + x^2 \varphi(\widehat{\mathbf{e}}_2) + x^3 \varphi(\widehat{\mathbf{e}}_3) = x^i \varphi(\widehat{\mathbf{e}}_i)
\end{aligned}$$

Thus we can write $\varphi(\vec{x}) = x^i \varphi(\widehat{\mathbf{e}}_i)$ and by defining the quantity $\varphi(\widehat{\mathbf{e}}_i) = a_i$ as a tensor we obtain $\varphi(\vec{x}) = x^i a_i$. Note that if we change basis from $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$ to $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ then the components of $\vec{x}$ also must change. Letting $\overline{x}^i$ denote the components of $\vec{x}$ with respect to the new basis, we would have

$$\vec{x} = \overline{x}^i \vec{E}_i \quad \text{and} \quad \varphi(\vec{x}) = \varphi(\overline{x}^i \vec{E}_i) = \overline{x}^i \varphi(\vec{E}_i).$$

The linear form φ defines a new tensor $\overline{a}_i = \varphi(\vec{E}_i)$ so that $\varphi(\vec{x}) = \overline{x}^i \overline{a}_i$. Whenever there is a definite relation between the basis vectors $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$ and $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$, say,

$$\vec{E}_i = \frac{\partial x^j}{\partial \overline{x}^i} \, \widehat{\mathbf{e}}_j,$$

then there exists a definite relation between the tensors a_i and $\overline{a}_i$. This relation is

$$\overline{a}_i = \varphi(\vec{E}_i) = \varphi(\frac{\partial x^j}{\partial \overline{x}^i} \, \widehat{\mathbf{e}}_j) = \frac{\partial x^j}{\partial \overline{x}^i} \varphi(\widehat{\mathbf{e}}_j) = \frac{\partial x^j}{\partial \overline{x}^i} a_j.$$

This is the transformation law for an absolute covariant tensor of rank or order one.

The above idea is now extended to higher order tensors.

Definition: (Bilinear form) A bilinear form in $\vec{x}$ and $\vec{y}$ is a scalar function $\varphi(\vec{x}, \vec{y})$ with two vector arguments, which satisfies the linearity properties:

$$\begin{aligned}
&(i) \quad \varphi(\vec{x}_1 + \vec{x}_2, \vec{y}_1) = \varphi(\vec{x}_1, \vec{y}_1) + \varphi(\vec{x}_2, \vec{y}_1) \\
&(ii) \quad \varphi(\vec{x}_1, \vec{y}_1 + \vec{y}_2) = \varphi(\vec{x}_1, \vec{y}_1) + \varphi(\vec{x}_1, \vec{y}_2) \\
&(iii) \quad \varphi(\mu \vec{x}_1, \vec{y}_1) = \mu \varphi(\vec{x}_1, \vec{y}_1) \\
&(iv) \quad \varphi(\vec{x}_1, \mu \vec{y}_1) = \mu \varphi(\vec{x}_1, \vec{y}_1)
\end{aligned} \tag{1.3.68}$$

for arbitrary vectors $\vec{x}_1, \vec{x}_2, \vec{y}_1, \vec{y}_2$ in the vector space V and for all real numbers μ.

Note in the definition of a bilinear form that the scalar function φ is linear in both the arguments $\vec{x}$ and $\vec{y}$. An example of a bilinear form is the dot product relation

$$\varphi(\vec{x},\vec{y}) = \vec{x}\cdot\vec{y} \tag{1.3.69}$$

where both $\vec{x}$ and $\vec{y}$ belong to the same vector space V.

The definition of a bilinear form suggests how multilinear forms can be defined.

Definition: (Multilinear forms) A multilinear form of degree M or a M degree linear form in the vector arguments

$$\vec{x}_1, \vec{x}_2, \dots, \vec{x}_M$$

is a scalar function

$$\varphi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_M)$$

of M vector arguments which satisfies the property that it is a linear form in each of its arguments. That is, φ must satisfy for each $j = 1, 2, \dots, M$ the properties:

$$\begin{aligned} &(i) \quad \varphi(\vec{x}_1, \dots, \vec{x}_{j1} + \vec{x}_{j2}, \dots \vec{x}_M) = \varphi(\vec{x}_1, \dots, \vec{x}_{j1}, \dots, \vec{x}_M) + \varphi(\vec{x}_1, \dots, \vec{x}_{j2}, \dots, \vec{x}_M) \\ &(ii) \qquad \varphi(\vec{x}_1, \dots, \mu\vec{x}_j, \dots, \vec{x}_M) = \mu\varphi(\vec{x}_1, \dots, \vec{x}_j, \dots, \vec{x}_M) \end{aligned} \tag{1.3.70}$$

for all arbitrary vectors $\vec{x}_1, \dots, \vec{x}_M$ in the vector space V and all real numbers μ.

An example of a third degree multilinear form or trilinear form is the triple scalar product

$$\varphi(\vec{x},\vec{y},\vec{z}) = \vec{x}\cdot(\vec{y}\times\vec{z}). \tag{1.3.71}$$

Note that multilinear forms are independent of the coordinate system selected and depend only upon the vector arguments. In a three dimensional vector space we select the basis vectors $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$ and represent all vectors with respect to this basis set. For example, if $\vec{x},\vec{y},\vec{z}$ are three vectors we can represent these vectors in the component forms

$$\vec{x} = x^i\,\widehat{\mathbf{e}}_i, \qquad \vec{y} = y^j\,\widehat{\mathbf{e}}_j, \qquad \vec{z} = z^k\,\widehat{\mathbf{e}}_k \tag{1.3.72}$$

where we have employed the summation convention on the repeated indices i, j and k. Substituting equations (1.3.72) into equation (1.3.71) we obtain

$$\varphi(x^i\,\widehat{\mathbf{e}}_i, y^j\,\widehat{\mathbf{e}}_j, z^k\,\widehat{\mathbf{e}}_k) = x^i y^j z^k \varphi(\widehat{\mathbf{e}}_i, \widehat{\mathbf{e}}_j, \widehat{\mathbf{e}}_k), \tag{1.3.73}$$

since φ is linear in all its arguments. By defining the tensor quantity

$$\varphi(\widehat{\mathbf{e}}_i, \widehat{\mathbf{e}}_j, \widehat{\mathbf{e}}_k) = e_{ijk} \tag{1.3.74}$$

(See exercise 1.1, problem 15) the trilinear form, given by equation (1.3.71), with vectors from equations (1.3.72), can be expressed as

$$\varphi(\vec{x},\vec{y},\vec{z}) = e_{ijk}x^i y^j z^k, \quad i,j,k = 1,2,3. \tag{1.3.75}$$

The coefficients e_{ijk} of the trilinear form is called a third order tensor. It is the familiar permutation symbol considered earlier.

In a multilinear form of degree M, $\varphi(\vec{x},\vec{y},\ldots,\vec{z})$, the M arguments can be represented in a component form with respect to a set of basis vectors $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$. Let these vectors have components $x^i, y^i, z^i, i = 1,2,3$ with respect to the selected basis vectors. We then can write

$$\vec{x} = x^i\,\widehat{\mathbf{e}}_i, \qquad \vec{y} = y^j\,\widehat{\mathbf{e}}_j, \qquad \vec{z} = z^k\,\widehat{\mathbf{e}}_k.$$

Substituting these vectors into the M degree multilinear form produces

$$\varphi(x^i\,\widehat{\mathbf{e}}_i, y^j\,\widehat{\mathbf{e}}_j, \ldots, z^k\,\widehat{\mathbf{e}}_k) = x^i y^j \cdots z^k \varphi(\widehat{\mathbf{e}}_i, \widehat{\mathbf{e}}_j, \ldots, \widehat{\mathbf{e}}_k). \tag{1.3.76}$$

Consequently, the multilinear form defines a set of coefficients

$$a_{ij\ldots k} = \varphi(\widehat{\mathbf{e}}_i, \widehat{\mathbf{e}}_j, \ldots, \widehat{\mathbf{e}}_k) \tag{1.3.77}$$

which are referred to as the components of a tensor of order M. The tensor is thus created by the multilinear form and has M indices if φ is of degree M.

Note that if we change to a different set of basis vectors, say, $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ the multilinear form defines a new tensor

$$\overline{a}_{ij\ldots k} = \varphi(\vec{E}_i, \vec{E}_j, \ldots, \vec{E}_k). \tag{1.3.78}$$

This new tensor has a bar over it to distinguish it from the previous tensor. A definite relation exists between the new and old basis vectors and consequently there exists a definite relation between the components of the barred and unbarred tensors components. Recall that if we are given a set of transformation equations

$$y^i = y^i(x^1, x^2, x^3), i = 1,2,3, \tag{1.3.79}$$

from rectangular to generalized curvilinear coordinates, we can express the basis vectors in the new system by the equations

$$\vec{E}_i = \frac{\partial y^j}{\partial x^i}\,\widehat{\mathbf{e}}_j, \quad i = 1,2,3. \tag{1.3.80}$$

For example, see equations (1.3.11) with $y^1 = x, y^2 = y, y^3 = z, x^1 = u, x^2 = v, x^3 = w$. Substituting equations (1.3.80) into equations (1.3.78) we obtain

$$\overline{a}_{ij\ldots k} = \varphi(\frac{\partial y^\alpha}{\partial x^i}\,\widehat{\mathbf{e}}_\alpha, \frac{\partial y^\beta}{\partial x^j}\,\widehat{\mathbf{e}}_\beta, \ldots, \frac{\partial y^\gamma}{\partial x^k}\,\widehat{\mathbf{e}}_\gamma).$$

By the linearity property of φ, this equation is expressible in the form

$$\overline{a}_{ij\ldots k} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\cdots\frac{\partial y^\gamma}{\partial x^k}\varphi(\widehat{\mathbf{e}}_\alpha, \widehat{\mathbf{e}}_\beta, \ldots, \widehat{\mathbf{e}}_\gamma)$$

$$\overline{a}_{ij\ldots k} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\cdots\frac{\partial y^\gamma}{\partial x^k}a_{\alpha\beta\ldots\gamma}$$

This is the familiar transformation law for a covariant tensor of degree M. By selecting reciprocal basis vectors the corresponding transformation laws for contravariant vectors can be determined.

The above examples illustrate that tensors can be considered as quantities derivable from multilinear forms defined on some vector space.

Dual Tensors

The e-permutation symbol is often used to generate new tensors from given tensors. For $T_{i_1 i_2 \ldots i_m}$ a skew-symmetric tensor, we define the tensor

$$\hat{T}^{j_1 j_2 \ldots j_{n-m}} = \frac{1}{m!} e^{j_1 j_2 \ldots j_{n-m} i_1 i_2 \ldots i_m} T_{i_1 i_2 \ldots i_m} \qquad m \leq n \tag{1.3.81}$$

as the dual tensor associated with $T_{i_1 i_2 \ldots i_m}$. Note that the e-permutation symbol or alternating tensor has a weight of $+1$ and consequently the dual tensor will have a higher weight than the original tensor.

The e-permutation symbol has the following properties

$$\begin{aligned} e^{i_1 i_2 \ldots i_N} e_{i_1 i_2 \ldots i_N} &= N! \\ e^{i_1 i_2 \ldots i_N} e_{j_1 j_2 \ldots j_N} &= \delta^{i_1 i_2 \ldots i_N}_{j_1 j_2 \ldots j_N} \\ e_{k_1 k_2 \ldots k_m i_1 i_2 \ldots i_{N-m}} e^{j_1 j_2 \ldots j_m i_1 i_2 \ldots i_{N-m}} &= (N-m)! \delta^{j_1 j_2 \ldots j_m}_{k_1 k_2 \ldots k_m} \\ \delta^{j_1 j_2 \ldots j_m}_{k_1 k_2 \ldots k_m} T_{j_1 j_2 \ldots j_m} &= m! T_{k_1 k_2 \ldots k_m}. \end{aligned} \tag{1.3.82}$$

Using the above properties we can solve for the skew-symmetric tensor in terms of the dual tensor. We find

$$T_{i_1 i_2 \ldots i_m} = \frac{1}{(n-m)!} e_{i_1 i_2 \ldots i_m j_1 j_2 \ldots j_{n-m}} \hat{T}^{j_1 j_2 \ldots j_{n-m}}. \tag{1.3.83}$$

For example, if A_{ij} $i,j = 1,2,3$ is a skew-symmetric tensor, we may associate with it the dual tensor

$$V^i = \frac{1}{2!} e^{ijk} A_{jk},$$

which is a first order tensor or vector. Note that A_{ij} has the components

$$\begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} \tag{1.3.84}$$

and consequently, the components of the vector $\vec{V}$ are

$$(V^1, V^2, V^3) = (A_{23}, A_{31}, A_{12}). \tag{1.3.85}$$

Note that the vector components have a cyclic order to the indices which comes from the cyclic properties of the e-permutation symbol.

As another example, consider the fourth order skew-symmetric tensor A_{ijkl}, $i,j,k,l = 1,\ldots,n$. We can associate with this tensor any of the dual tensor quantities

$$\begin{aligned} V &= \frac{1}{4!} e^{ijkl} A_{ijkl} \\ V^i &= \frac{1}{4!} e^{ijklm} A_{jklm} \\ V^{ij} &= \frac{1}{4!} e^{ijklmn} A_{klmn} \\ V^{ijk} &= \frac{1}{4!} e^{ijklmnp} A_{lmnp} \\ V^{ijkl} &= \frac{1}{4!} e^{ijklmnpr} A_{mnpr} \end{aligned} \tag{1.3.86}$$

Applications of dual tensors can be found in section 2.2.

EXERCISE 1.3

▶ **1.**

(a) From the transformation law for the second order tensor $\overline{g}_{ij} = g_{ab}\dfrac{\partial x^a}{\partial \overline{x}^i}\dfrac{\partial x^b}{\partial \overline{x}^j}$ solve for the g_{ab} in terms of $\overline{g}_{ij}$.

(b) Show that if g_{ij} is symmetric in one coordinate system it is symmetric in all coordinate systems.

(c) Let $\overline{g} = det(\overline{g}_{ij})$ and $g = det(g_{ij})$ and show that $\overline{g} = gJ^2(\frac{x}{\overline{x}})$ and consequently $\sqrt{\overline{g}} = \sqrt{g}J(\dfrac{x}{\overline{x}})$. This shows that g is a scalar invariant of weight 2 and $\sqrt{g}$ is a scalar invariant of weight 1.

▶ **2.** For

$$g_{ij} = \frac{\partial y^m}{\partial x^i}\frac{\partial y^m}{\partial x^j} \quad \text{show that} \quad g^{ij} = \frac{\partial x^i}{\partial y^m}\frac{\partial x^j}{\partial y^m}$$

▶ **3.** Show that in a curvilinear coordinate system which is orthogonal we have:

$$(a) \qquad g = det(g_{ij}) = g_{11}g_{22}g_{33}$$

$$(b) \qquad g_{mn} = g^{mn} = 0 \qquad \text{for } m \neq n$$

$$(c) \qquad g^{NN} = \frac{1}{g_{NN}} \quad \text{for } N = 1,2,3 \quad \text{(no summation on N)}$$

▶ **4.** Show that $g = det(g_{ij}) = \left|\dfrac{\partial y^i}{\partial x^j}\right|^2 = J^2$, where J is the Jacobian.

▶ **5.** Define the quantities $h_1 = h_u = |\dfrac{\partial \vec{r}}{\partial u}|$, $h_2 = h_v = |\dfrac{\partial \vec{r}}{\partial v}|$, $h_3 = h_w = |\dfrac{\partial \vec{r}}{\partial w}|$ and construct the unit vectors

$$\widehat{\mathbf{e}}_u = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u}, \qquad \widehat{\mathbf{e}}_v = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial v}, \qquad \widehat{\mathbf{e}}_w = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial w}.$$

(a) Assume the coordinate system is orthogonal and show that

$$g_{11} = h_1^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$g_{22} = h_2^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2,$$

$$g_{33} = h_3^2 = \left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2.$$

(b) Show that $d\vec{r}$ can be expressed in the form $d\vec{r} = h_1\,\widehat{\mathbf{e}}_u\,du + h_2\,\widehat{\mathbf{e}}_v\,dv + h_3\,\widehat{\mathbf{e}}_w\,dw$.

(c) Show that the volume of the elemental parallelepiped having $d\vec{r}$ as diagonal can be represented

$$d\tau = \sqrt{g}\,dudvdw = J\,dudvdw = \frac{\partial(x,y,z)}{\partial(u,v,w)}\,dudvdw.$$

Hint:

$$|\vec{A}\cdot(\vec{B}\times\vec{C})| = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

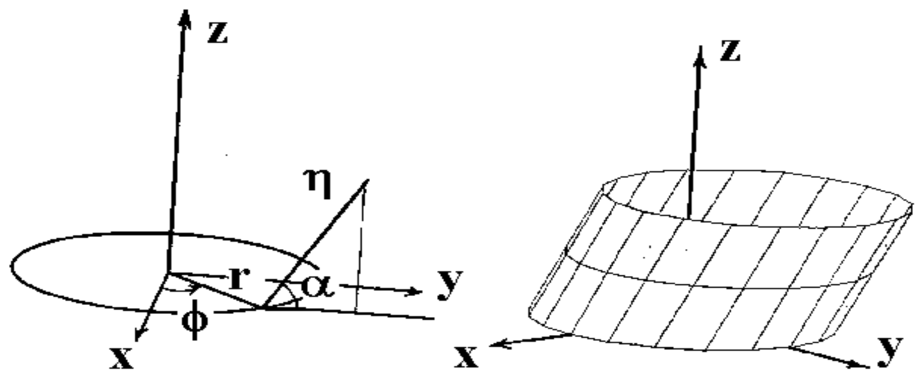

Figure 1.3-18 Oblique cylindrical coordinates.

▶ **6.** For the change $d\vec{r}$ given in problem 5, show the elemental parallelepiped with diagonal $d\vec{r}$ has:

(a) the element of area $dS_1 = \sqrt{g_{22}g_{33} - g_{23}^2}\, dvdw$ in the u =constant surface.

(b) The element of area $dS_2 = \sqrt{g_{33}g_{11} - g_{13}^2}\, dudw$ in the v =constant surface.

(c) the element of area $dS_3 = \sqrt{g_{11}g_{22} - g_{12}^2}\, dudv$ in the w =constant surface.

(d) What do the above elements of area reduce to in the special case the curvilinear coordinates are orthogonal? Hint:
$$\begin{aligned}|\vec{A} \times \vec{B}| &= \sqrt{(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B})} \\ &= \sqrt{(\vec{A} \cdot \vec{A})(\vec{B} \cdot \vec{B}) - (\vec{A} \cdot \vec{B})(\vec{A} \cdot \vec{B})}\end{aligned}.$$

▶ **7.** In Cartesian coordinates you are given the affine transformation. $\overline{x}_i = \ell_{ij}x_j$ where
$$\overline{x}_1 = \frac{1}{15}(5x_1 - 14x_2 + 2x_3), \qquad \overline{x}_2 = -\frac{1}{3}(2x_1 + x_2 + 2x_3), \qquad \overline{x}_3 = \frac{1}{15}(10x_1 + 2x_2 - 11x_3)$$

(a) Show the transformation is orthogonal.

(b) A vector $\vec{A}(x_1, x_2, x_3)$ in the unbarred system has the components
$$A_1 = (x_1)^2, \qquad A_2 = (x_2)^2 \qquad A_3 = (x_3)^2.$$
Find the components of this vector in the barred system of coordinates.

▶ **8.** Calculate the metric and conjugate metric tensors in cylindrical coordinates (r, θ, z).

▶ **9.** Calculate the metric and conjugate metric tensors in spherical coordinates (ρ, θ, ϕ).

▶ **10.** Calculate the metric and conjugate metric tensors in parabolic cylindrical coordinates (ξ, η, z).

▶ **11.** Calculate the metric and conjugate metric components in elliptic cylindrical coordinates (ξ, η, z).

▶ **12.** Calculate the metric and conjugate metric components for the oblique cylindrical coordinates (r, ϕ, η), illustrated in figure 1.3-18, where $x = r\cos\phi$, $y = r\sin\phi + \eta\cos\alpha$, $z = \eta\sin\alpha$ and α is a parameter $0 < \alpha \leq \frac{\pi}{2}$. Note: When $\alpha = \frac{\pi}{2}$ cylindrical coordinates result.

► **13.** Calculate the metric and conjugate metric tensor associated with the toroidal surface coordinates (ξ, η) illustrated in the figure 1.3-19, where

$$
\begin{aligned}
x &= (a + b\cos\xi)\cos\eta \qquad & a > b > 0 \\
y &= (a + b\cos\xi)\sin\eta \qquad & 0 < \xi < 2\pi \\
z &= b\sin\xi \qquad & 0 < \eta < 2\pi
\end{aligned}
$$

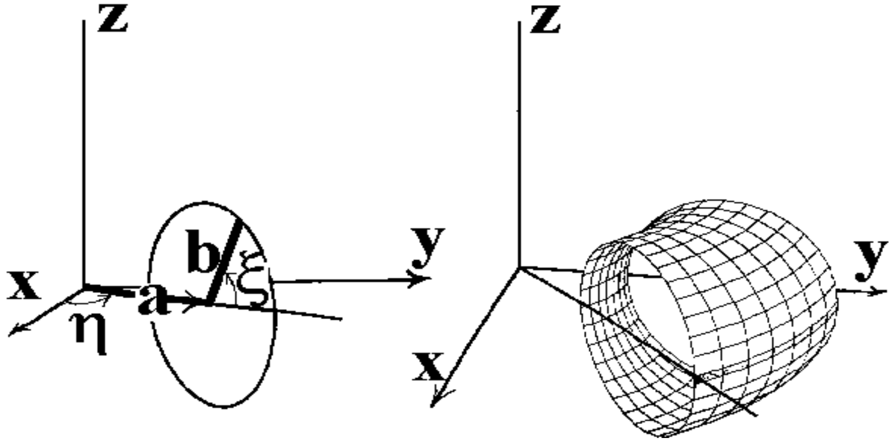

Figure 1.3-19. Toroidal surface coordinates

► **14.** Calculate the metric and conjugate metric tensor associated with the spherical surface coordinates (θ, ϕ), illustrated in the figure 1.3-20, where

$$
\begin{aligned}
x &= a\sin\theta\cos\phi \qquad & a > 0 \quad \text{is constant} \\
y &= a\sin\theta\sin\phi \qquad & 0 < \phi < 2\pi \\
z &= a\cos\theta \qquad & 0 < \theta < \frac{\pi}{2}
\end{aligned}
$$

► **15.** Consider $g_{ij},\ i, j = 1, 2$

(a) Show that $g^{11} = \dfrac{g_{22}}{\Delta}, \qquad g^{12} = g^{21} = \dfrac{-g_{12}}{\Delta}, \qquad g^{22} = \dfrac{g_{11}}{\Delta}$ where $\Delta = g_{11}g_{22} - g_{12}g_{21}$.

(b) Use the results in part (a) and verify that $g_{ij}g^{ik} = \delta_j^k, \quad i, j, k = 1, 2.$

► **16.** Let A_x, A_y, A_z denote the constant components of a vector in Cartesian coordinates. Using the transformation laws (1.2.42) and (1.2.47) to find the contravariant and covariant components of this vector upon changing to (a) cylindrical coordinates (r, θ, z). (b) spherical coordinates (ρ, θ, ϕ) and (c) Parabolic cylindrical coordinates.

► **17.** Find the relationship which exists between the given associated tensors.

$$
\begin{aligned}
&(a) \quad A^{pqk}_{r.} \quad \text{and} \quad A^{pq}_{rs} \qquad && (c) \quad A^{i.j.}_{.l.m} \quad \text{and} \quad A^{.s.p}_{r.t.} \\
&(b) \quad A^{p}_{.mrs} \quad \text{and} \quad A^{pq}_{..rs} \qquad && (d) \quad A_{mnk} \quad \text{and} \quad A^{ij}_{..k}
\end{aligned}
$$

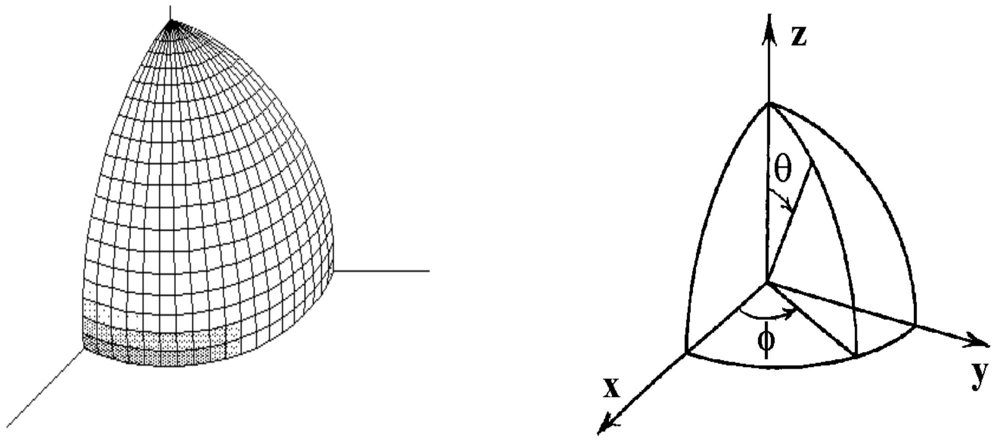

Figure 1.3-20. Spherical surface coordinates

► **18.** Given the fourth order tensor $C_{ikmp} = \lambda\delta_{ik}\delta_{mp} + \mu(\delta_{im}\delta_{kp} + \delta_{ip}\delta_{km}) + \nu(\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km})$ where λ, μ and ν are scalars and δ_{ij} is the Kronecker delta. Show that under an orthogonal transformation of rotation of axes with $\overline{x}_i = \ell_{ij}x_j$ where $\ell_{rs}\ell_{is} = \ell_{mr}\ell_{mi} = \delta_{ri}$ the components of the above tensor are unaltered. Any tensor whose components are unaltered under an orthogonal transformation is called an 'isotropic' tensor. Another way of stating this problem is to say "Show C_{ikmp} is an isotropic tensor."

► **19.** Assume A_{ijl} is a third order covariant tensor and B^{pqmn} is a fourth order contravariant tensor. Prove that $A_{ikl}B^{klmn}$ is a mixed tensor of order three, with one covariant and two contravariant indices.

► **20.** Assume that T_{mnrs} is an absolute tensor. Show that if $T_{ijkl} + T_{ijlk} = 0$ in the coordinate system x^r then $\overline{T}_{ijkl} + \overline{T}_{ijlk} = 0$ in any other coordinate system $\overline{x}^r$.

► **21.** Show that

$$\epsilon_{ijk}\epsilon_{rst} = \begin{vmatrix} g_{ir} & g_{is} & g_{it} \\ g_{jr} & g_{js} & g_{jt} \\ g_{kr} & g_{ks} & g_{kt} \end{vmatrix}$$

Hint: See problem 38, Exercise 1.1

► **22.** Determine if the tensor equation $\epsilon_{mnp}\epsilon_{mij} + \epsilon_{mnj}\epsilon_{mpi} = \epsilon_{mni}\epsilon_{mpj}$ is true or false. Justify your answer.

► **23.** Prove the epsilon identity $g^{ij}\epsilon_{ipt}\epsilon_{jrs} = g_{pr}g_{ts} - g_{ps}g_{tr}$. Hint: See problem 38, Exercise 1.1

► **24.** Let A^{rs} denote a skew-symmetric contravariant tensor and let $c_r = \frac{1}{2}\epsilon_{rmn}A^{mn}$ where $\epsilon_{rmn} = \sqrt{g}e_{rmn}$. Show that c_r are the components of a covariant tensor. Write out all the components.

► **25.** Let A_{rs} denote a skew-symmetric covariant tensor and let $c^r = \frac{1}{2}\epsilon^{rmn}A_{mn}$ where $\epsilon^{rmn} = \frac{1}{\sqrt{g}}e^{rmn}$. Show that c^r are the components of a contravariant tensor. Write out all the components.

► **26.** Let $A_{pq}B_r^{qs} = C_{pr}^s$ where B_r^{qs} is a relative tensor of weight ω_1 and C_{pr}^s is a relative tensor of weight ω_2. Prove that A_{pq} is a relative tensor of weight $(\omega_2 - \omega_1)$.

► **27.** When A_j^i is an absolute tensor prove that $\sqrt{g}A_j^i$ is a relative tensor of weight $+1$.

► **28.** When A_j^i is an absolute tensor prove that $\frac{1}{\sqrt{g}}A_j^i$ is a relative tensor of weight -1.

► **29.**

(a) Show e^{ijk} is a relative tensor of weight $+1$.

(b) Show $\epsilon^{ijk} = \frac{1}{\sqrt{g}}e^{ijk}$ is an absolute tensor. Hint: See example 1.1-25.

► **30.** The equation of a surface can be represented by an equation of the form $\Phi(x^1, x^2, x^3) = constant$. Show that a unit normal vector to the surface can be represented by the vector

$$n^i = \frac{g^{ij}\frac{\partial\Phi}{\partial x^j}}{(g^{mn}\frac{\partial\Phi}{\partial x^m}\frac{\partial\Phi}{\partial x^n})^{\frac{1}{2}}}.$$

► **31.** Assume that $\overline{g}_{ij} = \lambda g_{ij}$ with λ a nonzero constant. Find and calculate $\overline{g}^{ij}$ in terms of g^{ij}.

► **32.** Determine if the following tensor equation is true. Justify your answer.

$$\epsilon_{rjk}A_i^r + \epsilon_{irk}A_j^r + \epsilon_{ijr}A_k^r = \epsilon_{ijk}A_r^r.$$

Hint: See problem 21, Exercise 1.1.

► **33.** Show that for C_i and C^i associated tensors, and $C^i = \epsilon^{ijk}A_jB_k$, then $C_i = \epsilon_{ijk}A^jB^k$

► **34.** Prove that ϵ^{ijk} and ϵ_{ijk} are associated tensors. Hint: Consider the determinant of g_{ij}.

► **35.** Show $\epsilon^{ijk}A_iB_jC_k = \epsilon_{ijk}A^iB^jC^k$.

► **36.** Let T_j^i, $i,j = 1,2,3$ denote a second order mixed tensor. Show that the given quantities are scalar invariants.

$$(i) \quad I_1 = T_i^i$$
$$(ii) \quad I_2 = \frac{1}{2}\left[(T_i^i)^2 - T_m^iT_i^m\right]$$
$$(iii) \quad I_3 = det|T_j^i|$$

► **37.**

(a) Assume A^{ij} and B^{ij}, $i,j = 1,2,3$ are absolute contravariant tensors, and determine if the inner product $C^{ik} = A^{ij}B^{jk}$ is an absolute tensor?

(b) Assume that the condition $\dfrac{\partial\overline{x}^j}{\partial x^n}\dfrac{\partial\overline{x}^j}{\partial x^m} = \delta_{nm}$ is satisfied, and determine whether the inner product in part (a) is a tensor?

(c) Consider only transformations which are a rotation and translation of axes $\overline{y}_i = \ell_{ij}y_j + b_i$, where ℓ_{ij} are direction cosines for the rotation of axes. Show that $\dfrac{\partial\overline{y}_j}{\partial y_n}\dfrac{\partial\overline{y}_j}{\partial y_m} = \delta_{nm}$

▶ **38.** For A_{ijk} a Cartesian tensor, determine if a contraction on the indices i and j is allowed. That is, determine if the quantity $A_k = A_{iik}$, (summation on i) is a tensor. Hint: See part(c) of the previous problem.

▶ **39.** Prove the e-δ identity $e^{ijk}e_{imn} = \delta^j_m\delta^k_n - \delta^j_n\delta^k_m$.

▶ **40.** Consider the vector V_k, $k = 1,2,3$ and define the matrix (a_{ij}) having the elements $a_{ij} = e_{ijk}V_k$, where e_{ijk} is the $e-$permutation symbol.

(a) Solve for V_i in terms of a_{mn} by multiplying both sides of the given equation by e^{ijl} and note the $e - \delta$ identity allows us to simplify the result.

(b) Sum the given expression on k and then assign values to the free indices (i,j=1,2,3) and compare your results with part (a).

(c) Is a_{ij} symmetric, skew-symmetric, or neither?

▶ **41.** It can be shown that the continuity equation of fluid dynamics can be expressed in the tensor form

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}\left(\sqrt{g}\varrho V^r\right) + \frac{\partial \varrho}{\partial t} = 0,$$

where ϱ is the density of the fluid, t is time, V^r, with $r = 1,2,3$ are the velocity components and $g = |g_{ij}|$ is the determinant of the metric tensor. Employing the summation convention and replacing the tensor components of velocity by their physical components, express the continuity equation in

(a) Cartesian coordinates (x,y,z) with physical components V_x, V_y, V_z.

(b) Cylindrical coordinates (r,θ,z) with physical components V_r, V_θ, V_z.

(c) Spherical coordinates (ρ,θ,ϕ) with physical components V_ρ, V_θ, V_ϕ.

▶ **42.** Let x^1, x^2, x^3 denote a set of skewed coordinates with respect to the Cartesian coordinates y^1, y^2, y^3. Assume that $\vec{E}_1, \vec{E}_2, \vec{E}_3$ are unit vectors in the directions of the x^1, x^2 and x^3 axes respectively. If the unit vectors satisfy the relations

$$\begin{aligned} \vec{E}_1\cdot\vec{E}_1 &= 1 & \qquad \vec{E}_1\cdot\vec{E}_2 &= \cos\theta_{12} \\ \vec{E}_2\cdot\vec{E}_2 &= 1 & \qquad \vec{E}_1\cdot\vec{E}_3 &= \cos\theta_{13} \\ \vec{E}_3\cdot\vec{E}_3 &= 1 & \qquad \vec{E}_2\cdot\vec{E}_3 &= \cos\theta_{23}, \end{aligned}$$

then calculate the metrices g_{ij} and conjugate metrices g^{ij}.

▶ **43.** Let A_{ij}, $i,j = 1,2,3,4$ denote the skew-symmetric second rank tensor

$$A_{ij} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix},$$

where a,b,c,d,e,f are complex constants. Calculate the components of the dual tensor

$$V^{ij} = \frac{1}{2}e^{ijkl}A_{kl}.$$

▶ **44.** In Cartesian coordinates the vorticity tensor at a point in a fluid medium is defined

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial V_j}{\partial x^i} - \frac{\partial V_i}{\partial x^j}\right)$$

where V_i are the velocity components of the fluid at the point. The vorticity vector at a point in a fluid medium in Cartesian coordinates is defined by $\omega^i = \frac{1}{2}e^{ijk}\omega_{jk}$. Show that these tensors are dual tensors.

▶ **45.** Write out the relation between each of the components of the dual tensors

$$\hat{T}^{ij} = \frac{1}{2}e^{ijkl}T_{kl} \quad i,j,k,l = 1,2,3,4$$

and show that if $ijkl$ is an even permutation of 1234, then $\hat{T}^{ij} = T_{kl}$.

▶ **46.** Consider the general affine transformation $\bar{x}_i = a_{ij}x_j$ where $(x^1, x^2, x^3) = (x, y, z)$ with inverse transformation $x_i = b_{ij}\bar{x}_j$. Determine (a) the image of the plane $Ax + By + Cz + D = 0$ under this transformation and (b) the image of a second degree conic section

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0.$$

▶ **47.** Using a multilinear form of degree M, derive the transformation law for a contravariant vector of degree M.

▶ **48.** Let g denote the determinant of g_{ij} and show that $\dfrac{\partial g}{\partial x^k} = g g^{ij}\dfrac{\partial g_{ij}}{\partial x^k}$.

▶ **49.** We have shown that for a rotation of xyz axes with respect to a set of fixed $\bar{x}\bar{y}\bar{z}$ axes, the derivative of a vector $\vec{A}$ with respect to an observer on the barred axes is given by

$$\left.\frac{d\vec{A}}{dt}\right|_f = \left.\frac{d\vec{A}}{dt}\right|_r + \vec{\omega} \times \vec{A}.$$

Introduce the operators

$$D_f\vec{A} = \left.\frac{d\vec{A}}{dt}\right|_f = \text{derivative in fixed system}$$

$$D_r\vec{A} = \left.\frac{d\vec{A}}{dt}\right|_r = \text{derivative in rotating system}$$

(a) Show that $D_f\vec{A} = (D_r + \vec{\omega}\times)\vec{A}$.

(b) Consider the special case that the vector $\vec{A}$ is the position vector $\vec{r}$. Show that $D_f\vec{r} = (D_r + \vec{\omega}\times)\vec{r}$ produces $\left.\vec{V}\right|_f = \left.\vec{V}\right|_r + \vec{\omega} \times \vec{r}$ where $\left.\vec{V}\right|_f$ represents the velocity of a particle relative to the fixed system and $\left.\vec{V}\right|_r$ represents the velocity of a particle with respect to the rotating system of coordinates.

(c) Show that $\left.\vec{a}\right|_f = \left.\vec{a}\right|_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$ where $\left.\vec{a}\right|_f$ represents the acceleration of a particle relative to the fixed system and $\left.\vec{a}\right|_r$ represents the acceleration of a particle with respect to the rotating system.

(d) Show in the special case $\vec{\omega}$ is a constant that

$$\left.\vec{a}\right|_f = 2\vec{\omega} \times \vec{V} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

where $\vec{V}$ is the velocity of the particle relative to the rotating system. The term $2\vec{\omega} \times \vec{V}$ is referred to as the Coriolis acceleration and the term $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is referred to as the centripetal acceleration.