PART 2: INTRODUCTION TO CONTINUUM MECHANICS

In the following sections we develop some applications of tensor calculus in the areas of dynamics, elasticity, fluids and electricity and magnetism. We begin by first developing generalized expressions for the vector operations of gradient, divergence, and curl. Also generalized expressions for other vector operators are considered in order that tensor equations can be converted to vector equations. We construct a table to aid in the translating of generalized tensor equations to vector form and vice versa.

The basic equations of continuum mechanics are developed in the later sections. These equations are developed in both Cartesian and generalized tensor form and then converted to vector form.

§2.1 TENSOR NOTATION FOR SCALAR AND VECTOR QUANTITIES

We consider the tensor representation of some vector expressions. Our goal is to develop the ability to convert vector equations to tensor form as well as being able to represent tensor equations in vector form. In this section the basic equations of continuum mechanics are represented using both a vector notation and the indicial notation which focuses attention on the tensor components. In order to move back and forth between these notations, the representation of vector quantities in tensor form is now considered.

Gradient

For $\Phi = \Phi(x^1, x^2, \dots, x^N)$ a scalar function of the coordinates $x^i, i = 1, \dots, N$, the gradient of Φ is defined as the covariant vector

$$\Phi_{,i} = \frac{\partial \Phi}{\partial x^i}, \quad i = 1, \dots, N. \tag{2.1.1}$$

The contravariant form of the gradient is

$$g^{im}\Phi_{,m}. \tag{2.1.2}$$

Note, if $C^i = g^{im}\Phi_{,m},\ i = 1,2,3$ are the tensor components of the gradient then in an orthogonal coordinate system we will have

$$C^1 = g^{11}\Phi_{,1}, \qquad C^2 = g^{22}\Phi_{,2}, \qquad C^3 = g^{33}\Phi_{,3}.$$

We note that in an orthogonal coordinate system that $g^{ii} = 1/h_i^2$, (no sum on i), $i = 1,2,3$ and hence replacing the tensor components by their equivalent physical components there results the equations

$$\frac{C(1)}{h_1} = \frac{1}{h_1^2}\frac{\partial \Phi}{\partial x^1}, \qquad \frac{C(2)}{h_2} = \frac{1}{h_2^2}\frac{\partial \Phi}{\partial x^2}, \qquad \frac{C(3)}{h_3} = \frac{1}{h_3^2}\frac{\partial \Phi}{\partial x^3}.$$

Simplifying, we find the physical components of the gradient are

$$C(1) = \frac{1}{h_1}\frac{\partial \Phi}{\partial x^1}, \qquad C(2) = \frac{1}{h_2}\frac{\partial \Phi}{\partial x^2}, \qquad C(3) = \frac{1}{h_3}\frac{\partial \Phi}{\partial x^3}.$$

These results are only valid when the coordinate system is orthogonal and $g_{ij} = 0$ for $i \neq j$ and $g_{ii} = h_i^2$, with $i = 1,2,3$, and where i is not summed.

Divergence

The divergence of a contravariant tensor A^r is obtained by taking the covariant derivative with respect to x^k and then performing a contraction. This produces

$$\text{div } A^r = A^r{}_{,r}. \tag{2.1.3}$$

Still another form for the divergence is obtained by simplifying the expression (2.1.3). The covariant derivative can be represented

$$A^r{}_{,k} = \frac{\partial A^r}{\partial x^k} + \left\{ {r \atop m\, k} \right\} A^m.$$

Upon contracting the indices r and k and using the result from Exercise 1.4, problem 13, we obtain

$$\begin{aligned} A^r{}_{,r} &= \frac{\partial A^r}{\partial x^r} + \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g})}{\partial x^m} A^m \\ A^r{}_{,r} &= \frac{1}{\sqrt{g}} \left(\sqrt{g} \frac{\partial A^r}{\partial x^r} + A^r \frac{\partial \sqrt{g}}{\partial x^r} \right) \\ A^r{}_{,r} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} \left(\sqrt{g} A^r \right). \end{aligned} \tag{2.1.4}$$

EXAMPLE 2.1-1. (Divergence) Find the representation of the divergence of a vector A^r in spherical coordinates (ρ, θ, ϕ). **Solution:** In spherical coordinates we have

$$\begin{gathered} x^1 = \rho, \quad x^2 = \theta, \quad x^3 = \phi \quad \text{with} \quad g_{ij} = 0 \quad \text{for} \quad i \neq j \quad \text{and} \\ g_{11} = h_1^2 = 1, \qquad g_{22} = h_2^2 = \rho^2, \qquad g_{33} = h_3^2 = \rho^2 \sin^2 \theta. \end{gathered}$$

The determinant of g_{ij} is $g = |g_{ij}| = \rho^4 \sin^2 \theta$ and $\sqrt{g} = \rho^2 \sin \theta$. Employing the relation (2.1.4) we find

$$\text{div } A^r = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial x^1} (\sqrt{g} A^1) + \frac{\partial}{\partial x^2} (\sqrt{g} A^2) + \frac{\partial}{\partial x^3} (\sqrt{g} A^3) \right].$$

In terms of the physical components this equation becomes

$$\text{div } A^r = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial \rho} (\sqrt{g} \frac{A(1)}{h_1}) + \frac{\partial}{\partial \theta} (\sqrt{g} \frac{A(2)}{h_2}) + \frac{\partial}{\partial \phi} (\sqrt{g} \frac{A(3)}{h_3}) \right].$$

By using the notation

$$A(1) = A_\rho, \qquad A(2) = A_\theta, \qquad A(3) = A_\phi$$

for the physical components, the divergence can be expressed in either of the forms:

$$\begin{aligned} \text{div } A^r &= \frac{1}{\rho^2 \sin \theta} \left[\frac{\partial}{\partial \rho} (\rho^2 \sin \theta A_\rho) + \frac{\partial}{\partial \theta} (\rho^2 \sin \theta \frac{A_\theta}{\rho}) + \frac{\partial}{\partial \phi} (\rho^2 \sin \theta \frac{A_\phi}{\rho \sin \theta}) \right] \quad \text{or} \\ \text{div } A^r &= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 A_\rho) + \frac{1}{\rho \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{\rho \sin \theta} \frac{\partial A_\phi}{\partial \phi}. \end{aligned}$$

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Curl

The contravariant components of the vector $\vec{C} = \text{curl}\ \vec{A}$ are represented

$$C^i = \epsilon^{ijk} A_{k,j}. \tag{2.1.5}$$

In expanded form this representation becomes:

$$\begin{aligned} C^1 &= \frac{1}{\sqrt{g}}\left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) \\ C^2 &= \frac{1}{\sqrt{g}}\left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right) \\ C^3 &= \frac{1}{\sqrt{g}}\left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right). \end{aligned} \tag{2.1.6}$$

EXAMPLE 2.1-2. (Curl) Find the representation for the components of curl $\vec{A}$ in spherical coordinates (ρ, θ, ϕ).

Solution: In spherical coordinates we have :$x^1 = \rho, \quad x^2 = \theta, \quad x^3 = \phi$ with $g_{ij} = 0$ for $i \neq j$ and

$$g_{11} = h_1^2 = 1, \qquad g_{22} = h_2^2 = \rho^2, \qquad g_{33} = h_3^2 = \rho^2 \sin^2\theta.$$

The determinant of g_{ij} is $g = |g_{ij}| = \rho^4 \sin^2\theta$ with $\sqrt{g} = \rho^2 \sin\theta$. The relations (2.1.6) are tensor equations representing the components of the vector curl $\vec{A}$. To find the components of curl $\vec{A}$ in spherical components we write the equations (2.1.6) in terms of their physical components. These equations take on the form:

$$\begin{aligned} \frac{C(1)}{h_1} &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\theta}(h_3 A(3)) - \frac{\partial}{\partial\phi}(h_2 A(2))\right] \\ \frac{C(2)}{h_2} &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\phi}(h_1 A(1)) - \frac{\partial}{\partial\rho}(h_3 A(3))\right] \\ \frac{C(3)}{h_3} &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}(h_2 A(2)) - \frac{\partial}{\partial\theta}(h_1 A(1))\right]. \end{aligned} \tag{2.1.7}$$

We employ the notations

$$C(1) = C_\rho, \quad C(2) = C_\theta, \quad C(3) = C_\phi, \quad A(1) = A_\rho, \quad A(2) = A_\theta, \quad A(3) = A_\phi$$

to denote the physical components, and find the components of the vector curl $\vec{A}$, in spherical coordinates, are expressible in the form:

$$\begin{aligned} C_\rho &= \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(\rho\sin\theta A_\phi) - \frac{\partial}{\partial\phi}(\rho A_\theta)\right] \\ C_\theta &= \frac{1}{\rho\sin\theta}\left[\frac{\partial}{\partial\phi}(A_\rho) - \frac{\partial}{\partial\rho}(\rho\sin\theta A_\phi)\right] \\ C_\phi &= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\theta) - \frac{\partial}{\partial\theta}(A_\rho)\right]. \end{aligned} \tag{2.1.8}$$

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Laplacian

The Laplacian $\nabla^2 U$ has the contravariant form

$$\nabla^2 U = g^{ij} U_{,ij} = (g^{ij} U_{,i})_{,j} = \left(g^{ij} \frac{\partial U}{\partial x^i} \right)_{,j}. \tag{2.1.9}$$

Expanding this expression produces the equations:

$$\begin{aligned}
\nabla^2 U &= \frac{\partial}{\partial x^j} \left(g^{ij} \frac{\partial U}{\partial x^i} \right) + g^{im} \frac{\partial U}{\partial x^i} \left\{ \begin{matrix} j \\ m\,j \end{matrix} \right\} \\
\nabla^2 U &= \frac{\partial}{\partial x^j} \left(g^{ij} \frac{\partial U}{\partial x^i} \right) + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} g^{ij} \frac{\partial U}{\partial x^i} \\
\nabla^2 U &= \frac{1}{\sqrt{g}} \left[\sqrt{g} \frac{\partial}{\partial x^j} \left(g^{ij} \frac{\partial U}{\partial x^i} \right) + g^{ij} \frac{\partial U}{\partial x^i} \frac{\partial \sqrt{g}}{\partial x^j} \right] \\
\nabla^2 U &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial U}{\partial x^i} \right).
\end{aligned} \tag{2.1.10}$$

In orthogonal coordinates we have $g^{ij} = 0$ for $i \neq j$ and

$$g_{11} = h_1^2, \qquad g_{22} = h_2^2, \qquad g_{33} = h_3^2$$

and so (2.1.10) when expanded reduces to the form

$$\nabla^2 U = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x^1} \left(\frac{h_2 h_3}{h_1} \frac{\partial U}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(\frac{h_1 h_3}{h_2} \frac{\partial U}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left(\frac{h_1 h_2}{h_3} \frac{\partial U}{\partial x^3} \right) \right]. \tag{2.1.11}$$

This representation is only valid in an orthogonal system of coordinates.

EXAMPLE 2.1-3. (Laplacian) Find the Laplacian in spherical coordinates.

Solution: Utilizing the results given in the previous example we find the Laplacian in spherical coordinates has the form

$$\nabla^2 U = \frac{1}{\rho^2 \sin\theta} \left[\frac{\partial}{\partial \rho} \left(\rho^2 \sin\theta \frac{\partial U}{\partial \rho} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin\theta} \frac{\partial U}{\partial \phi} \right) \right]. \tag{2.1.12}$$

This simplifies to

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial U}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{\rho^2} \frac{\partial U}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta} \frac{\partial^2 U}{\partial \phi^2}. \tag{2.1.13}$$

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The table 1 gives the vector and tensor representation for various quantities of interest.

VECTOR	GENERAL TENSOR	CARTESIAN TENSOR
$\vec{A}$	A^i or A_i	A_i
$\vec{A}\cdot\vec{B}$	$A^iB_i = g_{ij}A^iB^j = A_iB^i$ $A^iB_i = g^{ij}A_iB_j$	A_iB_i
$\vec{C}=\vec{A}\times\vec{B}$	$C^i = \frac{1}{\sqrt{g}}e^{ijk}A_jB_k$	$C_i = e_{ijk}A_jB_k$
$\nabla\,\Phi = \text{grad}\,\Phi$	$g^{im}\Phi_{,m}$	$\Phi_{,i} = \frac{\partial\Phi}{\partial x^i}$
$\nabla\cdot\vec{A} = \text{div}\,\vec{A}$	$g^{mn}A_{m,n} = A^r{}_{,r} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}\left(\sqrt{g}A^r\right)$	$A_{i,i} = \frac{\partial A_i}{\partial x^i}$
$\nabla\times\vec{A} = \vec{C} = \text{curl}\;\vec{A}$	$C^i = \epsilon^{ijk}A_{k,j}$	$C_i = e_{ijk}\frac{\partial A_k}{\partial x^j}$
$\nabla^2 U$	$g^{mn}U_{,mn} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\frac{\partial U}{\partial x^i}\right)$	$\frac{\partial}{\partial x^i}\left(\frac{\partial U}{\partial x^i}\right)$
$\vec{C} = (\vec{A}\cdot\nabla)\vec{B}$	$C^i = A^mB^i{}_{,m}$	$C_i = A_m\frac{\partial B_i}{\partial x^m}$
$\vec{C} = \vec{A}(\nabla\cdot\vec{B})$	$C^i = A^iB^j{}_{,j}$	$C_i = A_i\frac{\partial B_m}{\partial x^m}$
$\vec{C} = \nabla^2\vec{A}$	$C^i = g^{jm}A^i{}_{,mj}$ or $C_i = g^{jm}A_{i,mj}$	$C_i = \frac{\partial}{\partial x^m}\left(\frac{\partial A_i}{\partial x^m}\right)$
$\left(\vec{A}\cdot\nabla\right)\phi$	$g^{im}A^i\phi_{,m}$	$A_i\phi_{,i}$
$\nabla\left(\nabla\cdot\vec{A}\right)$	$g^{im}\left(A^r{}_{,r}\right)_{,m}$	$\frac{\partial^2 A_r}{\partial x_i\partial x_r}$
$\nabla\times\left(\nabla\times\vec{A}\right)$	$\epsilon_{ijk}g^{jm}\left(\epsilon^{kst}A_{t,s}\right)_{,m}$	$\frac{\partial^2 A_j}{\partial x_j\partial x_i} - \frac{\partial^2 A_i}{\partial x_j\partial x_j}$

Table 1 Vector and tensor representations.

EXAMPLE 2.1-4. (Maxwell's equations) In the study of electrodynamics there arises the following vectors and scalars:

$$\begin{aligned}
\vec{E} =&\text{Electric force vector, } [\vec{E}] = \text{Newton/coulomb}\\
\vec{B} =&\text{Magnetic force vector, } [\vec{B}] = \text{Weber/m}^2\\
\vec{D} =&\text{Displacement vector, } [\vec{D}] = \text{coulomb/m}^2\\
\vec{H} =&\text{Auxilary magnetic force vector, } [\vec{H}] = \text{ampere/m}\\
\vec{J} =&\text{Free current density, } [\vec{J}] = \text{ampere/m}^2\\
\varrho =&\text{free charge density, } [\varrho] = \text{coulomb/m}^3
\end{aligned}$$

The above quantities arise in the representation of the following laws:

Faraday's Law This law states the line integral of the electromagnetic force around a loop is proportional to the rate of flux of magnetic induction through the loop. This gives rise to the first electromagnetic field equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{or} \qquad \epsilon^{ijk} E_{k,j} = -\frac{\partial B^i}{\partial t}. \tag{2.1.15}$$

Ampere's Law This law states the line integral of the magnetic force vector around a closed loop is proportional to the sum of the current through the loop and the rate of flux of the displacement vector through the loop. This produces the second electromagnetic field equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{or} \qquad \epsilon^{ijk} H_{k,j} = J^i + \frac{\partial D^i}{\partial t}. \tag{2.1.16}$$

Gauss's Law for Electricity This law states that the flux of the electric force vector through a closed surface is proportional to the total charge enclosed by the surface. This results in the third electromagnetic field equation:

$$\nabla \cdot \vec{D} = \varrho \qquad \text{or} \qquad \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} D^i\right) = \varrho. \tag{2.1.17}$$

Gauss's Law for Magnetism This law states the magnetic flux through any closed volume is zero. This produces the fourth electromagnetic field equation:

$$\nabla \cdot \vec{B} = 0 \qquad \text{or} \qquad \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} B^i\right) = 0. \tag{2.1.18}$$

The four electromagnetic field equations are referred to as Maxwell's equations. These equations arise in the study of electrodynamics and can be represented in other forms. These other forms will depend upon such things as the material assumptions and units of measurements used. Note that the tensor equations (2.1.15) through (2.1.18) are representations of Maxwell's equations in a form which is independent of the coordinate system chosen.

In applications, the tensor quantities must be expressed in terms of their physical components. In a general orthogonal curvilinear coordinate system we will have

$$g_{11} = h_1^2, \quad g_{22} = h_2^2, \quad g_{33} = h_3^2, \quad \text{and} \quad g_{ij} = 0 \quad \text{for} \quad i \neq j.$$

This produces the result $\sqrt{g} = h_1 h_2 h_3$. Further, if we represent the physical components of

$$D_i, B_i, E_i, H_i \quad \text{by} \quad D(i), B(i), E(i), \text{ and } H(i)$$

the Maxwell equations can be represented by the equations in table 2. The tables 3, 4 and 5 are the representation of Maxwell's equations in rectangular, cylindrical, and spherical coordinates. These latter tables are special cases associated with the more general table 2.

$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^2}(h_3E(3)) - \frac{\partial}{\partial x^3}(h_2E(2))\right] = -\frac{1}{h_1}\frac{\partial B(1)}{\partial t}$$
$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^3}(h_1E(1)) - \frac{\partial}{\partial x^1}(h_3E(3))\right] = -\frac{1}{h_2}\frac{\partial B(2)}{\partial t}$$
$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(h_2E(2)) - \frac{\partial}{\partial x^2}(h_1E(1))\right] = -\frac{1}{h_3}\frac{\partial B(3)}{\partial t}$$

$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^2}(h_3H(3)) - \frac{\partial}{\partial x^3}(h_2H(2))\right] = \frac{J(1)}{h_1} + \frac{1}{h_1}\frac{\partial D(1)}{\partial t}$$
$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^3}(h_1H(1)) - \frac{\partial}{\partial x^1}(h_3H(3))\right] = \frac{J(2)}{h_2} + \frac{1}{h_2}\frac{\partial D(2)}{\partial t}$$
$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(h_2H(2)) - \frac{\partial}{\partial x^2}(h_1H(1))\right] = \frac{J(3)}{h_3} + \frac{1}{h_3}\frac{\partial D(3)}{\partial t}$$

$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(h_1h_2h_3\frac{D(1)}{h_1}\right) + \frac{\partial}{\partial x^2}\left(h_1h_2h_3\frac{D(2)}{h_2}\right) + \frac{\partial}{\partial x^3}\left(h_1h_2h_3\frac{D(3)}{h_3}\right)\right] = \varrho$$

$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(h_1h_2h_3\frac{B(1)}{h_1}\right) + \frac{\partial}{\partial x^2}\left(h_1h_2h_3\frac{B(2)}{h_2}\right) + \frac{\partial}{\partial x^3}\left(h_1h_2h_3\frac{B(3)}{h_3}\right)\right] = 0$$

Table 2 Maxwell's equations in generalized orthogonal coordinates.

Note that all the tensor components have been replaced by their physical components.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \qquad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \varrho$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t} \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

Here we have introduced the notations:

$$D_x = D(1) \quad B_x = B(1) \quad H_x = H(1) \quad J_x = J(1) \quad E_x = E(1)$$
$$D_y = D(2) \quad B_y = B(2) \quad H_y = H(2) \quad J_y = J(2) \quad E_y = E(2)$$
$$D_z = D(3) \quad B_z = B(3) \quad H_z = H(3) \quad J_z = J(3) \quad E_z = E(3)$$

with $x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad h_1 = h_2 = h_3 = 1$

Table 3 Maxwell's equations Cartesian coordinates

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\frac{\partial B_r}{\partial t} \qquad \frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = J_r + \frac{\partial D_r}{\partial t}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t} \qquad \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = J_\theta + \frac{\partial D_\theta}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -\frac{\partial B_z}{\partial t} \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = J_z + \frac{\partial D_z}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\theta}{\partial \theta} + \frac{\partial D_z}{\partial z} = \varrho \qquad \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

Here we have introduced the notations:

$$D_r = D(1) \quad B_r = B(1) \quad H_r = H(1) \quad J_r = J(1) \quad E_r = E(1)$$
$$D_\theta = D(2) \quad B_\theta = B(2) \quad H_\theta = H(2) \quad J_\theta = J(2) \quad E_\theta = E(2)$$
$$D_z = D(3) \quad B_z = B(3) \quad H_z = H(3) \quad J_z = J(3) \quad E_z = E(3)$$

with $x^1 = r, \quad x^2 = \theta, \quad x^3 = z, \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1.$

Table 4 Maxwell's equations in cylindrical coordinates.

$$\frac{1}{\rho\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi)-\frac{\partial E_\theta}{\partial\phi}\right]=-\frac{\partial B_\rho}{\partial t} \qquad \frac{1}{\rho\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta H_\phi)-\frac{\partial H_\theta}{\partial\phi}\right]=J_\rho+\frac{\partial D_\rho}{\partial t}$$

$$\frac{1}{\rho\sin\theta}\frac{\partial E_\rho}{\partial\phi}-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi)=-\frac{\partial B_\theta}{\partial t} \qquad \frac{1}{\rho\sin\theta}\frac{\partial H_\rho}{\partial\phi}-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)=J_\theta+\frac{\partial D_\theta}{\partial t}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\theta)-\frac{1}{\rho}\frac{\partial E_\rho}{\partial\theta}=-\frac{\partial B_\phi}{\partial t} \qquad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\theta)-\frac{1}{\rho}\frac{\partial H_\rho}{\partial\theta}=J_\phi+\frac{\partial D_\phi}{\partial t}$$

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 D_\rho)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta)+\frac{1}{\rho\sin\theta}\frac{\partial D_\phi}{\partial\phi}=\varrho$$

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 B_\rho)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta)+\frac{1}{\rho\sin\theta}\frac{\partial B_\phi}{\partial\phi}=0$$

Here we have introduced the notations:

$$\begin{array}{lllll} D_\rho=D(1) & B_\rho=B(1) & H_\rho=H(1) & J_\rho=J(1) & E_\rho=E(1)\\ D_\theta=D(2) & B_\theta=B(2) & H_\theta=H(2) & J_\theta=J(2) & E_\theta=E(2)\\ D_\phi=D(3) & B_\phi=B(3) & H_\phi=H(3) & J_\phi=J(3) & E_\phi=E(3)\end{array}$$

with $x^1=\rho,\quad x^2=\theta,\quad x^3=\phi,\quad h_1=1,\quad h_2=\rho,\quad h_3=\rho\sin\theta$

Table 5 Maxwell's equations spherical coordinates.

■

Eigenvalues and Eigenvectors of Symmetric Tensors

Consider the equation

$$T_{ij}A_j=\lambda A_i,\quad i,j=1,2,3, \tag{2.1.19}$$

where $T_{ij}=T_{ji}$ is symmetric, A_i are the components of a vector and λ is a scalar. Any nonzero solution A_i of equation (2.1.19) is called an eigenvector of the tensor T_{ij} and the associated scalar λ is called an eigenvalue. When expanded these equations have the form

$$\begin{aligned}(T_{11}-\lambda)A_1+\quad T_{12}A_2+\quad T_{13}A_3&=0\\ T_{21}A_1+(T_{22}-\lambda)A_2+\quad T_{23}A_3&=0\\ T_{31}A_1+\quad T_{32}A_2+(T_{33}-\lambda)A_3&=0.\end{aligned}$$

The condition for equation (2.1.19) to have a nonzero solution A_i is that the characteristic equation should be zero. This equation is found from the determinant equation

$$f(\lambda)=\begin{vmatrix}T_{11}-\lambda & T_{12} & T_{13}\\ T_{21} & T_{22}-\lambda & T_{23}\\ T_{31} & T_{32} & T_{33}-\lambda\end{vmatrix}=0, \tag{2.1.20}$$

which when expanded is a cubic equation of the form

$$f(\lambda) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0, \tag{2.1.21}$$

where I_1, I_2 and I_3 are invariants defined by the relations

$$\begin{aligned} I_1 &= T_{ii} \\ I_2 &= \frac{1}{2}T_{ii}T_{jj} - \frac{1}{2}T_{ij}T_{ij} \\ I_3 &= e_{ijk}T_{i1}T_{j2}T_{k3}. \end{aligned} \tag{2.1.22}$$

When T_{ij} is subjected to an orthogonal transformation, where $\bar{T}_{mn} = T_{ij}\ell_{im}\ell_{jn}$, then

$$\ell_{im}\ell_{jn}\left(T_{mn} - \lambda\,\delta_{mn}\right) = \bar{T}_{ij} - \lambda\,\delta_{ij} \quad \text{and} \quad \det\left(T_{mn} - \lambda\,\delta_{mn}\right) = \det\left(\bar{T}_{ij} - \lambda\,\delta_{ij}\right).$$

Hence, the eigenvalues of a second order tensor remain invariant under an orthogonal transformation.

If T_{ij} is real and symmetric then

- the eigenvalues of T_{ij} will be real, and
- the eigenvectors corresponding to distinct eigenvalues will be orthogonal.

Proof: To show a quantity is real we show that the conjugate of the quantity equals the given quantity. If (2.1.19) is satisfied, we multiply by the conjugate $\overline{A}_i$ and obtain

$$\overline{A}_i T_{ij} A_j = \lambda A_i \overline{A}_i. \tag{2.1.25}$$

The right hand side of this equation has the inner product $A_i\overline{A}_i$ which is real. It remains to show the left hand side of equation (2.1.25) is also real. Consider the conjugate of this left hand side and write

$$\overline{\overline{A}_i T_{ij} A_j} = A_i \overline{T}_{ij} \overline{A}_j = A_i T_{ji} \overline{A}_j = \overline{A}_i T_{ij} A_j.$$

Consequently, the left hand side of equation (2.1.25) is real and the eigenvalue λ can be represented as the ratio of two real quantities.

Assume that $\lambda_{(1)}$ and $\lambda_{(2)}$ are two distinct eigenvalues which produce the unit eigenvectors $\hat{L}_1$ and $\hat{L}_2$ with components ℓ_{i1} and $\ell_{i2}, i = 1, 2, 3$ respectively. We then have

$$T_{ij}\ell_{j1} = \lambda_{(1)}\ell_{i1} \qquad \text{and} \qquad T_{ij}\ell_{j2} = \lambda_{(2)}\ell_{i2}. \tag{2.1.26}$$

Consider the products

$$\begin{aligned} \lambda_{(1)}\ell_{i1}\ell_{i2} &= T_{ij}\ell_{j1}\ell_{i2}, \\ \lambda_{(2)}\ell_{i1}\ell_{i2} &= \ell_{i1}T_{ij}\ell_{j2} = \ell_{j1}T_{ji}\ell_{i2}. \end{aligned} \tag{2.1.27}$$

and subtract these equations. We find that

$$[\lambda_{(1)} - \lambda_{(2)}]\ell_{i1}\ell_{i2} = 0. \tag{2.1.28}$$

By hypothesis, $\lambda_{(1)}$ is different from $\lambda_{(2)}$ and consequently the inner product $\ell_{i1}\ell_{i2}$ must be zero. Therefore, the eigenvectors corresponding to distinct eigenvalues are orthogonal.

Therefore, associated with distinct eigenvalues $\lambda_{(i)}, i = 1, 2, 3$ there are unit eigenvectors

$$\hat{L}_{(i)} = \ell_{i1}\,\hat{\mathbf{e}}_1 + \ell_{i2}\,\hat{\mathbf{e}}_2 + \ell_{i3}\,\hat{\mathbf{e}}_3$$

with components $\ell_{im}, m = 1, 2, 3$ which are direction cosines and satisfy

$$\ell_{in}\ell_{im} = \delta_{mn} \qquad \text{and} \qquad \ell_{ij}\ell_{jm} = \delta_{im}. \tag{2.1.23}$$

The unit eigenvectors satisfy the relations

$$T_{ij}\ell_{j1} = \lambda_{(1)}\ell_{i1} \qquad T_{ij}\ell_{j2} = \lambda_{(2)}\ell_{i2} \qquad T_{ij}\ell_{j3} = \lambda_{(3)}\ell_{i3}$$

and can be written as the single equation

$$T_{ij}\ell_{jm} = \lambda_{(m)}\ell_{im}, \quad m = 1, 2, \mathit{or}\ 3 \quad \text{m not summed.}$$

Consider the transformation

$$\overline{x}_i = \ell_{ij}x_j \qquad \text{or} \qquad x_m = \ell_{mj}\overline{x}_j$$

which represents a rotation of axes, where ℓ_{ij} are the direction cosines from the eigenvectors of T_{ij}. This is a linear transformation where the ℓ_{ij} satisfy equation (2.1.23). Such a transformation is called an orthogonal transformation. In the new $\overline{x}$ coordinate system, called principal axes, we have

$$\overline{T}_{mn} = T_{ij}\frac{\partial x^i}{\partial \overline{x}^m}\frac{\partial x^j}{\partial \overline{x}^n} = T_{ij}\ell_{im}\ell_{jn} = \lambda_{(n)}\ell_{in}\ell_{im} = \lambda_{(n)}\delta_{mn} \quad \text{(no sum on } n\text{).} \tag{2.1.24}$$

This equation shows that in the barred coordinate system there are the components

$$\left(\overline{T}_{mn}\right) = \begin{bmatrix} \lambda_{(1)} & 0 & 0 \\ 0 & \lambda_{(2)} & 0 \\ 0 & 0 & \lambda_{(3)} \end{bmatrix}.$$

That is, along the principal axes the tensor components T_{ij} are transformed to the components $\overline{T}_{ij}$ where $\overline{T}_{ij} = 0$ for $i \neq j$. The elements $\overline{T}_{(i)(i)}$, i not summed, represent the eigenvalues of the transformation (2.1.19).

EXERCISE 2.1

► **1.** In cylindrical coordinates (r, θ, z) with $f = f(r, \theta, z)$ find the gradient of f.

► **2.** In cylindrical coordinates (r, θ, z) with $\vec{A} = \vec{A}(r, \theta, z)$ find div $\vec{A}$.

► **3.** In cylindrical coordinates (r, θ, z) for $\vec{A} = \vec{A}(r, \theta, z)$ find curl $\vec{A}$.

► **4.** In cylindrical coordinates (r, θ, z) for $f = f(r, \theta, z)$ find $\nabla^2 f$.

► **5.** In spherical coordinates (ρ, θ, ϕ) with $f = f(\rho, \theta, \phi)$ find the gradient of f.

► **6.** In spherical coordinates (ρ, θ, ϕ) with $\vec{A} = \vec{A}(\rho, \theta, \phi)$ find div $\vec{A}$.

► **7.** In spherical coordinates (ρ, θ, ϕ) for $\vec{A} = \vec{A}(\rho, \theta, \phi)$ find curl $\vec{A}$.

► **8.** In spherical coordinates (ρ, θ, ϕ) for $f = f(\rho, \theta, \phi)$ find $\nabla^2 f$.

► **9.** Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector of a variable point (x, y, z) in Cartesian coordinates. Let $r = |\vec{r}|$ denote the distance of this point from the origin. Find in terms of $\vec{r}$ and r:

$$(a) \quad \text{grad}\,(r) \qquad (b) \quad \text{grad}\,(r^m) \qquad (c) \quad \text{grad}\,(\frac{1}{r}) \qquad (d) \quad \text{grad}\,(\ln r) \qquad (e) \quad \text{grad}\,(\phi)$$

where $\phi = \phi(r)$ is an arbitrary function of r.

► **10.** Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector of a variable point (x, y, z) in Cartesian coordinates. Let $r = |\vec{r}|$ denote the distance of this point from the origin. Find:

$$(a) \quad \text{div}\ (\vec{r}) \quad (b) \quad \text{div}\ (r^m \vec{r}) \quad (c) \quad \text{div}\ (r^{-3}\vec{r}) \quad (d) \quad \text{div}\ (\phi\,\vec{r})$$

where $\phi = \phi(r)$ is an arbitrary function or r.

► **11.** Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector of a variable point (x, y, z) in Cartesian coordinates. Let $r = |\vec{r}|$ denote the distance of this point from the origin. Find: (a) curl $\vec{r}$ (b) curl $(\phi\,\vec{r})$ where $\phi = \phi(r)$ is an arbitrary function of r.

► **12.** Expand and simplify the representation for curl (curl $\vec{A}$).

► **13.** Show that the curl of the gradient is zero in generalized coordinates.

► **14.** Write out the physical components associated with the gradient of $\phi = \phi(x^1, x^2, x^3)$.

► **15.** Show that

$$g^{im} A_{i,m} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left[\sqrt{g} g^{im} A_m\right] = A^i_{\ ,i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left[\sqrt{g} A^i\right].$$

► **16.** Let $r = (\vec{r}\cdot\vec{r})^{1/2} = \sqrt{x^2+y^2+z^2)}$ and calculate (a) $\nabla^2(r)$ (b) $\nabla^2(1/r)$ (c) $\nabla^2(r^2)$ (d) $\nabla^2(1/r^2)$

► **17.** Given the tensor equations $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad i,j = 1,2,3$. Let $v(1), v(2), v(3)$ denote the physical components of v_1, v_2, v_3 and let $D(ij)$ denote the physical components associated with D_{ij}. Assume the coordinate system (x^1, x^2, x^3) is orthogonal with metric coefficients $g_{(i)(i)} = h_i^2,\ i = 1,2,3$ and $g_{ij} = 0$ for $i \neq j$.

(a) Find expressions for the physical components $D(11), D(22)$ and $D(33)$ in terms of the physical components $v(i), i = 1,2,3$. Answer: $D(ii) = \dfrac{1}{h_i}\dfrac{\partial V(i)}{\partial x^i} + \sum_{j\neq i} \dfrac{V(j)}{h_i h_j}\dfrac{\partial h_i}{\partial x^j}$ no sum on i.

(b) Find expressions for the physical components $D(12), D(13)$ and $D(23)$ in terms of the physical components $v(i), i = 1,2,3$. Answer: $D(ij) = \dfrac{1}{2}\left[\dfrac{h_i}{h_j}\dfrac{\partial}{\partial x^j}\left(\dfrac{V(i)}{h_i}\right) + \dfrac{h_j}{h_i}\dfrac{\partial}{\partial x^i}\left(\dfrac{V(j)}{h_j}\right)\right]$

► **18.** Write out the tensor equations in problem 17 in Cartesian coordinates.

► **19.** Write out the tensor equations in problem 17 in cylindrical coordinates.

► **20.** Write out the tensor equations in problem 17 in spherical coordinates.

► **21.** Express the vector equation $(\lambda + 2\mu)\nabla\Phi - 2\mu\nabla\times\vec{\omega} + \vec{F} = \vec{0}$ in tensor form.

► **22.** Write out the equations in problem 21 for a generalized orthogonal coordinate system in terms of physical components.

► **23.** Write out the equations in problem 22 for cylindrical coordinates.

► **24.** Write out the equations in problem 22 for spherical coordinates.

► **25.** Use equation (2.1.4) to represent the divergence in parabolic cylindrical coordinates (ξ, η, z).

► **26.** Use equation (2.1.4) to represent the divergence in parabolic coordinates (ξ, η, ϕ).

► **27.** Use equation (2.1.4) to represent the divergence in elliptic cylindrical coordinates (ξ, η, z).

Change the given equations from a vector notation to a tensor notation.

► 28. $\vec{B} = \vec{v}\,\nabla\cdot\vec{A} + (\nabla\cdot\vec{v})\,\vec{A}$

► 29. $\dfrac{d}{dt}[\vec{A}\cdot(\vec{B}\times\vec{C})] = \dfrac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C}) + \vec{A}\cdot(\dfrac{d\vec{B}}{dt}\times\vec{C}) + \vec{A}\cdot(\vec{B}\times\dfrac{d\vec{C}}{dt})$

► 30. $\dfrac{d\vec{v}}{dt} = \dfrac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$

► 31. $\dfrac{1}{c}\dfrac{\partial\vec{H}}{\partial t} = -\text{curl}\ \vec{E}$

► 32. $\dfrac{d\vec{B}}{dt} - (\vec{B}\cdot\nabla)\vec{v} + \vec{B}(\nabla\cdot\vec{v}) = \vec{0}$

Change the given equations from a tensor notation to a vector notation.

► 33. $\epsilon^{ijk}B_{k,j} + F^i = 0$

► 34. $g_{ij}\epsilon^{jkl}B_{l,k} + F_i = 0$

► 35. $\dfrac{\partial \varrho}{\partial t} + (\varrho v_i), i = 0$

► 36. $\varrho(\dfrac{\partial v_i}{\partial t} + v_m \dfrac{\partial v_i}{\partial x^m}) = -\dfrac{\partial P}{\partial x^i} + \mu \dfrac{\partial^2 v_i}{\partial x^m \partial x^m} + F_i$

► **37.** The moment of inertia of an area or second moment of area is defined by $I_{ij} = \int\int_A (y_m y_m \delta_{ij} - y_i y_j)\, dA$ where dA is an element of area. Calculate the moment of inertia I_{ij}, $i, j = 1, 2$ for the triangle illustrated in the figure 2.1-1 and show that $I_{ij} = \begin{pmatrix} \frac{1}{12}bh^3 & -\frac{1}{24}b^2h^2 \\ -\frac{1}{24}b^2h^2 & \frac{1}{12}b^3h \end{pmatrix}$.

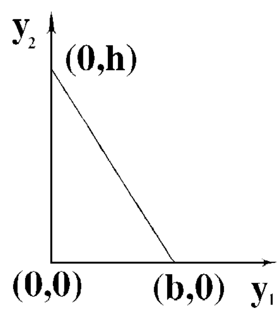

Figure 2.1-1 Moments of inertia for a triangle

► **38.** Use the results from problem 37 and rotate the axes in figure 2.1-1 through an angle θ to a barred system of coordinates.

(a) Show that in the barred system of coordinates

$$\overline{I}_{11} = \left(\frac{I_{11}+I_{22}}{2}\right) + \left(\frac{I_{11}-I_{22}}{2}\right)\cos 2\theta + I_{12}\sin 2\theta$$

$$\overline{I}_{12} = \overline{I}_{21} = -\left(\frac{I_{11}-I_{22}}{2}\right)\sin 2\theta + I_{12}\cos 2\theta$$

$$\overline{I}_{22} = \left(\frac{I_{11}+I_{22}}{2}\right) - \left(\frac{I_{11}-I_{22}}{2}\right)\cos 2\theta - I_{12}\sin 2\theta$$

(b) For what value of θ will $\overline{I}_{11}$ have a maximum value?

(c) Show that when $\overline{I}_{11}$ is a maximum, we will have $\overline{I}_{22}$ a minimum and $\overline{I}_{12} = \overline{I}_{21} = 0$.

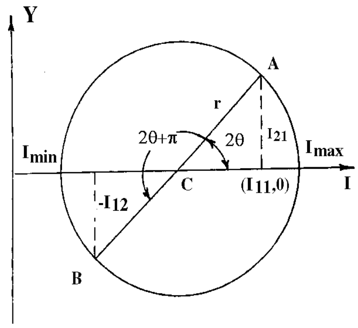

Figure 2.1-2 Mohr's circle

► **39.** Otto Mohr[1] gave the following physical interpretation to the results obtained in problem 38:

• Plot the points $A(I_{11}, I_{12})$ and $B(I_{22}, -I_{12})$ as illustrated in the figure 2.1-2

• Draw the line $\overline{AB}$ and calculate the point C where this line intersects the I axes. Show the point C has the coordinates

$$(\frac{I_{11}+I_{22}}{2}, 0)$$

• Calculate the radius of the circle with center at the point C and with diagonal $\overline{AB}$ and show this radius is

$$r = \sqrt{\left(\frac{I_{11}-I_{22}}{2}\right)^2 + I_{12}^2}$$

• Show the maximum and minimum values of I occur where the constructed circle intersects the I axes. Show that $I_{max} = \overline{I}_{11} = \dfrac{I_{11}+I_{22}}{2} + r \qquad I_{min} = \overline{I}_{22} = \dfrac{I_{11}+I_{22}}{2} - r.$

► **40.** Show directly that the eigenvalues of the symmetric matrix $I_{ij} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$ are $\lambda_1 = I_{max}$ and $\lambda_2 = I_{min}$ where I_{max} and I_{min} are given in problem 39.

► **41.** Find the principal axes and moments of inertia for the triangle given in problem 37 and summarize your results from problems 37,38,39, and 40.

► **42.** Verify for orthogonal coordinates the relations

$$\left[\nabla \times \vec{A}\right] \cdot \hat{e}_{(i)} = \sum_{k=1}^{3} \frac{e_{(i)jk}}{h_1h_2h_3} h_{(i)} \frac{\partial (h_{(k)}A(k))}{\partial x_j}$$

or

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3} \begin{vmatrix} h_1\,\hat{\mathbf{e}}_1 & h_2\,\hat{\mathbf{e}}_2 & h_3\,\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1A(1) & h_2A(2) & h_3A(3) \end{vmatrix}.$$

► **43.** Verify for orthogonal coordinates the relation

$$\left[\nabla \times (\nabla \times \vec{A})\right] \cdot \hat{e}_{(i)} = \sum_{m=1}^{3} e_{(i)jr}e_{rsm} \frac{h_{(i)}}{h_1h_2h_3} \frac{\partial}{\partial x_j} \left[\frac{h_{(r)}^2}{h_1h_2h_3} \frac{\partial (h_{(m)}A(m))}{\partial x_s}\right]$$

[1]Christian Otto Mohr (1835-1918) German civil engineer.

▶ **44.** Verify for orthogonal coordinates the relation

$$\left[\nabla\left(\nabla\cdot\vec{A}\right)\right]\cdot\hat{e}_{(i)}=\frac{1}{h_{(i)}}\frac{\partial}{\partial x_{(i)}}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A(1))}{\partial x_1}+\frac{\partial(h_1h_3A(2))}{\partial x_2}+\frac{\partial(h_1h_2A(3))}{\partial x_3}\right]\right\}$$

▶ **45.** Verify the relation

$$\left[(\vec{A}\cdot\nabla)\vec{B}\right]\cdot\hat{\mathbf{e}}_{(i)}=\sum_{k=1}^{3}\frac{A(k)}{h_{(k)}}\frac{\partial B(i)}{\partial x_k}+\sum_{k\neq i}\frac{B(k)}{h_kh_{(i)}}\left(A(i)\frac{\partial h_{(i)}}{\partial x_k}-A(k)\frac{\partial h_k}{\partial x_{(i)}}\right)$$

▶ **46.** The Gauss divergence theorem is written

$$\iiint_V\left(\frac{\partial F^1}{\partial x}+\frac{\partial F^2}{\partial y}+\frac{\partial F^3}{\partial z}\right)d\tau=\iint_S\left(n_1F^1+n_2F^2+n_3F^3\right)d\sigma$$

where V is the volume within a simple closed surface S. Here it is assumed that $F^i=F^i(x,y,z)$ are continuous functions with continuous first order derivatives throughout V and n_i are the direction cosines of the outward normal to S, $d\tau$ is an element of volume and $d\sigma$ is an element of surface area.

(a) Show that in a Cartesian coordinate system

$$F^i_{,i}=\frac{\partial F^1}{\partial x}+\frac{\partial F^2}{\partial y}+\frac{\partial F^3}{\partial z}$$

and that the tensor form of this theorem is $\iiint_V F^i_{,i}\,d\tau=\iint_S F^in_i\,d\sigma.$

(b) Write the vector form of this theorem.

(c) Show that if we define

$$u_r=\frac{\partial u}{\partial x^r},\quad v_r=\frac{\partial v}{\partial x^r}\quad\text{and}\quad F_r=g_{rm}F^m=uv_r$$

then $F^i_{,i}=g^{im}F_{i,m}=g^{im}(uv_{i,m}+u_mv_i)$

(d) Show that another form of the Gauss divergence theorem is

$$\iiint_V g^{im}u_mv_i\,d\tau=\iint_S uv_mn^m\,d\sigma-\iiint_V ug^{im}v_{i,m}\,d\tau$$

Write out the above equation in Cartesian coordinates.

▶ **47.** Find the eigenvalues and eigenvectors associated with the matrix $A=\begin{pmatrix}1&1&2\\1&2&1\\2&1&1\end{pmatrix}$.

Show that the eigenvectors are orthogonal.

▶ **48.** Find the eigenvalues and eigenvectors associated with the matrix $A=\begin{pmatrix}1&2&1\\2&1&0\\1&0&1\end{pmatrix}$.

Show that the eigenvectors are orthogonal.

▶ **49.** Find the eigenvalues and eigenvectors associated with the matrix $A=\begin{pmatrix}1&1&0\\1&1&1\\0&1&1\end{pmatrix}$.

Show that the eigenvectors are orthogonal.

▶ **50.** The harmonic and biharmonic functions or potential functions occur in the mathematical modeling of many physical problems. Any solution of Laplace's equation $\nabla^2\Phi=0$ is called a harmonic function and any solution of the biharmonic equation $\nabla^4\Phi=0$ is called a biharmonic function.

(a) Expand the Laplace equation in Cartesian, cylindrical and spherical coordinates.

(b) Expand the biharmonic equation in two dimensional Cartesian and polar coordinates.

Hint: Consider $\nabla^4\Phi=\nabla^2(\nabla^2\Phi)$. In Cartesian coordinates $\nabla^2\Phi=\Phi_{,ii}$ and $\nabla^4\Phi=\Phi_{,iijj}$.