§1.4 DERIVATIVE OF A TENSOR

In this section we develop some additional operations associated with tensors. Historically, one of the basic problems of the tensor calculus was to try and find a tensor quantity which is a function of the metric tensor g_{ij} and some of its derivatives $\dfrac{\partial g_{ij}}{\partial x^m}$, $\dfrac{\partial^2 g_{ij}}{\partial x^m \partial x^n}$, ... A solution of this problem is the fourth order Riemann Christoffel tensor R_{ijkl} to be developed shortly. In order to understand how this tensor was arrived at, we must first develop some preliminary relationships involving Christoffel symbols.

Christoffel Symbols

Let us consider the metric tensor g_{ij} which we know satisfies the transformation law

$$\overline{g}_{\alpha\beta} = g_{ab}\frac{\partial x^a}{\partial \overline{x}^\alpha}\frac{\partial x^b}{\partial \overline{x}^\beta}.$$

Define the quantity

$$(\alpha,\beta,\gamma) = \frac{\partial \overline{g}_{\alpha\beta}}{\partial \overline{x}^\gamma} = \frac{\partial g_{ab}}{\partial x^c}\frac{\partial x^c}{\partial \overline{x}^\gamma}\frac{\partial x^a}{\partial \overline{x}^\alpha}\frac{\partial x^b}{\partial \overline{x}^\beta} + g_{ab}\frac{\partial^2 x^a}{\partial \overline{x}^\alpha \partial \overline{x}^\gamma}\frac{\partial x^b}{\partial \overline{x}^\beta} + g_{ab}\frac{\partial x^a}{\partial \overline{x}^\alpha}\frac{\partial^2 x^b}{\partial \overline{x}^\beta \partial \overline{x}^\gamma}$$

and form the combination of terms $\frac{1}{2}\left[(\alpha,\beta,\gamma)+(\beta,\gamma,\alpha)-(\gamma,\alpha,\beta)\right]$ to obtain the result

$$\frac{1}{2}\left[\frac{\partial \overline{g}_{\alpha\beta}}{\partial \overline{x}^\gamma} + \frac{\partial \overline{g}_{\beta\gamma}}{\partial \overline{x}^\alpha} - \frac{\partial \overline{g}_{\gamma\alpha}}{\partial \overline{x}^\beta}\right] = \frac{1}{2}\left[\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b}\right]\frac{\partial x^a}{\partial \overline{x}^\alpha}\frac{\partial x^b}{\partial \overline{x}^\beta}\frac{\partial x^c}{\partial \overline{x}^\gamma} + g_{ab}\frac{\partial x^b}{\partial \overline{x}^\beta}\frac{\partial^2 x^a}{\partial \overline{x}^\alpha \partial \overline{x}^\gamma}. \tag{1.4.1}$$

In this equation the combination of derivatives occurring inside the brackets is called a Christoffel symbol of the first kind and is defined by the notation

$$[ac,b] = [ca,b] = \frac{1}{2}\left[\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^b}\right]. \tag{1.4.2}$$

The equation (1.4.1) defines the transformation for a Christoffel symbol of the first kind and can be expressed as

$$\overline{[\alpha\ \gamma,\beta]} = [ac,b]\frac{\partial x^a}{\partial \overline{x}^\alpha}\frac{\partial x^b}{\partial \overline{x}^\beta}\frac{\partial x^c}{\partial \overline{x}^\gamma} + g_{ab}\frac{\partial^2 x^a}{\partial \overline{x}^\alpha \partial \overline{x}^\gamma}\frac{\partial x^b}{\partial \overline{x}^\beta}. \tag{1.4.3}$$

Observe that the Christoffel symbol of the first kind $[ac,b]$ does not transform like a tensor. However, it is symmetric in the indices a and c.

At this time it is convenient to use the equation (1.4.3) to develop an expression for the second derivative term which occurs in that equation as this second derivative term arises in some of our future considerations. To solve for this second derivative we can multiply equation (1.4.3) by $\dfrac{\partial \overline{x}^\beta}{\partial x^d} g^{de}$ and simplify the result to the form

$$\frac{\partial^2 x^e}{\partial \overline{x}^\alpha \partial \overline{x}^\gamma} = -g^{de}[ac,d]\frac{\partial x^a}{\partial \overline{x}^\alpha}\frac{\partial x^c}{\partial \overline{x}^\gamma} + \overline{[\alpha\,\gamma,\beta]}\frac{\partial \overline{x}^\beta}{\partial x^d} g^{de}. \tag{1.4.4}$$

The transformation $g^{de} = \overline{g}^{\lambda\mu}\dfrac{\partial x^d}{\partial \overline{x}^\lambda}\dfrac{\partial x^e}{\partial \overline{x}^\mu}$ allows us to express the equation (1.4.4) in the form

$$\frac{\partial^2 x^e}{\partial \overline{x}^\alpha \partial \overline{x}^\gamma} = -g^{de}[ac,d]\frac{\partial x^a}{\partial \overline{x}^\alpha}\frac{\partial x^c}{\partial \overline{x}^\gamma} + \overline{g}^{\beta\mu}\overline{[\alpha\,\gamma,\beta]}\frac{\partial x^e}{\partial \overline{x}^\mu}. \tag{1.4.5}$$

Define the Christoffel symbol of the second kind as

$$\left\{ {i \atop j\,k} \right\} = \left\{ {i \atop k\,j} \right\} = g^{i\alpha}[jk,\alpha] = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{k\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\alpha} \right). \tag{1.4.6}$$

This Christoffel symbol of the second kind is symmetric in the indices j and k and from equation (1.4.5) we see that it satisfies the transformation law

$$\overline{\left\{ {\mu \atop \alpha\,\gamma} \right\}} \frac{\partial x^e}{\partial \overline{x}^\mu} = \left\{ {e \atop a\,c} \right\} \frac{\partial x^a}{\partial \overline{x}^\alpha} \frac{\partial x^c}{\partial \overline{x}^\gamma} + \frac{\partial^2 x^e}{\partial \overline{x}^\alpha \partial \overline{x}^\gamma}. \tag{1.4.7}$$

Observe that the Christoffel symbol of the second kind does not transform like a tensor quantity. We can use the relation defined by equation (1.4.7) to express the second derivative of the transformation equations in terms of the Christoffel symbols of the second kind. At times it will be convenient to represent the Christoffel symbols with a subscript to indicate the metric from which they are calculated. Thus, an alternative notation for $\overline{\left\{ {i \atop j\,k} \right\}}$ is the notation $\left\{ {i \atop j\,k} \right\}_{\overline{g}}$.

EXAMPLE 1.4-1. (Christoffel symbols) Solve for the Christoffel symbol of the first kind in terms of the Christoffel symbol of the second kind.

Solution: By the definition from equation (1.4.6) we have

$$\left\{ {i \atop j\,k} \right\} = g^{i\alpha}[jk,\alpha].$$

We multiply this equation by $g_{\beta i}$ and find

$$g_{\beta i} \left\{ {i \atop j\,k} \right\} = \delta^\alpha_\beta [jk,\alpha] = [jk,\beta]$$

and so

$$[jk,\alpha] = g_{\alpha i} \left\{ {i \atop j\,k} \right\} = g_{\alpha 1} \left\{ {1 \atop j\,k} \right\} + \cdots + g_{\alpha N} \left\{ {N \atop j\,k} \right\}.$$

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EXAMPLE 1.4-2. (Christoffel symbols of first kind)

Derive formulas to find the Christoffel symbols of the first kind in a generalized orthogonal coordinate system with metric coefficients

$$g_{ij} = 0 \quad \text{for} \quad i \neq j \qquad \text{and} \qquad g_{(i)(i)} = h^2_{(i)}, \quad i = 1, 2, 3$$

where i is not summed.

Solution: In an orthogonal coordinate system where $g_{ij} = 0$ for $i \neq j$ we observe that

$$[ab,c] = \frac{1}{2} \left(\frac{\partial g_{ac}}{\partial x^b} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^c} \right). \tag{1.4.8}$$

Here there are $3^3 = 27$ quantities to calculate. We consider the following cases:

CASE I Let $a = b = c = i$, then the equation (1.4.8) simplifies to

$$[ab,c] = [ii,i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i} \quad \text{(no summation on } i). \tag{1.4.9}$$

From this equation we can calculate any of the Christoffel symbols

$$[11,1], \quad [22,2], \quad \text{or} \quad [33,3].$$

CASE II Let $a = b = i \neq c$, then the equation (1.4.8) simplifies to the form

$$[ab,c] = [ii,c] = -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^c} \quad \text{(no summation on } i \text{ and } i \neq c). \tag{1.4.10}$$

since, $g_{ic} = 0$ for $i \neq c$. This equation shows how we may calculate any of the six Christoffel symbols

$$[11,2], \quad [11,3], \quad [22,1], \quad [22,3], \quad [33,1], \quad [33,2].$$

CASE III Let $a = c = i \neq b$, and noting that $g_{ib} = 0$ for $i \neq b$, it can be verified that the equation (1.4.8) simplifies to the form

$$[ab,c] = [ib,i] = [bi,i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^b} \quad \text{(no summation on } i \text{ and } i \neq b). \tag{1.4.11}$$

From this equation we can calculate any of the twelve Christoffel symbols

$$\begin{array}{ll} [12,1] = [21,1] & [31,3] = [13,3] \\ [32,3] = [23,3] & [21,2] = [12,2] \\ [13,1] = [31,1] & [23,2] = [32,2] \end{array}$$

CASE IV Let $a \neq b \neq c$ and show that the equation (1.4.8) reduces to

$$[ab,c] = 0, \qquad (a \neq b \neq c.)$$

This represents the six Christoffel symbols

$$[12,3] = [21,3] = [23,1] = [32,1] = [31,2] = [13,2] = 0.$$

From the Cases I,II,III,IV all twenty seven Christoffel symbols of the first kind can be determined. In practice, only the nonzero Christoffel symbols are listed.

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EXAMPLE 1.4-3. (Christoffel symbols of the first kind) Find the nonzero Christoffel symbols of the first kind in cylindrical coordinates.

Solution: From the results of example 1.4-2 we find that for $x^1 = r, \quad x^2 = \theta, \quad x^3 = z$ and

$$g_{11} = 1, \qquad g_{22} = (x^1)^2 = r^2, \qquad g_{33} = 1$$

the nonzero Christoffel symbols of the first kind in cylindrical coordinates are:

$$[22,1] = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = -x^1 = -r$$

$$[21,2] = [12,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = x^1 = r.$$

■

EXAMPLE 1.4-4. (Christoffel symbols of the second kind)

Find formulas for the calculation of the Christoffel symbols of the second kind in a generalized orthogonal coordinate system with metric coefficients

$$g_{ij} = 0 \quad \text{for} \quad i \neq j \qquad \text{and} \qquad g_{(i)(i)} = h_{(i)}^2, \quad i = 1,2,3$$

where i is not summed.

Solution: By definition we have

$$\left\{ {i \atop j\,k} \right\} = g^{im}[jk,m] = g^{i1}[jk,1] + g^{i2}[jk,2] + g^{i3}[jk,3] \tag{1.4.12}$$

By hypothesis the coordinate system is orthogonal and so

$$g^{ij} = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad g^{ii} = \frac{1}{g_{ii}} \quad i \text{ not summed.}$$

The only nonzero term in the equation (1.4.12) occurs when $m = i$ and consequently

$$\left\{ {i \atop j\,k} \right\} = g^{ii}[jk,i] = \frac{[jk,i]}{g_{ii}} \quad \text{no summation on } i. \tag{1.4.13}$$

We can now consider the four cases considered in the example 1.4-2.

CASE I Let $j = k = i$ and show

$$\left\{ {i \atop i\,i} \right\} = \frac{[ii,i]}{g_{ii}} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^i} = \frac{1}{2} \frac{\partial}{\partial x^i} \ln g_{ii} \quad \text{no summation on } i. \tag{1.4.14}$$

CASE II Let $k = j \neq i$ and show

$$\left\{ {i \atop j\,j} \right\} = \frac{[jj,i]}{g_{ii}} = \frac{-1}{2g_{ii}} \frac{\partial g_{jj}}{\partial x^i} \quad \text{no summation on } i \text{ or } j. \tag{1.4.15}$$

CASE III Let $i = j \neq k$ and verify that

$$\left\{ {j \atop j\,k} \right\} = \left\{ {j \atop k\,j} \right\} = \frac{[jk,j]}{g_{jj}} = \frac{1}{2g_{jj}} \frac{\partial g_{jj}}{\partial x^k} = \frac{1}{2} \frac{\partial}{\partial x^k} \ln g_{jj} \quad \text{no summation on } i \text{ or } j. \tag{1.4.16}$$

CASE IV For the case $i \neq j \neq k$ we find

$$\left\{ {i \atop j\,k} \right\} = \frac{[jk,i]}{g_{ii}} = 0, \quad i \neq j \neq k \quad \text{no summation on } i.$$

The above cases represent all 27 terms.

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EXAMPLE 1.4-5. (Notation) In the case of cylindrical coordinates we can use the above relations and find the nonzero Christoffel symbols of the second kind:

$$\left\{ {1 \atop 2\,2} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial x^1} = -x^1 = -r$$
$$\left\{ {2 \atop 1\,2} \right\} = \left\{ {2 \atop 2\,1} \right\} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^1} = \frac{1}{x^1} = \frac{1}{r}$$

Note 1: The notation for the above Christoffel symbols are based upon the assumption that $x^1 = r, x^2 = \theta$ and $x^3 = z$. However, in tensor calculus the choice of the coordinates can be arbitrary. We could just as well have defined $x^1 = z, x^2 = r$ and $x^3 = \theta$. In this latter case, the numbering system of the Christoffel symbols changes. To avoid confusion, an alternate method of writing the Christoffel symbols is to use coordinates in place of the integers 1,2 and 3. For example, in cylindrical coordinates we can write

$$\left\{ {\theta \atop r\,\theta} \right\} = \left\{ {\theta \atop \theta\,r} \right\} = \frac{1}{r} \quad \text{and} \quad \left\{ {r \atop \theta\,\theta} \right\} = -r.$$

If we define $x^1 = r, x^2 = \theta, x^3 = z$, then the nonzero Christoffel symbols are written as

$$\left\{ {2 \atop 1\,2} \right\} = \left\{ {2 \atop 2\,1} \right\} = \frac{1}{r} \quad \text{and} \quad \left\{ {1 \atop 2\,2} \right\} = -r.$$

In contrast, if we define $x^1 = z, x^2 = r, x^3 = \theta$, then the nonzero Christoffel symbols are written

$$\left\{ {3 \atop 2\,3} \right\} = \left\{ {3 \atop 3\,2} \right\} = \frac{1}{r} \quad \text{and} \quad \left\{ {2 \atop 3\,3} \right\} = -r.$$

Note 2: Some textbooks use the notation $\Gamma_{a,bc}$ for Christoffel symbols of the first kind and $\Gamma^d_{bc} = g^{da}\Gamma_{a,bc}$ for Christoffel symbols of the second kind. This notation is not used in these notes since the notation suggests that the Christoffel symbols are third order tensors, which is not true. The Christoffel symbols of the first and second kind are not tensors. This fact is clearly illustrated by the transformation equations (1.4.3) and (1.4.7).

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Covariant Differentiation

Let A_i denote a covariant tensor of rank 1 which obeys the transformation law

$$\overline{A}_\alpha = A_i \frac{\partial x^i}{\partial \overline{x}^\alpha}. \tag{1.4.17}$$

Differentiate this relation with respect to $\overline{x}^\beta$ and show

$$\frac{\partial \overline{A}_\alpha}{\partial \overline{x}^\beta} = A_i \frac{\partial^2 x^i}{\partial \overline{x}^\alpha \partial \overline{x}^\beta} + \frac{\partial A_i}{\partial x^j} \frac{\partial x^j}{\partial \overline{x}^\beta} \frac{\partial x^i}{\partial \overline{x}^\alpha}. \tag{1.4.18}$$

Now use the relation from equation (1.4.7) to eliminate the second derivative term from (1.4.18) and express it in the form

$$\frac{\partial \overline{A}_\alpha}{\partial \overline{x}^\beta} = A_i \left[\overline{\left\{ {\sigma \atop \alpha\,\beta} \right\}} \frac{\partial x^i}{\partial \overline{x}^\sigma} - \left\{ {i \atop j\,k} \right\} \frac{\partial x^j}{\partial \overline{x}^\alpha} \frac{\partial x^k}{\partial \overline{x}^\beta} \right] + \frac{\partial A_i}{\partial x^j} \frac{\partial x^j}{\partial \overline{x}^\beta} \frac{\partial x^i}{\partial \overline{x}^\alpha}. \tag{1.4.19}$$

Employing the equation (1.4.17), with α replaced by σ, the equation (1.4.19) is expressible in the form

$$\frac{\partial \overline{A}_\alpha}{\partial \overline{x}^\beta} - \overline{A}_\sigma \overline{\left\{ \begin{matrix} \sigma \\ \alpha\,\beta \end{matrix} \right\}} = \frac{\partial A_j}{\partial x^k} \frac{\partial x^j}{\partial \overline{x}^\alpha} \frac{\partial x^k}{\partial \overline{x}^\beta} - A_i \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \frac{\partial x^j}{\partial \overline{x}^\alpha} \frac{\partial x^k}{\partial \overline{x}^\beta} \tag{1.4.20}$$

or alternatively

$$\left[\frac{\partial \overline{A}_\alpha}{\partial \overline{x}^\beta} - \overline{A}_\sigma \overline{\left\{ \begin{matrix} \sigma \\ \alpha\,\beta \end{matrix} \right\}} \right] = \left[\frac{\partial A_j}{\partial x^k} - A_i \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \right] \frac{\partial x^j}{\partial \overline{x}^\alpha} \frac{\partial x^k}{\partial \overline{x}^\beta}. \tag{1.4.21}$$

Define the quantity

$$A_{j,k} = \frac{\partial A_j}{\partial x^k} - A_i \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \tag{1.4.22}$$

as the covariant derivative of A_j with respect to x^k. The equation (1.4.21) demonstrates that the covariant derivative of a covariant tensor produces a second order tensor which satisfies the transformation law

$$\overline{A}_{\alpha,\beta} = A_{j,k} \frac{\partial x^j}{\partial \overline{x}^\alpha} \frac{\partial x^k}{\partial \overline{x}^\beta}. \tag{1.4.23}$$

Other notations frequently used to denote the covariant derivative are:

$$A_{j,k} = A_{j;k} = A_{j/k} = \nabla_k A_j = A_j|_k. \tag{1.4.24}$$

In the special case where g_{ij} are constants the Christoffel symbols of the second kind are zero, and consequently the covariant derivative reduces to $A_{j,k} = \frac{\partial A_j}{\partial x^k}$. That is, under the special circumstances where the Christoffel symbols of the second kind are zero, the covariant derivative reduces to an ordinary derivative.

Covariant Derivative of Contravariant Tensor

A contravariant tensor A^i obeys the transformation law $\overline{A}^i = A^\alpha \frac{\partial \overline{x}^i}{\partial x^\alpha}$ which can be expressed in the form

$$A^i = \overline{A}^\alpha \frac{\partial x^i}{\partial \overline{x}^\alpha} \tag{1.4.24}$$

by interchanging the barred and unbarred quantities. We write the transformation law in the form of equation (1.4.24) in order to make use of the second derivative relation from the previously derived equation (1.4.7). Differentiate equation (1.4.24) with respect to x^j to obtain the relation

$$\frac{\partial A^i}{\partial x^j} = \overline{A}^\alpha \frac{\partial^2 x^i}{\partial \overline{x}^\alpha \partial \overline{x}^\beta} \frac{\partial \overline{x}^\beta}{\partial x^j} + \frac{\partial \overline{A}^\alpha}{\partial \overline{x}^\beta} \frac{\partial \overline{x}^\beta}{\partial x^j} \frac{\partial x^i}{\partial \overline{x}^\alpha}. \tag{1.4.25}$$

Changing the indices in equation (1.4.25) and substituting for the second derivative term, using the relation from equation (1.4.7), produces the equation

$$\frac{\partial A^i}{\partial x^j} = \overline{A}^\alpha \left[\overline{\left\{ \begin{matrix} \sigma \\ \alpha\,\beta \end{matrix} \right\}} \frac{\partial x^i}{\partial \overline{x}^\sigma} - \left\{ \begin{matrix} i \\ m\,k \end{matrix} \right\} \frac{\partial x^m}{\partial \overline{x}^\alpha} \frac{\partial x^k}{\partial \overline{x}^\beta} \right] \frac{\partial \overline{x}^\beta}{\partial x^j} + \frac{\partial \overline{A}^\alpha}{\partial \overline{x}^\beta} \frac{\partial \overline{x}^\beta}{\partial x^j} \frac{\partial x^i}{\partial \overline{x}^\alpha}. \tag{1.4.26}$$

Applying the relation found in equation (1.4.24), with i replaced by m, together with the relation

$$\frac{\partial \overline{x}^\beta}{\partial x^j} \frac{\partial x^k}{\partial \overline{x}^\beta} = \delta^k_j,$$

we simplify equation (1.4.26) to the form

$$\left[\frac{\partial A^i}{\partial x^j} + \left\{ \begin{matrix} i \\ m\, j \end{matrix} \right\} A^m\right] = \left[\frac{\partial \overline{A}^\sigma}{\partial \overline{x}^\beta} + \overline{\left\{ \begin{matrix} \sigma \\ \alpha\, \beta \end{matrix} \right\}}\overline{A}^\alpha\right] \frac{\partial \overline{x}^\beta}{\partial x^j}\frac{\partial x^i}{\partial \overline{x}^\sigma}. \tag{1.4.27}$$

Define the quantity

$$A^i{}_{,j} = \frac{\partial A^i}{\partial x^j} + \left\{ \begin{matrix} i \\ m\, j \end{matrix} \right\} A^m \tag{1.4.28}$$

as the covariant derivative of the contravariant tensor A^i. The equation (1.4.27) demonstrates that a covariant derivative of a contravariant tensor will transform like a mixed second order tensor and

$$A^i{}_{,j} = \overline{A}^\sigma{}_{,\beta}\frac{\partial \overline{x}^\beta}{\partial x^j}\frac{\partial x^i}{\partial \overline{x}^\sigma}. \tag{1.4.29}$$

Again it should be observed that for the condition where g_{ij} are constants we have $A^i{}_{,j} = \dfrac{\partial A^i}{\partial x^j}$ and the covariant derivative of a contravariant tensor reduces to an ordinary derivative in this special case.

In a similar manner the covariant derivative of second rank tensors can be derived. We find these derivatives have the forms:

$$\begin{aligned} A_{ij,k} &= \frac{\partial A_{ij}}{\partial x^k} - A_{\sigma j}\left\{ \begin{matrix} \sigma \\ i\, k \end{matrix} \right\} - A_{i\sigma}\left\{ \begin{matrix} \sigma \\ j\, k \end{matrix} \right\} \\ A^i_{j\,,k} &= \frac{\partial A^i_j}{\partial x^k} + A^\sigma_j\left\{ \begin{matrix} i \\ \sigma\, k \end{matrix} \right\} - A^i_\sigma\left\{ \begin{matrix} \sigma \\ j\, k \end{matrix} \right\} \\ A^{ij}{}_{,k} &= \frac{\partial A^{ij}}{\partial x^k} + A^{\sigma j}\left\{ \begin{matrix} i \\ \sigma\, k \end{matrix} \right\} + A^{i\sigma}\left\{ \begin{matrix} j \\ \sigma\, k \end{matrix} \right\}. \end{aligned} \tag{1.4.30}$$

In general, the covariant derivative of a mixed tensor

$$A^{ij\ldots k}_{lm\ldots p}$$

of rank n has the form

$$\begin{aligned} A^{ij\ldots k}_{lm\ldots p,q} = \frac{\partial A^{ij\ldots k}_{lm\ldots p}}{\partial x^q} &+ A^{\sigma j\ldots k}_{lm\ldots p}\left\{ \begin{matrix} i \\ \sigma\, q \end{matrix} \right\} + A^{i\sigma\ldots k}_{lm\ldots p}\left\{ \begin{matrix} j \\ \sigma\, q \end{matrix} \right\} + \cdots + A^{ij\ldots\sigma}_{lm\ldots p}\left\{ \begin{matrix} k \\ \sigma\, q \end{matrix} \right\} \\ &- A^{ij\ldots k}_{\sigma m\ldots p}\left\{ \begin{matrix} \sigma \\ l\, q \end{matrix} \right\} - A^{ij\ldots k}_{l\sigma\ldots p}\left\{ \begin{matrix} \sigma \\ m\, q \end{matrix} \right\} - \cdots - A^{ij\ldots k}_{lm\ldots\sigma}\left\{ \begin{matrix} \sigma \\ p\, q \end{matrix} \right\} \end{aligned} \tag{1.4.31}$$

and this derivative is a tensor of rank $n+1$. Note the pattern of the $+$ signs for the contravariant indices and the $-$ signs for the covariant indices.

Observe that the covariant derivative of an nth order tensor produces an $n+1$st order tensor, the indices of these higher order tensors can also be raised and lowered by multiplication by the metric or conjugate metric tensor. For example we can write

$$g^{im}A_{jk}|_m = A_{jk}|^i \quad \text{and} \quad g^{im}A^{jk}|_m = A^{jk}|^i$$

Rules for Covariant Differentiation

The rules for covariant differentiation are the same as for ordinary differentiation. That is:

(i) The covariant derivative of a sum is the sum of the covariant derivatives.

(ii) The covariant derivative of a product of tensors is the first times the covariant derivative of the second plus the second times the covariant derivative of the first.

(iii) Higher derivatives are defined as derivatives of derivatives. Be careful in calculating higher order derivatives as in general

$$A_{i,jk} \neq A_{i,kj}.$$

EXAMPLE 1.4-6. (Covariant differentiation) Calculate the second covariant derivative $A_{i,jk}$.

Solution: The covariant derivative of A_i is

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_\sigma \left\{ \begin{matrix} \sigma \\ i\, j \end{matrix} \right\}.$$

By definition, the second covariant derivative is the covariant derivative of a covariant derivative and hence

$$A_{i,jk} = (A_{i,j})_{,k} = \frac{\partial}{\partial x^k}\left[\frac{\partial A_i}{\partial x^j} - A_\sigma \left\{ \begin{matrix} \sigma \\ i\, j \end{matrix} \right\}\right] - A_{m,j}\left\{ \begin{matrix} m \\ i\, k \end{matrix} \right\} - A_{i,m}\left\{ \begin{matrix} m \\ j\, k \end{matrix} \right\}.$$

Simplifying this expression one obtains

$$\begin{aligned} A_{i,jk} = & \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial A_\sigma}{\partial x^k}\left\{ \begin{matrix} \sigma \\ i\, j \end{matrix} \right\} - A_\sigma \frac{\partial}{\partial x^k}\left\{ \begin{matrix} \sigma \\ i\, j \end{matrix} \right\} \\ & - \left[\frac{\partial A_m}{\partial x^j} - A_\sigma \left\{ \begin{matrix} \sigma \\ m\, j \end{matrix} \right\}\right]\left\{ \begin{matrix} m \\ i\, k \end{matrix} \right\} - \left[\frac{\partial A_i}{\partial x^m} - A_\sigma \left\{ \begin{matrix} \sigma \\ i\, m \end{matrix} \right\}\right]\left\{ \begin{matrix} m \\ j\, k \end{matrix} \right\}. \end{aligned}$$

Rearranging terms, the second covariant derivative can be expressed in the form

$$\begin{aligned} A_{i,jk} = & \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial A_\sigma}{\partial x^k}\left\{ \begin{matrix} \sigma \\ i\, j \end{matrix} \right\} - \frac{\partial A_m}{\partial x^j}\left\{ \begin{matrix} m \\ i\, k \end{matrix} \right\} - \frac{\partial A_i}{\partial x^m}\left\{ \begin{matrix} m \\ j\, k \end{matrix} \right\} \\ & - A_\sigma \left[\frac{\partial}{\partial x^k}\left\{ \begin{matrix} \sigma \\ i\, j \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ i\, m \end{matrix} \right\}\left\{ \begin{matrix} m \\ j\, k \end{matrix} \right\} - \left\{ \begin{matrix} m \\ i\, k \end{matrix} \right\}\left\{ \begin{matrix} \sigma \\ m\, j \end{matrix} \right\}\right]. \end{aligned} \tag{1.4.32}$$

■

Riemann Christoffel Tensor

Utilizing the equation (1.4.32), it is left as an exercise to show that

$$A_{i,jk} - A_{i,kj} = A_\sigma R^\sigma_{ijk}$$

where

$$R^\sigma_{ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix} \sigma \\ i\,k \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} \sigma \\ i\,j \end{Bmatrix} + \begin{Bmatrix} m \\ i\,k \end{Bmatrix}\begin{Bmatrix} \sigma \\ m\,j \end{Bmatrix} - \begin{Bmatrix} m \\ i\,j \end{Bmatrix}\begin{Bmatrix} \sigma \\ m\,k \end{Bmatrix} \tag{1.4.33}$$

is called the Riemann Christoffel tensor. The covariant form of this tensor is

$$R_{hjkl} = g_{ih} R^i_{jkl}. \tag{1.4.34}$$

It is an easy exercise to show that this covariant form can be expressed in either of the forms

$$R_{injk} = \frac{\partial}{\partial x^j}[nk,i] - \frac{\partial}{\partial x^k}[nj,i] + [ik,s]\begin{Bmatrix} s \\ n\,j \end{Bmatrix} - [ij,s]\begin{Bmatrix} s \\ n\,k \end{Bmatrix}$$

or $$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}\right) + g^{\alpha\beta}\left([jk,\beta][il,\alpha] - [jl,\beta][ik,\alpha]\right).$$

From these forms we find that the Riemann Christoffel tensor is skew symmetric in the first two indices and the last two indices as well as being symmetric in the interchange of the first pair and last pairs of indices and consequently

$$R_{jikl} = -R_{ijkl} \qquad R_{ijlk} = -R_{ijkl} \qquad R_{klij} = R_{ijkl}.$$

In a two dimensional space there are only four components of the Riemann Christoffel tensor to consider. These four components are either $+R_{1212}$ or $-R_{1212}$ since they are all related by

$$R_{1212} = -R_{2112} = R_{2121} = -R_{1221}.$$

In a Cartesian coordinate system $R_{hijk} = 0$. The Riemann Christoffel tensor is important because it occurs in differential geometry and relativity which are two areas of interest to be considered later. Additional properties of this tensor are found in the exercises of section 1.5.

Physical Interpretation of Covariant Differentiation

In a system of generalized coordinates (x^1, x^2, x^3) we can construct the basis vectors $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$. These basis vectors change with position. That is, each basis vector is a function of the coordinates at which they are evaluated. We can emphasize this dependence by writing

$$\vec{E}_i = \vec{E}_i(x^1, x^2, x^3) = \frac{\partial \vec{r}}{\partial x^i} \qquad i = 1, 2, 3.$$

Associated with these basis vectors we have the reciprocal basis vectors

$$\vec{E}^i = \vec{E}^i(x^1, x^2, x^3), \qquad i = 1, 2, 3$$

which are also functions of position. A vector $\vec{A}$ can be represented in terms of contravariant components as

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3 = A^j\vec{E}_j \tag{1.4.35}$$

or it can be represented in terms of covariant components as

$$\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3 = A_j\vec{E}^j. \tag{1.4.36}$$

A change in the vector $\vec{A}$ is represented as

$$d\vec{A} = \frac{\partial \vec{A}}{\partial x^k}\, dx^k$$

where from equation (1.4.35) we find

$$\frac{\partial \vec{A}}{\partial x^k} = A^j \frac{\partial \vec{E}_j}{\partial x^k} + \frac{\partial A^j}{\partial x^k}\vec{E}_j \tag{1.4.37}$$

or alternatively from equation (1.4.36) we may write

$$\frac{\partial \vec{A}}{\partial x^k} = A_j \frac{\partial \vec{E}^j}{\partial x^k} + \frac{\partial A_j}{\partial x^k}\vec{E}^j. \tag{1.4.38}$$

We define the covariant derivative of the covariant components as

$$A_{i,k} = \frac{\partial \vec{A}}{\partial x^k} \cdot \vec{E}_i = \frac{\partial A_i}{\partial x^k} + A_j \frac{\partial \vec{E}^j}{\partial x^k} \cdot \vec{E}_i. \tag{1.4.39}$$

The covariant derivative of the contravariant components are defined by the relation

$$A^i{}_{,k} = \frac{\partial \vec{A}}{\partial x^k} \cdot \vec{E}^i = \frac{\partial A^i}{\partial x^k} + A^j \frac{\partial \vec{E}_j}{\partial x^k} \cdot \vec{E}^i. \tag{1.4.40}$$

Introduce the notation

$$\frac{\partial \vec{E}_j}{\partial x^k} = \left\{ {m \atop j\,k} \right\} \vec{E}_m \qquad \text{and} \qquad \frac{\partial \vec{E}^j}{\partial x^k} = -\left\{ {j \atop m\,k} \right\} \vec{E}^m. \tag{1.4.41}$$

We then have

$$\vec{E}^i \cdot \frac{\partial \vec{E}_j}{\partial x^k} = \left\{ {m \atop j\,k} \right\} \vec{E}_m \cdot \vec{E}^i = \left\{ {m \atop j\,k} \right\} \delta^i_m = \left\{ {i \atop j\,k} \right\} \tag{1.4.42}$$

and

$$\vec{E}_i \cdot \frac{\partial \vec{E}^j}{\partial x^k} = -\begin{Bmatrix} j \\ m\,k \end{Bmatrix} \vec{E}^m \cdot \vec{E}_i = -\begin{Bmatrix} j \\ m\,k \end{Bmatrix} \delta_i^m = -\begin{Bmatrix} j \\ i\,k \end{Bmatrix}. \tag{1.4.43}$$

Then equations (1.4.39) and (1.4.40) become

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \begin{Bmatrix} j \\ i\,k \end{Bmatrix} A_j$$
$$A^i{}_{,k} = \frac{\partial A^i}{\partial x^k} + \begin{Bmatrix} i \\ j\,k \end{Bmatrix} A^j,$$

which is consistent with our earlier definitions from equations (1.4.22) and (1.4.28). Here the first term of the covariant derivative represents the rate of change of the tensor field as we move along a coordinate curve. The second term in the covariant derivative represents the change in the local basis vectors as we move along the coordinate curves. This is the physical interpretation associated with the Christoffel symbols of the second kind.

We make the observation that the derivatives of the basis vectors in equations (1.4.39) and (1.4.40) are related since

$$\vec{E}_i \cdot \vec{E}^j = \delta_i^j$$

and consequently

$$\frac{\partial}{\partial x^k}(\vec{E}_i \cdot \vec{E}^j) = \vec{E}_i \cdot \frac{\partial \vec{E}^j}{\partial x^k} + \frac{\partial \vec{E}_i}{\partial x^k} \cdot \vec{E}^j = 0$$
$$\text{or} \qquad \vec{E}_i \cdot \frac{\partial \vec{E}^j}{\partial x^k} = -\vec{E}^j \cdot \frac{\partial \vec{E}_i}{\partial x^k}$$

Hence we can express equation (1.4.39) in the form

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - A_j \vec{E}^j \cdot \frac{\partial \vec{E}_i}{\partial x^k}. \tag{1.4.44}$$

We write the first equation in (1.4.41) in the form

$$\frac{\partial \vec{E}_j}{\partial x^k} = \begin{Bmatrix} m \\ j\,k \end{Bmatrix} g_{im} \vec{E}^i = [jk, i] \vec{E}^i \tag{1.4.45}$$

and consequently

$$\begin{aligned} \frac{\partial \vec{E}_j}{\partial x^k} \cdot \vec{E}^m &= \begin{Bmatrix} i \\ j\,k \end{Bmatrix} \vec{E}_i \cdot \vec{E}^m = \begin{Bmatrix} i \\ j\,k \end{Bmatrix} \delta_i^m = \begin{Bmatrix} m \\ j\,k \end{Bmatrix} \\ \text{and} \qquad \frac{\partial \vec{E}_j}{\partial x^k} \cdot \vec{E}_m &= [jk, i] \vec{E}^i \cdot \vec{E}_m = [jk, i] \delta_m^i = [jk, m]. \end{aligned} \tag{1.4.46}$$

These results also reduce the equations (1.4.40) and (1.4.44) to our previous forms for the covariant derivatives.

The equations (1.4.41) are representations of the vectors $\frac{\partial \vec{E}_i}{\partial x^k}$ and $\frac{\partial \vec{E}^j}{\partial x^k}$ in terms of the basis vectors and reciprocal basis vectors of the space. The covariant derivative relations then take into account how these vectors change with position and affect changes in the tensor field.

The Christoffel symbols in equations (1.4.46) are symmetric in the indices j and k since

$$\frac{\partial \vec{E}_j}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\frac{\partial \vec{r}}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial \vec{r}}{\partial x^k}\right) = \frac{\partial \vec{E}_k}{\partial x^j}. \tag{1.4.47}$$

The equations (1.4.46) and (1.4.47) enable us to write

$$
\begin{aligned}
[jk,m] =&\vec{E}_m \cdot \frac{\partial \vec{E}_j}{\partial x^k} = \frac{1}{2}\left[\vec{E}_m \cdot \frac{\partial \vec{E}_j}{\partial x^k} + \vec{E}_m \cdot \frac{\partial \vec{E}_k}{\partial x^j}\right]\\
=&\frac{1}{2}\left[\frac{\partial}{\partial x^k}\left(\vec{E}_m \cdot \vec{E}_j\right) + \frac{\partial}{\partial x^j}\left(\vec{E}_m \cdot \vec{E}_k\right) - \vec{E}_j \cdot \frac{\partial \vec{E}_m}{\partial x^k} - \vec{E}_k \cdot \frac{\partial \vec{E}_m}{\partial x^j}\right]\\
=&\frac{1}{2}\left[\frac{\partial}{\partial x^k}\left(\vec{E}_m \cdot \vec{E}_j\right) + \frac{\partial}{\partial x^j}\left(\vec{E}_m \cdot \vec{E}_k\right) - \vec{E}_j \cdot \frac{\partial \vec{E}_k}{\partial x^m} - \vec{E}_k \cdot \frac{\partial \vec{E}_j}{\partial x^m}\right]\\
=&\frac{1}{2}\left[\frac{\partial}{\partial x^k}\left(\vec{E}_m \cdot \vec{E}_j\right) + \frac{\partial}{\partial x^j}\left(\vec{E}_m \cdot \vec{E}_k\right) - \frac{\partial}{\partial x^m}\left(\vec{E}_j \cdot \vec{E}_k\right)\right]\\
=&\frac{1}{2}\left[\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right] = [kj,m]
\end{aligned}
$$

which again agrees with our previous result.

For future reference we make the observation that if the vector $\vec{A}$ is represented in the form $\vec{A} = A^j \vec{E}_j$, involving contravariant components, then we may write

$$
\begin{aligned}
d\vec{A} = \frac{\partial \vec{A}}{\partial x^k}\,dx^k =& \left(\frac{\partial A^j}{\partial x^k}\vec{E}_j + A^j \frac{\partial \vec{E}_j}{\partial x^k}\right) dx^k\\
=& \left(\frac{\partial A^j}{\partial x^k}\vec{E}_j + A^j \begin{Bmatrix} i \\ j\,k \end{Bmatrix}\vec{E}_i\right) dx^k\\
=& \left(\frac{\partial A^j}{\partial x^k} + \begin{Bmatrix} j \\ m\,k \end{Bmatrix} A^m\right)\vec{E}_j\,dx^k = A^j{}_{,k}\,dx^k\,\vec{E}_j.
\end{aligned}
\tag{1.4.48}
$$

Similarly, if the vector $\vec{A}$ is represented in the form $\vec{A} = A_j \vec{E}^j$ involving covariant components it is left as an exercise to show that

$$
d\vec{A} = A_{j,k}\,dx^k\,\vec{E}^j \tag{1.4.49}
$$

Ricci's Theorem

Ricci's theorem states that the covariant derivative of the metric tensor vanishes and $g_{ik,l} = 0$.

Proof: We have

$$
\begin{aligned}
g_{ik,l} =& \frac{\partial g_{ik}}{\partial x^l} - \begin{Bmatrix} m \\ k\,l \end{Bmatrix} g_{im} - \begin{Bmatrix} m \\ i\,l \end{Bmatrix} g_{mk}\\
g_{ik,l} =& \frac{\partial g_{ik}}{\partial x^l} - [kl,i] - [il,k]\\
g_{ik,l} =& \frac{\partial g_{ik}}{\partial x^l} - \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right] - \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^k}\right] = 0.
\end{aligned}
$$

Because of Ricci's theorem the components of the metric tensor can be regarded as constants during covariant differentiation.

EXAMPLE 1.4-7. (Covariant differentiation) Show that $\delta^i_{j,k} = 0$.

Solution

$$
\delta^i_{j,k} = \frac{\partial \delta^i_j}{\partial x^k} + \delta^\sigma_j \begin{Bmatrix} i \\ \sigma\,k \end{Bmatrix} - \delta^i_\sigma \begin{Bmatrix} \sigma \\ j\,k \end{Bmatrix} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix} - \begin{Bmatrix} i \\ j\,k \end{Bmatrix} = 0.
$$

■

EXAMPLE 1.4-8. (Covariant differentiation) Show that $g^{ij}{}_{,k} = 0$.

Solution: Since $g_{ij}g^{jk} = \delta_i^k$ we take the covariant derivative of this expression and find

$$(g_{ij}g^{jk})_{,l} = \delta^k_{i,l} = 0$$
$$g_{ij}g^{jk}{}_{,l} + g_{ij,l}g^{jk} = 0.$$

But $g_{ij,l} = 0$ by Ricci's theorem and hence $g_{ij}g^{jk}{}_{,l} = 0$. We multiply this expression by g^{im} and obtain

$$g^{im}g_{ij}g^{jk}{}_{,l} = \delta_j^m g^{jk}{}_{,l} = g^{mk}_{,l} = 0$$

which demonstrates that the covariant derivative of the conjugate metric tensor is also zero.

■

EXAMPLE 1.4-9. (Covariant differentiation) Some additional examples of covariant differentiation are:

$$(i) \quad (g_{il}A^l)_{,k} = g_{il}A^l{}_{,k} = A_{i,k}$$
$$(ii) \quad (g_{im}g_{jn}A^{ij})_{,k} = g_{im}g_{jn}A^{ij}_{,k} = A_{mn,k}$$

■

<u>**Intrinsic or Absolute Differentiation**</u>

The intrinsic or absolute derivative of a covariant vector A_i taken along a curve $x^i = x^i(t), i = 1, \ldots, N$ is defined as the inner product of the covariant derivative with the tangent vector to the curve. The intrinsic derivative is represented

$$\begin{aligned}
\frac{\delta A_i}{\delta t} &= A_{i,j}\frac{dx^j}{dt} \\
\frac{\delta A_i}{\delta t} &= \left[\frac{\partial A_i}{\partial x^j} - A_\alpha \begin{Bmatrix} \alpha \\ i\, j \end{Bmatrix}\right]\frac{dx^j}{dt} \\
\frac{\delta A_i}{\delta t} &= \frac{dA_i}{dt} - A_\alpha \begin{Bmatrix} \alpha \\ i\, j \end{Bmatrix}\frac{dx^j}{dt}.
\end{aligned} \tag{1.4.50}$$

Similarly, the absolute or intrinsic derivative of a contravariant tensor A^i is represented

$$\frac{\delta A^i}{\delta t} = A^i{}_{,j}\frac{dx^j}{dt} = \frac{dA^i}{dt} + \begin{Bmatrix} i \\ j\, k \end{Bmatrix} A^k \frac{dx^j}{dt}.$$

The intrinsic or absolute derivative is used to differentiate sums and products in the same manner as used in ordinary differentiation. Also if the coordinate system is Cartesian the intrinsic derivative becomes an ordinary derivative.

The intrinsic derivative of higher order tensors is similarly defined as an inner product of the covariant derivative with the tangent vector to the given curve. For example,

$$\frac{\delta A^{ij}_{klm}}{\delta t} = A^{ij}_{klm,p}\frac{dx^p}{dt}$$

is the intrinsic derivative of the fifth order mixed tensor A^{ij}_{klm}.

EXAMPLE 1.4-10. (Generalized velocity and acceleration) Let t denote time and let $x^i = x^i(t)$ for $i = 1, \ldots, N$, denote the position vector of a particle in the generalized coordinates $(x^1, \ldots, x^N)$. From the transformation equations (1.2.30), the position vector of the same particle in the barred system of coordinates, $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N)$, is

$$\overline{x}^i = \overline{x}^i(x^1(t), x^2(t), \ldots, x^N(t)) = \overline{x}^i(t), \quad i = 1, \ldots, N.$$

The generalized velocity is $v^i = \frac{dx^i}{dt}$, $i = 1, \ldots, N$. The quantity v^i transforms as a tensor since by definition

$$\overline{v}^i = \frac{d\overline{x}^i}{dt} = \frac{\partial \overline{x}^i}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial \overline{x}^i}{\partial x^j} v^j. \tag{1.4.51}$$

Let us now find an expression for the generalized acceleration. Write equation (1.4.51) in the form

$$v^j = \overline{v}^i \frac{\partial x^j}{\partial \overline{x}^i} \tag{1.4.52}$$

and differentiate with respect to time to obtain

$$\frac{dv^j}{dt} = \overline{v}^i \frac{\partial^2 x^j}{\partial \overline{x}^i \partial \overline{x}^k}\frac{d\overline{x}^k}{dt} + \frac{d\overline{v}^i}{dt}\frac{\partial x^j}{\partial \overline{x}^i} \tag{1.4.53}$$

The equation (1.4.53) demonstrates that $\frac{dv^i}{dt}$ does not transform like a tensor. From the equation (1.4.7) previously derived, we change indices and write equation (1.4.53) in the form

$$\frac{dv^j}{dt} = \overline{v}^i \frac{d\overline{x}^k}{dt}\left[\overline{\begin{Bmatrix} \sigma \\ i\,k \end{Bmatrix}}\frac{\partial x^j}{\partial \overline{x}^\sigma} - \begin{Bmatrix} j \\ a\,c \end{Bmatrix}\frac{\partial x^a}{\partial \overline{x}^i}\frac{\partial x^c}{\partial \overline{x}^k}\right] + \frac{\partial x^j}{\partial \overline{x}^i}\frac{d\overline{v}^i}{dt}.$$

Rearranging terms we find

$$\frac{\partial v^j}{\partial x^k}\frac{dx^k}{dt} + \begin{Bmatrix} j \\ a\,c \end{Bmatrix}\left(\frac{\partial x^a}{\partial \overline{x}^i}\overline{v}^i\right)\left(\frac{\partial x^c}{\partial \overline{x}^k}\frac{d\overline{x}^k}{dt}\right) = \frac{\partial x^j}{\partial \overline{x}^i}\frac{\partial \overline{v}^i}{\partial \overline{x}^k}\frac{d\overline{x}^k}{dt} + \overline{\begin{Bmatrix} \sigma \\ i\,k \end{Bmatrix}}\overline{v}^i \frac{\partial x^j}{\partial \overline{x}^\sigma}\frac{d\overline{x}^k}{dt} \qquad \text{or}$$

$$\left[\frac{\partial v^j}{\partial x^k} + \begin{Bmatrix} j \\ a\,k \end{Bmatrix} v^a\right]\frac{dx^k}{dt} = \left[\frac{\partial \overline{v}^\sigma}{\partial \overline{x}^k} + \overline{\begin{Bmatrix} \sigma \\ i\,k \end{Bmatrix}}\overline{v}^i\right]\frac{d\overline{x}^k}{dt}\frac{\partial x^j}{\partial \overline{x}^\sigma}$$

$$\frac{\delta v^j}{\delta t} = \frac{\delta \overline{v}^\sigma}{\delta t}\frac{\partial x^j}{\partial \overline{x}^\sigma}.$$

The above equation illustrates that the intrinsic derivative of the velocity is a tensor quantity. This derivative is called the generalized acceleration and is denoted

$$f^i = \frac{\delta v^i}{\delta t} = v^i_{\ ,j}\frac{dx^j}{dt} = \frac{dv^i}{dt} + \begin{Bmatrix} i \\ m\,n \end{Bmatrix} v^m v^n = \frac{d^2 x^i}{dt^2} + \begin{Bmatrix} i \\ m\,n \end{Bmatrix}\frac{dx^m}{dt}\frac{dx^n}{dt}, \quad i = 1, \ldots, N \tag{1.4.54}$$

To summarize, we have shown that if

$x^i = x^i(t), \quad i = 1, \ldots, N$ is the generalized position vector, then

$v^i = \dfrac{dx^i}{dt}, \quad i = 1, \ldots, N$ is the generalized velocity, and

$f^i = \dfrac{\delta v^i}{\delta t} = v^i_{\ ,j}\dfrac{dx^j}{dt}, \quad i = 1, \ldots, N$ is the generalized acceleration.

■

Parallel Vector Fields

Let $y^i = y^i(t)$, $i = 1,2,3$ denote a space curve C in a Cartesian coordinate system and let Y^i define a constant vector in this system. Construct at each point of the curve C the vector Y^i. This produces a field of parallel vectors along the curve C. What happens to the curve and the field of parallel vectors when we transform to an arbitrary coordinate system using the transformation equations

$$y^i = y^i(x^1, x^2, x^3), \quad i = 1,2,3$$

with inverse transformation

$$x^i = x^i(y^1, y^2, y^3), \quad i = 1,2,3?$$

The space curve C in the new coordinates is obtained directly from the transformation equations and can be written

$$x^i = x^i(y^1(t), y^2(t), y^3(t)) = x^i(t), \quad i = 1,2,3.$$

The field of parallel vectors Y^i become X^i in the new coordinates where

$$Y^i = X^j \frac{\partial y^i}{\partial x^j}. \tag{1.4.55}$$

Since the components of Y^i are constants, their derivatives will be zero and consequently we obtain by differentiating the equation (1.4.55), with respect to the parameter t, that the field of parallel vectors X^i must satisfy the differential equation

$$\frac{dX^j}{dt}\frac{\partial y^i}{\partial x^j} + X^j \frac{\partial^2 y^i}{\partial x^j \partial x^m}\frac{dx^m}{dt} = \frac{dY^i}{dt} = 0. \tag{1.4.56}$$

Changing symbols in the equation (1.4.7) and setting the Christoffel symbol to zero in the Cartesian system of coordinates, we represent equation (1.4.7) in the form

$$\frac{\partial^2 y^i}{\partial x^j \partial x^m} = \begin{Bmatrix} \alpha \\ j\,m \end{Bmatrix} \frac{\partial y^i}{\partial x^\alpha}$$

and consequently, the equation (1.4.56) can be reduced to the form

$$\frac{\delta X^j}{\delta t} = \frac{dX^j}{dt} + \begin{Bmatrix} j \\ k\,m \end{Bmatrix} X^k \frac{dx^m}{dt} = 0. \tag{1.4.57}$$

The equation (1.4.57) is the differential equation which must be satisfied by a parallel field of vectors X^i along an arbitrary curve $x^i(t)$.

EXERCISE 1.4

▶ **1.** Find the nonzero Christoffel symbols of the first and second kind in cylindrical coordinates $(x^1, x^2, x^3) = (r, \theta, z)$, where $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$

▶ **2.** Find the nonzero Christoffel symbols of the first and second kind in spherical coordinates $(x^1, x^2, x^3) = (\rho, \theta, \phi)$, where $x = \rho\sin\theta\cos\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\theta.$

▶ **3.** Find the nonzero Christoffel symbols of the first and second kind in parabolic cylindrical coordinates $(x^1, x^2, x^3) = (\xi, \eta, z)$, where $x = \xi\eta, \quad y = \frac{1}{2}(\xi^2 - \eta^2), \quad z = z.$

▶ **4.** Find the nonzero Christoffel symbols of the first and second kind in parabolic coordinates $(x^1, x^2, x^3) = (\xi, \eta, \phi)$, where $x = \xi\eta\cos\phi, \quad y = \xi\eta\sin\phi, \quad z = \frac{1}{2}(\xi^2 - \eta^2).$

▶ **5.** Find the nonzero Christoffel symbols of the first and second kind in elliptic cylindrical coordinates $(x^1, x^2, x^3) = (\xi, \eta, z)$, where $x = \cosh\xi\cos\eta, \quad y = \sinh\xi\sin\eta, \quad z = z.$

▶ **6.** Find the nonzero Christoffel symbols of the first and second kind for the oblique cylindrical coordinates $(x^1, x^2, x^3) = (r, \phi, \eta)$, where $x = r\cos\phi, \quad y = r\sin\phi + \eta\cos\alpha, \quad z = \eta\sin\alpha$ with $0 < \alpha < \frac{\pi}{2}$ and α constant. Hint: See figure 1.3-18 and exercise 1.3, problem 12.

▶ **7.** Show $[ij, k] + [kj, i] = \dfrac{\partial g_{ik}}{\partial x^j}$.

▶ **8.**

(a) Let $\left\{ {r \atop s\,t} \right\} = g^{ri}[st, i]$ and solve for the Christoffel symbol of the first kind in terms of the Christoffel symbol of the second kind.

(b) Assume $[st, i] = g_{ni}\left\{ {n \atop s\,t} \right\}$ and solve for the Christoffel symbol of the second kind in terms of the Christoffel symbol of the first kind.

▶ **9.**

(a) Write down the transformation law satisfied by the fourth order tensor $\epsilon_{ijk,m}$.

(b) Show that $\epsilon_{ijk,m} = 0$ in all coordinate systems.

(c) Show that $(\sqrt{g})_{,k} = 0$.

▶ **10.** Show $\epsilon^{ijk}{}_{,m} = 0$.

▶ **11.** Calculate the second covariant derivative $A_{i\,,kj}$.

▶ **12.** The gradient of a scalar field $\phi(x^1, x^2, x^3)$ is the vector $\operatorname{grad}\phi = \vec{E}^i \dfrac{\partial\phi}{\partial x^i}$.

(a) Find the physical components associated with the covariant components $\phi_{\,,i}$

(b) Show the directional derivative of ϕ in a direction A^i is $\dfrac{d\phi}{dA} = \dfrac{A^i \phi_{,i}}{(g_{mn}A^m A^n)^{1/2}}$.

▶ **13.**

(a) Show $\sqrt{g}$ is a relative scalar of weight $+1$.

(b) Use the results from problem 9(c) and problem 44, Exercise 1.4, to show that
$$(\sqrt{g})_{,k} = \frac{\partial \sqrt{g}}{\partial x^k} - \left\{ {m \atop k\,m} \right\} \sqrt{g} = 0.$$

(c) Show that $\left\{ {m \atop k\,m} \right\} = \frac{\partial}{\partial x^k} \ln(\sqrt{g}) = \frac{1}{2g}\frac{\partial g}{\partial x^k}$.

▶ **14.** Use the result from problem 9(b) to show $\left\{ {m \atop k\,m} \right\} = \frac{\partial}{\partial x^k} \ln(\sqrt{g}) = \frac{1}{2g}\frac{\partial g}{\partial x^k}$.

Hint: Expand the covariant derivative $\epsilon_{rst,p}$ and then substitute $\epsilon_{rst} = \sqrt{g} e_{rst}$. Simplify by inner multiplication with $\frac{e^{rst}}{\sqrt{g}}$ and note the Exercise 1.1, problem 26.

▶ **15.** Calculate the covariant derivative $A^i_{\ ,m}$ and then contract on m and i to show that
$$A^i_{\ ,i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right).$$

▶ **16.** Show $\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\right) + \left\{ {i \atop p\,q} \right\} g^{pq} = 0$. Hint: See problem 14.

▶ **17.** Prove that the covariant derivative of a sum equals the sum of the covariant derivatives.

Hint: Assume $C_i = A_i + B_i$ and write out the covariant derivative for $C_{i,j}$.

▶ **18.** Let $C^i_j = A^i B_j$ and prove that the covariant derivative of a product equals the first term times the covariant derivative of the second term plus the second term times the covariant derivative of the first term.

▶ **19.** Start with the transformation law $\bar{A}_{ij} = A_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\beta}{\partial \bar{x}^j}$ and take an ordinary derivative of both sides with respect to $\bar{x}^k$ and hence derive the relation for $A_{ij,k}$ given in (1.4.30).

▶ **20.** Start with the transformation law $A^{ij} = \bar{A}^{\alpha\,\beta}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}$ and take an ordinary derivative of both sides with respect to x^k and hence derive the relation for $A^{ij}_{\ \ ,k}$ given in (1.4.30).

▶ **21.** Find the covariant derivatives of

$$(a)\quad A^{ijk} \qquad (b)\quad A^{ij}_{\ \ k} \qquad (c)\quad A^i_{jk} \qquad (d)\quad A_{ijk}$$

▶ **22.** Find the intrinsic derivative along the curve $x^i = x^i(t), \quad i = 1, \ldots, N$ for

$$(a)\quad A^{ijk} \qquad (b)\quad A^{ij}_{\ \ k} \qquad (c)\quad A^i_{jk} \qquad (d)\quad A_{ijk}$$

▶ **23.**

(a) Assume $\vec{A} = A^i \vec{E}_i$ and show that $d\vec{A} = A^i_{\ ,k}\, dx^k\, \vec{E}_i$.

(b) Assume $\vec{A} = A_i \vec{E}^i$ and show that $d\vec{A} = A_{i,k}\, dx^k\, \vec{E}^i$.

▶ **24.** (parallel vector field) Imagine a vector field $A^i = A^i(x^1, x^2, x^3)$ which is a function of position. Assume that at all points along a curve $x^i = x^i(t), i = 1, 2, 3$ the vector field points in the same direction, we would then have a parallel vector field or homogeneous vector field. Assume $\vec{A}$ is a constant, then $d\vec{A} = \frac{\partial \vec{A}}{\partial x^k}\, dx^k = 0$. Show that for a parallel vector field the condition $A_{i,k} = 0$ must be satisfied.

▶ **25.** Show that $\dfrac{\partial [ik, n]}{\partial x^j} = g_{n\sigma} \dfrac{\partial}{\partial x^j} \left\{ \begin{matrix} \sigma \\ i\,k \end{matrix} \right\} + ([nj, \sigma] + [\sigma j, n]) \left\{ \begin{matrix} \sigma \\ i\,k \end{matrix} \right\}$.

▶ **26.** Show $A_{r,s} - A_{s,r} = \dfrac{\partial A_r}{\partial x^s} - \dfrac{\partial A_s}{\partial x^r}$.

▶ **27.** In cylindrical coordinates you are given the contravariant vector components

$$A^1 = r \qquad A^2 = \cos\theta \qquad A^3 = z\sin\theta$$

(a) Find the physical components A_r, A_θ, and A_z.

(b) Denote the physical components of $A^i_{\ ,j}$, $i, j = 1, 2, 3$, by $\begin{matrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz}. \end{matrix}$

Find these physical components.

▶ **28.** Find the covariant form of the contravariant tensor $C^i = \epsilon^{ijk} A_{k,j}$.
Express your answer in terms of $A^k_{\ ,j}$.

▶ **29.** In Cartesian coordinates let x denote the magnitude of the position vector x_i. Show that (a) $x_{\ ,j} = \dfrac{1}{x} x_j$ (b) $x_{\ ,ij} = \dfrac{1}{x}\delta_{ij} - \dfrac{1}{x^3} x_i x_j$ (c) $x_{\ ,ii} = \dfrac{2}{x}$. (d) Let $U = \dfrac{1}{x}$, $x \neq 0$, and show that $U_{\ ,ij} = \dfrac{-\delta_{ij}}{x^3} + \dfrac{3x_i x_j}{x^5}$ and $U_{\ ,ii} = 0$.

▶ **30.** Consider a two dimensional space with element of arc length squared

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 \quad \text{and metric} \quad g_{ij} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

where u^1, u^2 are surface coordinates.
(a) Find formulas to calculate the Christoffel symbols of the first kind.
(b) Find formulas to calculate the Christoffel symbols of the second kind.

▶ **31.** Find the metric tensor and Christoffel symbols of the first and second kind associated with the two dimensional space describing points on a cylinder of radius a. Let $u^1 = \theta$ and $u^2 = z$ denote surface coordinates where

$$x = a\cos\theta = a\cos u^1$$
$$y = a\sin\theta = a\sin u^1$$
$$z = z = u^2$$

▶ **32.** Find the metric tensor and Christoffel symbols of the first and second kind associated with the two dimensional space describing points on a sphere of radius a. Let $u^1 = \theta$ and $u^2 = \phi$ denote surface coordinates where

$$x = a\sin\theta\cos\phi = a\sin u^1 \cos u^2$$
$$y = a\sin\theta\sin\phi = a\sin u^1 \sin u^2$$
$$z = a\cos\theta = a\cos u^1$$

▶ **33.** Find the metric tensor and Christoffel symbols of the first and second kind associated with the two dimensional space describing points on a torus having the parameters a and b and surface coordinates $u^1 = \xi, \quad u^2 = \eta$. illustrated in the figure 1.3-19. The points on the surface of the torus are given in terms of the surface coordinates by the equations

$$x = (a + b\cos\xi)\cos\eta$$
$$y = (a + b\cos\xi)\sin\eta$$
$$z = b\sin\xi$$

▶ **34.** Prove that $e_{ijk}a^m b^j c^k u^i_{\ ,m} + e_{ijk}a^i b^m c^k u^j_{\ ,m} + e_{ijk}a^i b^j c^m u^k_{\ ,m} = u^r_{\ ,r} e_{ijk} a^i b^j c^k$. Hint: See Exercise 1.3, problem 32 and Exercise 1.1, problem 21.

▶ **35.** Calculate the second covariant derivative $A^i_{\ ,jk}$.

▶ **36.** Show that $\sigma^{ij}_{\ ,j} = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^j}\left(\sqrt{g}\sigma^{ij}\right) + \sigma^{mn}\left\{ {i \atop m\, n} \right\}$

▶ **37.** Find the contravariant, covariant and physical components of velocity and acceleration in (a) Cartesian coordinates and (b) cylindrical coordinates.

▶ **38.** Find the contravariant, covariant and physical components of velocity and acceleration in spherical coordinates.

▶ **39.** In spherical coordinates (ρ, θ, ϕ) show that the acceleration components can be represented in terms of the velocity components as

$$f_\rho = \dot v_\rho - \frac{v_\theta^2 + v_\phi^2}{\rho}, \qquad f_\theta = \dot v_\theta + \frac{v_\rho v_\theta}{\rho} - \frac{v_\phi^2}{\rho\tan\theta}, \qquad f_\phi = \dot v_\phi + \frac{v_\rho v_\phi}{\rho} + \frac{v_\theta v_\phi}{\rho\tan\theta}$$

Hint: Calculate $\dot v_\rho, \dot v_\theta, \dot v_\phi$.

▶ **40.** The divergence of a vector A^i is $A^i_{\ ,i}$. That is, perform a contraction on the covariant derivative $A^i_{\ ,j}$ to obtain $A^i_{\ ,i}$. Calculate the divergence in (a) Cartesian coordinates (b) cylindrical coordinates and (c) spherical coordinates.

▶ **41.** If S is a scalar invariant of weight one and A^i_{jk} is a third order relative tensor of weight W, show that $S^{-W} A^i_{jk}$ is an absolute tensor.

▶ **42.** Let $\bar{Y}^i$,$i = 1, 2, 3$ denote the components of a field of parallel vectors along the curve $\overline{C}$ defined by the equations $\overline{y}^i = \bar{y}^i(t), i = 1, 2, 3$ in a space with metric tensor $\bar{g}_{ij}$, $i, j = 1, 2, 3$. Assume that $\bar{Y}^i$ and $\frac{d\bar{y}^i}{dt}$ are unit vectors such that at each point of the curve $\bar{C}$ we have

$$\bar{g}_{ij}\bar{Y}^i \frac{d\bar{y}^j}{dt} = \cos\theta = \text{Constant}.$$

(i.e. The field of parallel vectors makes a constant angle θ with the tangent to each point of the curve $\bar{C}$.) Show that if $\bar{Y}^i$ and $\bar{y}^i(t)$ undergo a transformation $x^i = x^i(\bar{y}^1, \bar{y}^2, \bar{y}^3)$, $i = 1, 2, 3$ then the transformed vector $X^m = \bar{Y}^i \frac{\partial x^m}{\partial \bar{y}^j}$ makes a constant angle with the tangent vector to the transformed curve C given by $x^i = x^i(\bar{y}^1(t), \bar{y}^2(t), \bar{y}^3(t))$.

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▶ **43.** Let J denote the Jacobian determinant $|\dfrac{\partial x^i}{\partial \overline{x}^j}|$. Differentiate J with respect to x^m and show that

$$\frac{\partial J}{\partial x^m} = J\overline{\left\{ \begin{matrix} \alpha \\ \alpha\, p \end{matrix} \right\}}\frac{\partial \overline{x}^p}{\partial x^m} - J\left\{ \begin{matrix} r \\ r\, m \end{matrix} \right\}.$$

Hint: See Exercise 1.1, problem 27 and (1.4.7).

▶ **44.** Assume that ϕ is a relative scalar of weight W so that $\overline{\phi} = J^W\,\phi$. Differentiate this relation with respect to $\overline{x}^k$. Use the result from problem 43 to obtain the transformation law:

$$\left[\frac{\partial\overline{\phi}}{\partial\overline{x}^k} - W\overline{\left\{ \begin{matrix} \alpha \\ \alpha\, k \end{matrix} \right\}}\overline{\phi}\right] = J^W\left[\frac{\partial\phi}{\partial x^m} - W\left\{ \begin{matrix} r \\ m\, r \end{matrix} \right\}\phi\right]\frac{\partial x^m}{\partial\overline{x}^k}.$$

The quantity inside the brackets is called the covariant derivative of a relative scalar of weight W. The covariant derivative of a relative scalar of weight W is defined as

$$\phi_{\,,k} = \frac{\partial\phi}{\partial x^k} - W\left\{ \begin{matrix} r \\ k\, r \end{matrix} \right\}\phi$$

and this definition has an extra term involving the weight.

It can be shown that similar results hold for relative tensors of weight W. For example, the covariant derivative of first and second order relative tensors of weight W have the forms

$$T^i_{\,,k} = \frac{\partial T^i}{\partial x^k} + \left\{ \begin{matrix} i \\ k\, m \end{matrix} \right\}T^m - W\left\{ \begin{matrix} r \\ k\, r \end{matrix} \right\}T^i$$

$$T^i_{j\,,k} = \frac{\partial T^i_j}{\partial x^k} + \left\{ \begin{matrix} i \\ k\, \sigma \end{matrix} \right\}T^\sigma_j - \left\{ \begin{matrix} \sigma \\ j\, k \end{matrix} \right\}T^i_\sigma - W\left\{ \begin{matrix} r \\ k\, r \end{matrix} \right\}T^i_j$$

When the weight term is zero these covariant derivatives reduce to the results given in our previous definitions.

▶ **45.** Let $\frac{dx^i}{dt} = v^i$ denote a generalized velocity and define the scalar function of kinetic energy T of a particle with mass m as

$$T = \frac{1}{2}m\,g_{ij}\,v^i\,v^j = \frac{1}{2}m\,g_{ij}\,\dot{x}^i\,\dot{x}^j.$$

Show that the intrinsic derivative of T is the same as an ordinary derivative of T. (i.e. Show that $\frac{\delta T}{\delta T} = \frac{dT}{dt}$.)

► **46.** Verify the relations

$$\frac{\partial g_{ij}}{\partial x^k} = -g_{mj}\, g_{ni}\, \frac{\partial g^{nm}}{\partial x^k}$$
$$\frac{\partial g^{in}}{\partial x^k} = -g^{mn}\, g^{ij}\, \frac{\partial g_{jm}}{\partial x^k}$$

► **47.** Assume that B^{ijk} is an absolute tensor. Is the quantity $T^{jk} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}B^{ijk}\right)$ a tensor? Justify your answer. If your answer is "no", explain your answer and determine if there any conditions you can impose upon B^{ijk} such that the above quantity will be a tensor?

► **48.** The e-permutation symbol can be used to define various vector products. Let A_i, B_i, C_i, D_i $i = 1, \dots, N$ denote vectors, then expand and verify the following products:

(a) In two dimensions

$$R = e_{ij}A_iB_j \quad \text{a scalar determinant.}$$
$$R_i = e_{ij}A_j \quad \text{a vector (rotation).}$$

(b) In three dimensions

$$S = e_{ijk}A_iB_jC_k \quad \text{a scalar determinant.}$$
$$S_i = e_{ijk}B_jC_k \quad \text{a vector cross product.}$$
$$S_{ij} = e_{ijk}C_k \quad \text{a skew-symmetric matrix}$$

(c) In four dimensions

$$T = e_{ijkm}A_iB_jC_kD_m \quad \text{a scalar determinant.}$$
$$T_i = e_{ijkm}B_jC_kD_m \quad \text{4-dimensional cross product.}$$
$$T_{ij} = e_{ijkm}C_kD_m \quad \text{skew-symmetric matrix.}$$
$$T_{ijk} = e_{ikm}D_m \quad \text{skew-symmetric tensor.}$$

with similar products in higher dimensions.

► **49.** Expand the curl operator for:

(a) Two dimensions $B = e_{ij}A_{j,i}$

(b) Three dimensions $B_i = e_{ijk}A_{k,j}$

(c) Four dimensions $B_{ij} = e_{ijkm}A_{m,k}$