§2.3 BASIC EQUATIONS OF CONTINUUM MECHANICS

Continuum mechanics is the study of how materials behave when subjected to external influences. External influences which affect the properties of a substance are such things as forces, temperature, chemical reactions, and electric phenomena. Examples of forces are gravitational forces, electromagnetic forces, and mechanical forces. Solids deform under external forces and so deformations are studied. Fluids move under external forces and so the velocity of the fluid is studied.

A material is considered to be a continuous media which is a collection of material points interconnected by internal forces (forces between the atoms making up the material). We concentrate upon the macroscopic properties rather than the microscopic properties of the material. We treat the material as a body which is homogeneous and continuous in its makeup.

In this introduction we will only consider solid media and liquid media. In general, most of the ideas and concepts developed in this section can be applied to any type of material which is assumed to be a collection of material points held together by some kind of internal forces.

An elastic material is one which deforms under applied forces in such a way that it will return to its original unloaded state when the applied forces are removed. When a linear relation exists between the applied forces and material displacements, then the material is called a linear elastic material. In contrast, a plastic material is one which deforms under applied forces in such a way that it does not return to its original state after removal of the applied forces. Plastic materials will always exhibit some permanent deformation after removal of the applied forces. An elastic material is called homogeneous if it has the same properties throughout. An isotropic material has the same properties, at a point, in all directions about the point.

In this introduction we develop the basic mathematical equations which describe how a continuum behaves when subjected to external forces. We shall discover that there exists a set of basic equations associated with all continuous material media. These basic equations are developed for linear elastic materials and applied to solids and fluids in later sections.

Introduction to Elasticity

Take a rubber band, which has a rectangular cross section, and mark on it a parallelepiped having a length ℓ, a width w and a height h, as illustrated in the figure 2.3-1.

Now apply a force F to both ends of the parallelepiped cross section on the rubber band and examine what happens to the parallelepiped. You will see that:

1. ℓ increases by an amount $\Delta\ell$.
2. w decreases by an amount Δw.
3. h decreases by an amount Δh.

There are many materials which behave in a manner very similar to the rubber band. Most materials, when subjected to tension forces will break if the change $\Delta\ell$ is only one or two percent of the original length. The above example introduces us to several concepts which arise in the study of materials when they are subjected to external forces. The first concept is that of strain which is defined as

$$strain = \frac{\text{change in length}}{\text{original length}}, \qquad \text{(dimensionless)}.$$

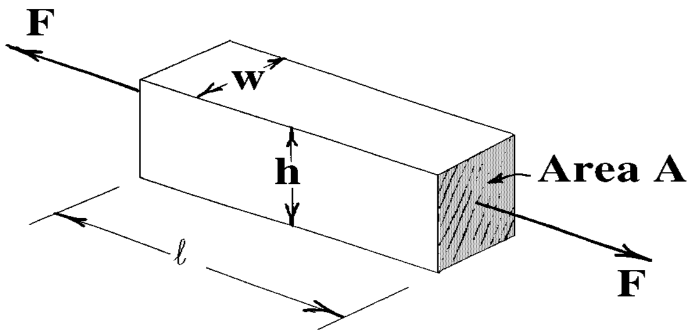

Figure 2.3-1. Section of a rubber band

When the force F is applied to our rubber band example there arises the strains

$$\frac{\Delta \ell}{\ell}, \qquad \frac{\Delta w}{w}, \qquad \frac{\Delta h}{h}.$$

The second concept introduced by our simple example is stress. Stress is defined as a force per unit area. In particular,

$$stress = \frac{\text{Force}}{\text{Area over which force acts}}, \qquad \text{with dimension of} \quad \frac{\text{force}}{\text{unit area}}.$$

We will be interested in studying stress and strain in homogeneous, isotropic materials which are in equilibrium with respect to the force system acting on the material.

Hooke's Law

For linear elastic materials, where the forces are all one dimensional, the stress and strains are related by Hooke's law which has two parts. The Hooke's law, part one, states that stress is proportional to strain in the stretch direction, where the Young's modulus E is the proportionality constant. This is written

$$\text{Hooke's law part 1} \qquad \frac{F}{A} = E\left(\frac{\Delta \ell}{\ell}\right). \tag{2.3.1}$$

A graph of stress vs strain is a straight line with slope E in the linear elastic range of the material.

The Hooke's law, part two, involves the fact that there is a strain contraction perpendicular to the stretch direction. The strain contraction is the same for both the width and height and is proportional to the strain in the stretch direction. The proportionality constant being the Poisson's ratio ν.

$$\text{Hooke's law part 2} \qquad \frac{\Delta w}{w} = \frac{\Delta h}{h} = -\nu \frac{\Delta \ell}{\ell}, \qquad 0 < \nu < \frac{1}{2}. \tag{2.3.2}$$

The proportionality constants E and ν depend upon the material being considered. The constant ν is called the Poisson's ratio and it is always a positive number which is less than one half. Some representative values for E and ν are as follows.

Various types of steel	$28\,(10)^6\,\text{psi} \le E \le 30\,(10)^6\,\text{psi}$	$0.26 \le \nu \le 0.31$
Various types of aluminium	$9.0\,(10)^6\,\text{psi} \le E \le 11.0\,(10)^6\,\text{psi}$	$0.3 \le \nu \le 0.35$

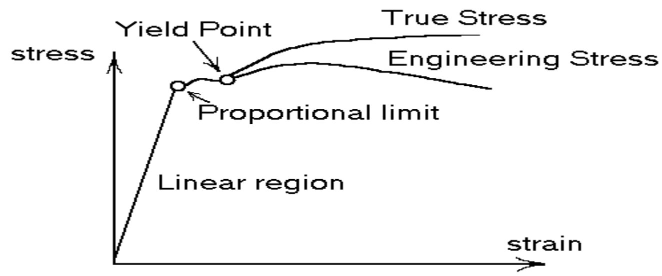

Figure 2.3-2. Typical Stress-strain curve.

Consider a typical stress-strain curve, such as the one illustrated in the figure 2.3-2, which is obtained by placing a material in the shape of a rod or wire in a machine capable of performing tensile straining at a low rate. The engineering stress is the tensile force F divided by the original cross sectional area A_0. Note that during a tensile straining the cross sectional area A of the sample is continually changing and getting smaller so that the actual stress will be larger than the engineering stress. Observe in the figure 2.3-2 that the stress-strain relation remains linear up to a point labeled the proportional limit. For stress-strain points in this linear region the Hooke's law holds and the material will return to its original shape when the loading is removed. For points beyond the proportional limit, but less than the yield point, the material no longer obeys Hooke's law. In this nonlinear region the material still returns to its original shape when the loading is removed. The region beyond the yield point is called the plastic region. At the yield point and beyond, there is a great deal of material deformation while the loading undergoes only small changes. For points in this plastic region, the material undergoes a permanent deformation and does not return to its original shape when the loading is removed. In the plastic region there usually occurs deformation due to slipping of atomic planes within the material. In this introductory section we will restrict our discussions of material stress-strain properties to the linear region.

EXAMPLE 2.3-1. (One dimensional elasticity) Consider a circular rod with cross sectional area A which is subjected to an external force F applied to both ends. The figure 2.3-3 illustrates what happens to the rod after the tension force F is applied. Consider two neighboring points P and Q on the rod, where P is at the point x and Q is at the point $x + \Delta x$. When the force F is applied to the rod it is stretched and P moves to P' and Q moves to Q'. We assume that when F is applied to the rod there is a displacement function $u = u(x,t)$ which describes how each point in the rod moves as a function of time t. If we know the displacement function $u = u(x,t)$ we would then be able to calculate the following distances in terms of the displacement function

$$\overline{PP'} = u(x,t), \qquad \overline{0P'} = x + u(x,t), \qquad \overline{QQ'} = u(x+\Delta x, t) \qquad \overline{0Q'} = x + \Delta x + u(x+\Delta x, t).$$

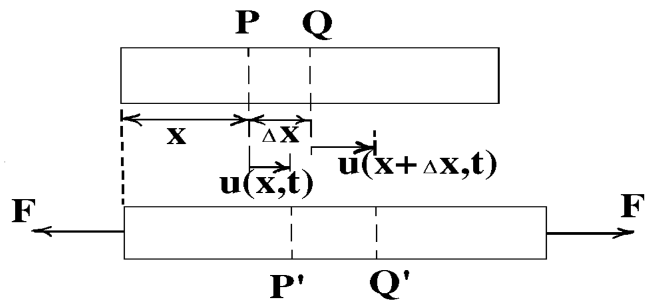

Figure 2.3-3. One dimensional rod subjected to tension force

The strain associated with the distance $\ell = \Delta x = \overline{PQ}$ is

$$\begin{aligned}
e =& \frac{\Delta \ell}{\ell} = \frac{\overline{P'Q'} - \overline{PQ}}{\overline{PQ}} = \frac{(\overline{0Q'} - \overline{0P'}) - (\overline{0Q} - \overline{0P})}{\overline{PQ}} \\
e =& \frac{[x + \Delta x + u(x + \Delta x, t) - (x + u(x,t))] - [(x + \Delta x) - x]}{\Delta x} \\
e =& \frac{u(x + \Delta x, t) - u(x,t)}{\Delta x}.
\end{aligned}$$

Use the Hooke's law part(i) and write

$$\frac{F}{A} = E\frac{u(x + \Delta x, t) - u(x,t)}{\Delta x}.$$

Taking the limit as Δx approaches zero we find that

$$\frac{F}{A} = E\frac{\partial u(x,t)}{\partial x}.$$

Hence, the stress is proportional to the spatial derivative of the displacement function.

■

Normal and Shearing Stresses

Let us consider a more general situation in which we have some material which can be described as having a surface area S which encloses a volume V. Assume that the density of the material is ϱ and the material is homogeneous and isotropic. Further assume that the material is subjected to the forces $\vec{b}$ and $\vec{t}^{(n)}$ where $\vec{b}$ is a body force per unit mass $[force/mass]$, and $\vec{t}^{(n)}$ is a surface traction per unit area $[force/area]$. The superscript (n) on the vector is to remind you that we will only be interested in the normal component of the surface forces. We will neglect body couples, surface couples, and concentrated forces or couples that act at a single point. If the forces described above are everywhere continuous we can calculate the resultant force $\vec{F}$ and resultant moment $\vec{M}$ acting on the material by constructing various surface and volume integrals which sum the forces acting upon the material. In particular, the resultant force $\vec{F}$ acting on our material can be described by the surface and volume integrals:

$$\vec{F} = \iint_S \vec{t}^{(n)}\, dS + \iiint_V \varrho \vec{b}\, d\tau \tag{2.3.3}$$

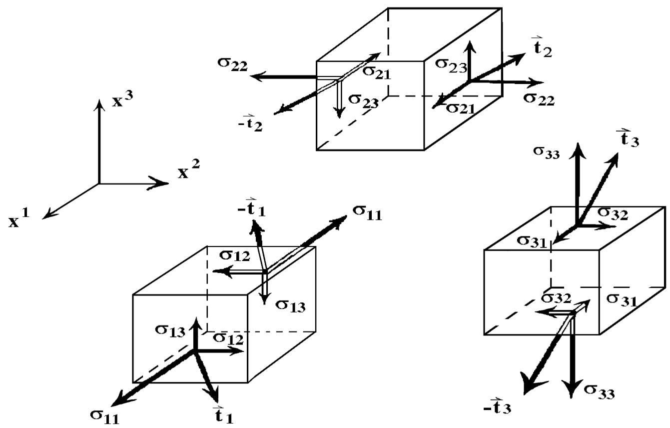

Figure 2.3-4. Stress vectors acting upon an element of volume

which is a summation of all the body forces and surface tractions acting upon our material. Here ϱ is the density of the material, dS is an element of surface area, and $d\tau$ is an element of volume.

The resultant moment $\vec{M}$ about the origin is similarly expressed as

$$\vec{M} = \iint_S \vec{r} \times \vec{t}^{(n)}\, dS + \iiint_V \varrho(\vec{r} \times \vec{b})\, d\tau. \tag{2.3.4}$$

The global motion of the material is governed by the Euler equations of motion.

- The time rate of change of linear momentum equals the resultant force or

$$\frac{d}{dt}\left[\iiint_V \varrho\vec{v}\, d\tau\right] = \vec{F} = \iint_S \vec{t}^{(n)}\, dS + \iiint_V \varrho\vec{b}\, d\tau. \tag{2.3.5}$$

  This is a statement concerning the conservation of linear momentum.

- The time rate of change of angular momentum equals the resultant moment or

$$\frac{d}{dt}\left[\iiint_V \varrho\vec{r} \times \vec{v}\, d\tau\right] = \vec{M} = \iint_S \vec{r} \times \vec{t}^{(n)}\, dS + \iiint_V \varrho(\vec{r} \times \vec{b})\, d\tau. \tag{2.3.6}$$

  This is a statement concerning conservation of angular momentum.

The Stress Tensor

Define the stress vectors

$$\begin{aligned} \vec{t}^1 &= \sigma^{11}\,\hat{\mathbf{e}}_1 + \sigma^{12}\,\hat{\mathbf{e}}_2 + \sigma^{13}\,\hat{\mathbf{e}}_3 \\ \vec{t}^2 &= \sigma^{21}\,\hat{\mathbf{e}}_1 + \sigma^{22}\,\hat{\mathbf{e}}_2 + \sigma^{23}\,\hat{\mathbf{e}}_3 \\ \vec{t}^3 &= \sigma^{31}\,\hat{\mathbf{e}}_1 + \sigma^{32}\,\hat{\mathbf{e}}_2 + \sigma^{33}\,\hat{\mathbf{e}}_3, \end{aligned} \tag{2.3.7}$$

where σ^{ij}, $i, j = 1, 2, 3$ is the stress tensor acting at each point of the material. The index i indicates the coordinate surface $x^i =$ a constant, upon which $\vec{t}^i$ acts. The second index j denotes the direction associated with the components of $\vec{t}^i$.

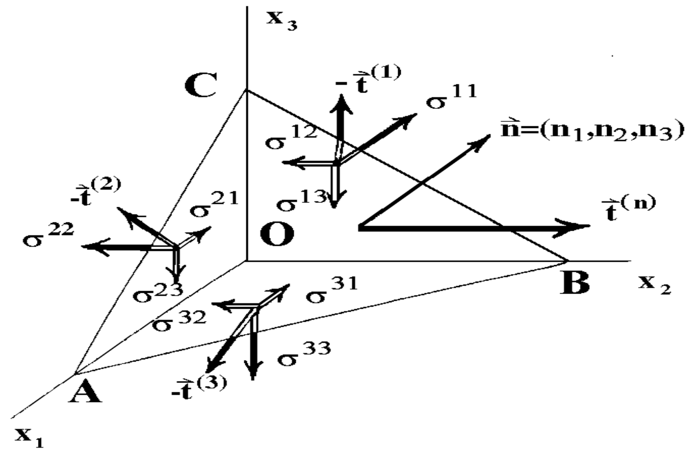

Figure 2.3-5. Stress distribution at a point

For $i = 1, 2, 3$ we adopt the convention of sketching the components of $\vec{t}^{\,i}$ in the positive directions if the exterior normal to the surface $x^i =$ constant also points in the positive direction. This gives rise to the figure 2.3-4 which illustrates the stress vectors acting upon an element of volume in rectangular Cartesian coordinates. The components $\sigma^{11}, \sigma^{22}, \sigma^{33}$ are called normal stresses while the components σ^{ij}, $i \neq j$ are called shearing stresses. The equations (2.3.7) can be written in the more compact form using the indicial notation as

$$\vec{t}^{\,i} = \sigma^{ij}\,\hat{\mathbf{e}}_j, \quad i, j = 1, 2, 3. \tag{2.3.8}$$

If we know the stress distribution at three orthogonal interfaces at a point P in a solid body, we can then determine the stress at the point P with respect to any plane passing through the point P. With reference to the figure 2.3-5, consider an arbitrary plane passing through the point P which lies within the material body being considered. Construct the elemental tetrahedron with orthogonal axes parallel to the $x^1 = x, x^2 = y$ and $x^3 = z$ axes. In this figure we have the following surface tractions:

$$\begin{array}{ll} -\vec{t}^{\,1} & \text{on the surface 0BC} \\ -\vec{t}^{\,2} & \text{on the surface 0AC} \\ -\vec{t}^{\,3} & \text{on the surface 0AB} \\ \vec{t}^{(n)} & \text{on the surface ABC} \end{array}$$

The superscript parenthesis n is to remind you that this surface traction depends upon the orientation of the plane ABC which is determined by a unit normal vector having the direction cosines n_1, n_2 and n_3.

Let

$$\Delta S_1 = \text{the surface area 0BC}$$
$$\Delta S_2 = \text{the surface area 0AC}$$
$$\Delta S_3 = \text{the surface area 0AB}$$
$$\Delta S = \text{the surface area ABC} .$$

These surface areas are related by the relations

$$\Delta S_1 = n_1 \Delta S, \qquad \Delta S_2 = n_2 \Delta S, \qquad \Delta S_3 = n_3 \Delta S \tag{2.3.9}$$

which can be thought of as projections of ΔS upon the planes x_i =constant for $i = 1, 2, 3$.

Cauchy Stress Law

Let $t^{j\,(n)}$ denote the components of the surface traction on the surface ABC. That is, we let

$$\vec{t}^{\,(n)} = t^{1\,(n)}\,\hat{\mathbf{e}}_1 + t^{2\,(n)}\,\hat{\mathbf{e}}_2 + t^{3\,(n)}\,\hat{\mathbf{e}}_3 = t^{j\,(n)}\,\hat{\mathbf{e}}_j. \tag{2.3.10}$$

It will be demonstrated that the components $t^{j\,(n)}$ of the surface traction forces $\vec{t}^{(n)}$ associated with a plane through P and having the unit normal with direction cosines n_1, n_2 and n_3, must satisfy the relations

$$t^{j\,(n)} = n_i\,\sigma^{ij}, \quad i, j = 1, 2, 3. \tag{2.3.11}$$

This relation is known as the Cauchy stress law.

Proof: Sum the forces acting on the elemental tetrahedron in the figure 2.3-5. If the body is in equilibrium, then the sum of these forces must equal zero or

$$(-\vec{t}^{\,1}\,\Delta S_1) + (-\vec{t}^{\,2}\,\Delta S_2) + (-\vec{t}^{\,3}\,\Delta S_3) + \vec{t}^{\,(n)}\,\Delta S = 0. \tag{2.3.12}$$

The relations in the equations (2.3.9) are used to simplify the sum of forces in the equation (2.3.12). It is readily verified that the sum of forces simplifies to

$$\vec{t}^{\,(n)} = n_1\vec{t}^{\,1} + n_2\vec{t}^{\,2} + n_3\vec{t}^{\,3} = n_i\vec{t}^{\,i}. \tag{2.3.13}$$

Substituting in the relations from equation (2.3.8) we find

$$\vec{t}^{\,(n)} = t^{j\,(n)}\,\hat{\mathbf{e}}_j = n_i\sigma^{ij}\,\hat{\mathbf{e}}_j, \quad i, j = 1, 2, 3 \tag{2.3.14}$$

or in component form

$$t^{j\,(n)} = n_i\sigma^{ij} \tag{2.3.15}$$

which is the Cauchy stress law.

Conservation of Linear Momentum

Let R denote a region in space where there exists a material volume with density ϱ having surface tractions and body forces acting upon it. Let v^i denote the velocity of the material volume and use Newton's second law to set the time rate of change of linear momentum equal to the forces acting upon the volume as in (2.3.5). We find

$$\frac{\delta}{\delta t}\left[\iiint_R \varrho v^j \, d\tau\right] = \iint_S \sigma^{ij} n_i \, dS + \iiint_R \varrho b^j \, d\tau.$$

Here $d\tau$ is an element of volume, dS is an element of surface area, b^j are body forces per unit mass, and σ^{ij} are the stresses. Employing the Gauss divergence theorem, the surface integral term is replaced by a volume integral and Newton's second law is expressed in the form

$$\iiint_R \left[\varrho f^j - \varrho b^j - \sigma^{ij}{}_{,i}\right] \, d\tau = 0, \tag{2.3.16}$$

where f^j is the acceleration from equation (1.4.54). Since R is an arbitrary region, the equation (2.3.16) implies that

$$\sigma^{ij}{}_{,i} + \varrho b^j = \varrho f^j. \tag{2.3.17}$$

This equation arises from a balance of linear momentum and represents the equations of motion for material in a continuum. If there is no velocity term, then equation (2.3.17) reduces to an equilibrium equation which can be written

$$\sigma^{ij}{}_{,i} + \varrho b^j = 0. \tag{2.3.18}$$

This equation can also be written in the covariant form

$$g^{si}\sigma_{ms,i} + \varrho b_m = 0,$$

which reduces to $\sigma_{ij,j} + \varrho b_i = 0$ in Cartesian coordinates. The equation (2.3.18) is an equilibrium equation and is one of our fundamental equations describing a continuum.

Conservation of Angular Momentum

The conservation of angular momentum equation (2.3.6) has the Cartesian tensors representation

$$\frac{d}{dt}\left[\iiint_R \varrho e_{ijk} x_j v_k \, d\tau\right] = \iint_S e_{ijk} x_j \sigma_{pk} n_p \, dS + \iiint_R \varrho e_{ijk} x_j b_k \, d\tau. \tag{2.3.19}$$

Employing the Gauss divergence theorem, the surface integral term is replaced by a volume integral to obtain

$$\iiint_R \left[e_{ijk}\varrho \frac{d}{dt}(x_j v_k) - e_{ijk}\left\{\varrho x_j b_k + \frac{\partial}{\partial x^p}(x_j \sigma_{pk})\right\}\right] \, d\tau = 0. \tag{2.3.20}$$

Since equation (2.3.20) must hold for all arbitrary volumes R we conclude that

$$e_{ijk}\varrho \frac{d}{dt}(x_j v_k) = e_{ijk}\left\{\varrho x_j b_k + x_j \frac{\partial \sigma_{pk}}{\partial x^p} + \sigma_{jk}\right\}$$

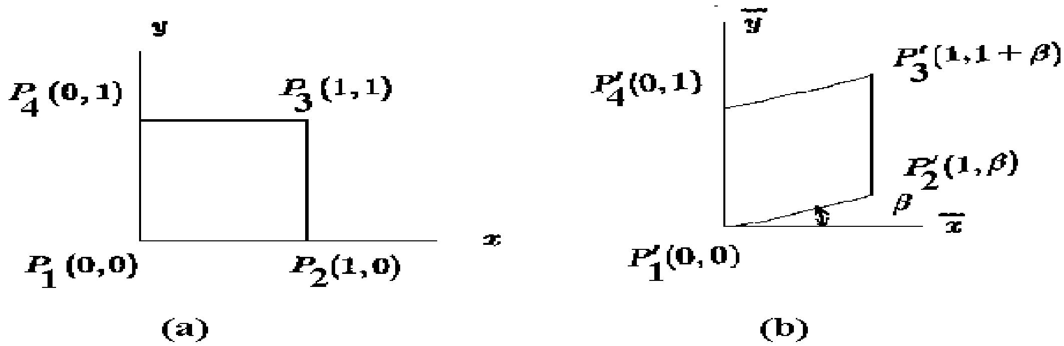

Figure 2.3-6. Shearing parallel to the y axis

which can be rewritten in the form

$$e_{ijk}\left[\sigma_{jk} + x_j(\frac{\partial \sigma_{pk}}{\partial x^p} + \varrho b_k - \varrho \frac{dv_k}{dt}) - \varrho v_j v_k\right] = 0. \tag{2.3.21}$$

In the equation (2.3.21) the middle term is zero because of the equation (2.3.17). Also the last term in (2.3.21) is zero because $e_{ijk}v_jv_k$ represents the cross product of a vector with itself. The equation (2.3.21) therefore reduces to

$$e_{ijk}\sigma_{jk} = 0, \tag{2.3.22}$$

which implies (see exercise 1.1, problem 22) that $\sigma_{ij} = \sigma_{ji}$ for all i and j. Thus, the conservation of angular momentum requires that the stress tensor be symmetric. Consequently, there are only 6 independent stress components to be determined. This is another fundamental law for a continuum.

Strain in Two Dimensions

Consider the matrix equation

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.3.23}$$

which can be used to transform points (x, y) to points $(\overline{x}, \overline{y})$. When this transformation is applied to the unit square illustrated in the figure 2.3-6(a) we obtain the geometry illustrated in the figure 2.3-6(b) which represents a shearing parallel to the y axis. If β is very small, we can use the approximation $\tan\beta \approx \beta$ and then this transformation can be thought of as a rotation of the element $\overline{P_1P_2}$ through an angle β to the position $\overline{P'_1P'_2}$ when the barred axes are placed atop the unbarred axes.

Similarly, the matrix equation

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.3.24}$$

can be used to represent a shearing of the unit square parallel to the x axis as illustrated in the figure 2.3-7(b).

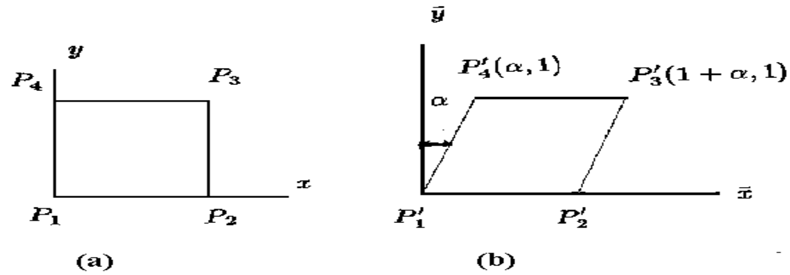

Figure 2.3-7. Shearing parallel to the x axis

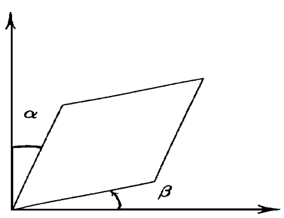

Figure 2.3-8. Shearing parallel to x and y axes

Again, if α is very small, we may use the approximation $\tan\alpha \approx \alpha$ and interpret α as an angular rotation of the element $\overline{P_1P_4}$ to the position $\overline{P_1'P_4'}$. Now let us multiply the matrices given in equations (2.3.23) and (2.3.24). Note that the order of multiplication is important as can be seen by an examination of the products

$$\begin{aligned} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \beta & 1+\alpha\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} &= \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+\alpha\beta & \alpha \\ \beta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \tag{2.3.25}$$

In equation (2.3.25) we will assume that the product $\alpha\beta$ is very, very small and can be neglected. Then the order of matrix multiplication will be immaterial and the transformation equation (2.3.25) will reduce to

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.3.26}$$

Applying this transformation to our unit square we obtain the simultaneous shearing parallel to both the x and y axes as illustrated in the figure 2.3-8.

This transformation can then be interpreted as the superposition of the two shearing elements depicted in the figure 2.3-9.

For comparison, we consider also the transformation equation

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.3.27}$$

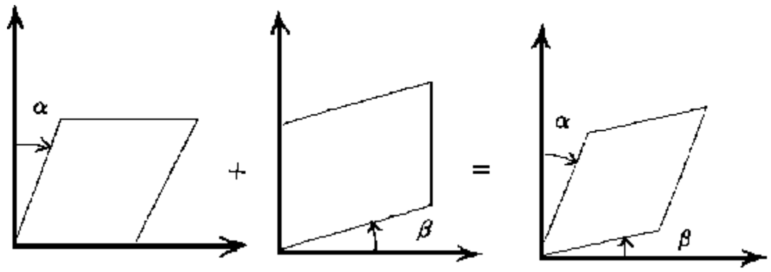

Figure 2.3-9. Superposition of shearing elements

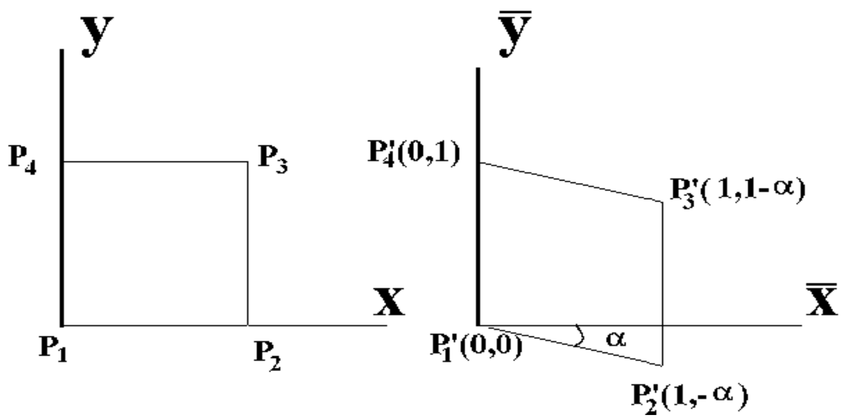

Figure 2.3-10. Rotation of element $\overline{P_1P_2}$

where α is very small. Applying this transformation to the unit square previously considered we obtain the results illustrated in the figure 2.3-10.

Note the difference in the direction of shearing associated with the transformation equations (2.3.27) and (2.3.23) illustrated in the figures 2.3-6 and 2.3-10. If the matrices appearing in the equations (2.3.24) and (2.3.27) are multiplied and we neglect product terms because α is assumed to be very small, we obtain the matrix equation

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{identity}} + \underbrace{\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{rotation}. \tag{2.3.28}$$

This can be interpreted as a superposition of the transformation equations (2.3.24) and (2.3.27) which represents a rotation of the unit square as illustrated in the figure 2.3-11.

The matrix on the right-hand side of equation (2.3.28) is referred to as a rotation matrix. The ideas illustrated by the above simple transformations will appear again when we consider the transformation of an arbitrary small element in a continuum when it under goes a strain. In particular, we will be interested in extracting the rigid body rotation from a deformed element and treating this rotation separately from the strain displacement.

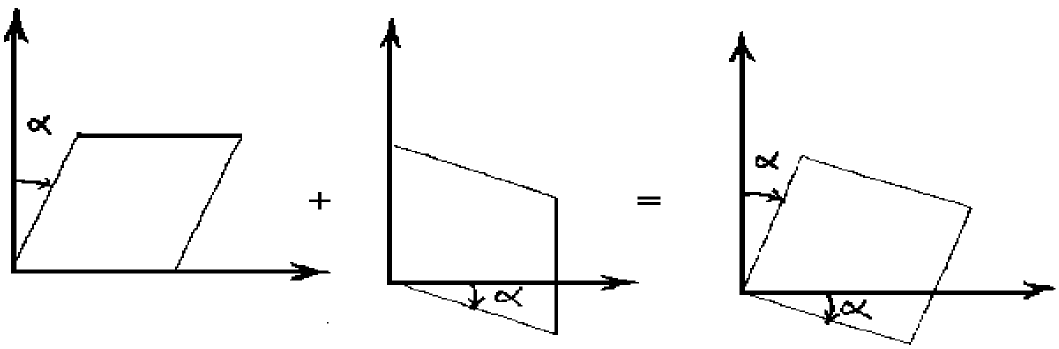

Figure 2.3-11. Rotation of unit square

Transformation of an Arbitrary Element

In two dimensions, we consider a rectangular element ABCD as illustrated in the figure 2.3-12. Let the points $ABCD$ have the coordinates

$$A(x,y), \qquad B(x+\Delta x, y), \qquad C(x, y+\Delta y), \qquad D(x+\Delta x, y+\Delta y) \tag{2.3.29}$$

and denote by

$$u = u(x,y), \qquad v = v(x,y)$$

the displacement field associated with each of the points in the material continuum when it undergoes a deformation. Assume that the deformation of the element $ABCD$ in figure 2.3-12 can be represented by the matrix equation

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.3.30}$$

where the coefficients $b_{ij}, i,j = 1,2,3$ are to be determined. Let us define $u = u(x,y)$ as the horizontal displacement of the point (x,y) and $v = v(x,y)$ as the vertical displacement of the same point. We can now express the displacement of each of the points A, B, C and D in terms of the displacement field $u = u(x,y)$ and $v = v(x,y)$. Consider first the displacement of the point A to A'. Here the coordinates (x,y) deform to the new coordinates

$$\overline{x} = x + u, \qquad \overline{y} = y + v.$$

That is, the coefficients b_{ij} must be chosen such that the equation

$$\begin{pmatrix} x+u \\ y+v \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.3.31}$$

is satisfied. We next examine the displacement of the point B to B'. This displacement is described by the coordinates $(x+\Delta x, y)$ transforming to $(\overline{x}, \overline{y})$, where

$$\overline{x} = x + \Delta x + u(x+\Delta x, y), \qquad \overline{y} = y + v(x+\Delta x, y). \tag{2.3.32}$$

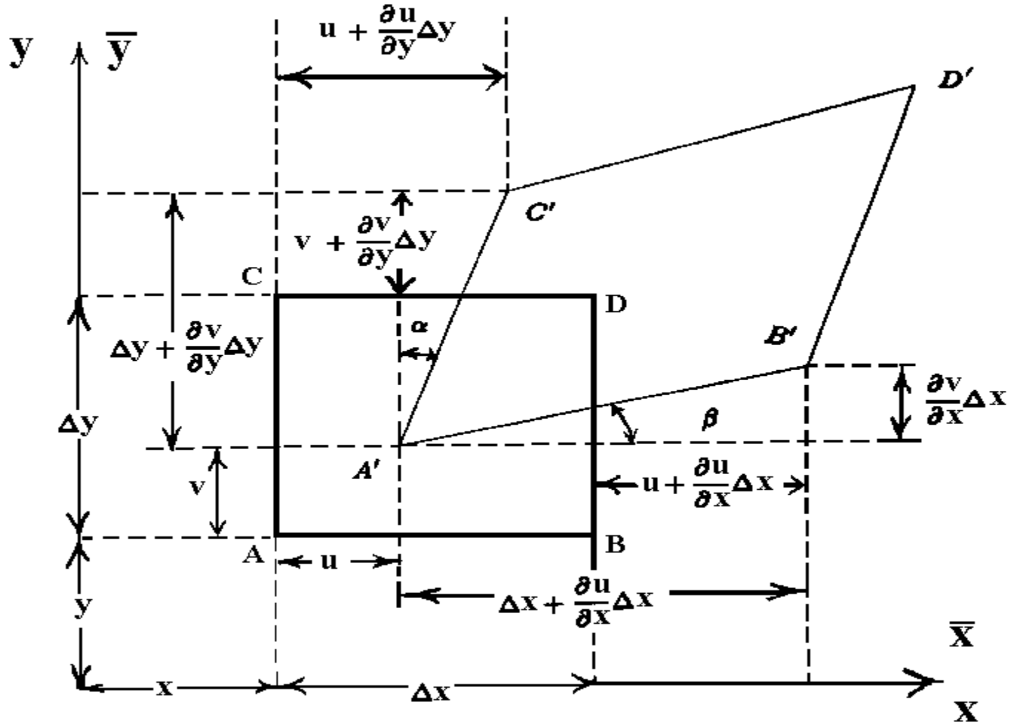

Figure 2.3-12. Displacement of element $ABCD$ to $A'B'C'D'$

Expanding u and v in (2.3.32) in Taylor series about the point (x, y) we find

$$\begin{aligned} \overline{x} &= x + \Delta x + u + \frac{\partial u}{\partial x}\Delta x + h.o.t. \\ \overline{y} &= y + v + \frac{\partial v}{\partial x}\Delta x + h.o.t., \end{aligned} \tag{2.3.33}$$

where $h.o.t.$ denotes higher order terms which have been neglected. The equations (2.3.33) require that the coefficients b_{ij} satisfy the matrix equation

$$\begin{pmatrix} x + u + \Delta x + \frac{\partial u}{\partial x}\Delta x \\ y + v + \frac{\partial v}{\partial x}\Delta x \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x + \Delta x \\ y \end{pmatrix}. \tag{2.3.34}$$

The displacement of the point C to C' is described by the coordinates $(x, y+\Delta y)$ transforming to $(\overline{x}, \overline{y})$ where

$$\overline{x} = x + u(x, y+\Delta y), \qquad \overline{y} = y + \Delta y + v(x, y+\Delta y). \tag{2.3.35}$$

Again we expand the displacement field components u and v in a Taylor series about the point (x, y) and find

$$\begin{aligned} \overline{x} &= x + u + \frac{\partial u}{\partial y}\Delta y + h.o.t. \\ \overline{y} &= y + \Delta y + v + \frac{\partial v}{\partial y}\Delta y + h.o.t. \end{aligned} \tag{2.3.36}$$

This equation implies that the coefficients b_{ij} must be chosen such that

$$\begin{pmatrix} x + u + \frac{\partial u}{\partial y}\Delta y \\ y + v + \Delta y + \frac{\partial v}{\partial y}\Delta y \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y + \Delta y \end{pmatrix}. \tag{2.3.37}$$

Finally, it can be verified that the point D with coordinates $(x+\Delta x, y+\Delta y)$ moves to the point D' with coordinates

$$\overline{x} = x + \Delta x + u(x+\Delta x, y+\Delta y), \qquad \overline{y} = y + \Delta y + v(x+\Delta x, y+\Delta y). \tag{2.3.38}$$

Expanding u and v in a Taylor series about the point (x, y) we find the coefficients b_{ij} must be chosen to satisfy the matrix equation

$$\begin{pmatrix} x + \Delta x + u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \\ y + \Delta y + v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x + \Delta x \\ y + \Delta y \end{pmatrix}. \tag{2.3.39}$$

The equations (2.3.31),(2.3.34),(2.3.37) and (2.3.39) give rise to the simultaneous equations

$$\begin{aligned} b_{11}x + b_{12}y &= x + u \\ b_{21}x + b_{22}y &= y + v \\ b_{11}(x+\Delta x) + b_{12}y &= x + u + \Delta x + \frac{\partial u}{\partial x}\Delta x \\ b_{21}(x+\Delta x) + b_{22}y &= y + v + \frac{\partial v}{\partial x}\Delta x \\ b_{11}x + b_{12}(y+\Delta y) &= x + u + \frac{\partial u}{\partial y}\Delta y \\ b_{21}x + b_{22}(y+\Delta y) &= y + v + \Delta y + \frac{\partial v}{\partial y}\Delta y \\ b_{11}(x+\Delta x) + b_{12}(y+\Delta y) &= x + \Delta x + u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \\ b_{21}(x+\Delta x) + b_{22}(y+\Delta y) &= y + \Delta y + v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y. \end{aligned} \tag{2.3.40}$$

It is readily verified that the system of equations (2.3.40) has the solution

$$\begin{aligned} b_{11} &= 1 + \frac{\partial u}{\partial x} \qquad & b_{12} &= \frac{\partial u}{\partial y} \\ b_{21} &= \frac{\partial v}{\partial x} \qquad & b_{22} &= 1 + \frac{\partial v}{\partial y}. \end{aligned} \tag{2.3.41}$$

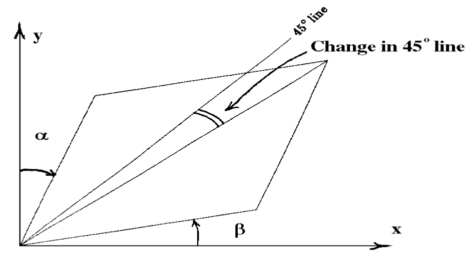

Figure 2.3-13. Change in 45° line

Hence the transformation equation (2.3.30) can be written as

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 1+\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & 1+\frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.3.42}$$

A physical interpretation associated with this transformation is obtained by writing it in the form:

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{identity}} + \underbrace{\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{strain matrix}} + \underbrace{\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{rotation matrix}}, \tag{2.3.43}$$

where

$$\begin{aligned} e_{11} &= \frac{\partial u}{\partial x} & e_{21} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ e_{12} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & e_{22} &= \frac{\partial v}{\partial y} \end{aligned} \tag{2.3.44}$$

are the elements of a symmetric matrix called the strain matrix and

$$\begin{aligned} \omega_{11} &= 0 & \omega_{12} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ \omega_{21} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & \omega_{22} &= 0 \end{aligned} \tag{2.3.45}$$

are the elements of a skew symmetric matrix called the rotation matrix.

The strain per unit length in the x-direction associated with the point A in the figure 2.3-12 is

$$e_{11} = \frac{\Delta x + \frac{\partial u}{\partial x}\Delta x - \Delta x}{\Delta x} = \frac{\partial u}{\partial x} \tag{2.3.46}$$

and the strain per unit length of the point A in the y direction is

$$e_{22} = \frac{\Delta y + \frac{\partial v}{\partial y}\Delta y - \Delta y}{\Delta y} = \frac{\partial v}{\partial y}. \tag{2.3.47}$$

These are the terms along the main diagonal in the strain matrix. The geometry of the figure 2.3-12 implies that

$$\tan\beta = \frac{\frac{\partial v}{\partial x}\Delta x}{\Delta x + \frac{\partial u}{\partial x}\Delta x}, \qquad \text{and} \qquad \tan\alpha = \frac{\frac{\partial u}{\partial y}\Delta y}{\Delta y + \frac{\partial v}{\partial y}\Delta y}. \tag{2.3.48}$$

For small derivatives associated with the displacements u and v it is assumed that the angles α and β are small and the equations (2.3.48) therefore reduce to the approximate equations

$$\tan\beta \approx \beta = \frac{\partial v}{\partial x} \qquad \tan\alpha \approx \alpha = \frac{\partial u}{\partial y}. \tag{2.3.49}$$

For a physical interpretation of these terms we consider the deformation of a small rectangular element which undergoes a shearing as illustrated in the figure 2.3-13.

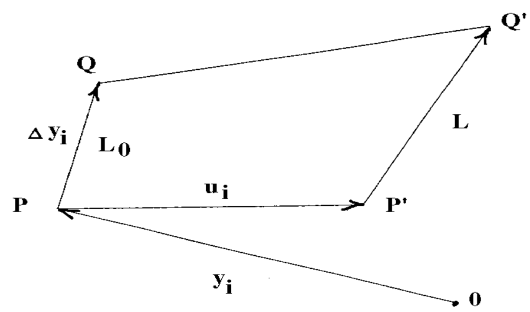

Figure 2.3-14. Displacement field due to state of strain

The quantity

$$\alpha + \beta = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 2e_{12} = 2e_{21} \tag{2.3.50}$$

is the change from a ninety degree angle due to the deformation and hence we can write $\frac{1}{2}(\alpha+\beta) = e_{12} = e_{21}$ as representing a change from a 45° angle due to the deformation. The quantities e_{21}, e_{12} are called the shear strains and the quantity

$$\gamma_{12} = 2e_{12} \tag{2.3.51}$$

is called the shear angle.

In the equation (2.3.45), the quantities $\omega_{21} = -\omega_{12}$ are the elements of the rigid body rotation matrix and are interpreted as angles associated with a rotation. The situation is analogous to the transformations and figures for the deformation of the unit square which was considered earlier.

Strain in Three Dimensions

The development of strain in three dimensions is approached from two different viewpoints. The first approach considers the derivation using Cartesian tensors and the second approach considers the derivation of strain using generalized tensors.

Cartesian Tensor Derivation of Strain.

Consider a material which is subjected to external forces such that all the points in the material undergo a deformation. Let (y_1, y_2, y_3) denote a set of orthogonal Cartesian coordinates, fixed in space, which is used to describe the deformations within the material. Further, let $u_i = u_i(y_1, y_2, y_3), i = 1, 2, 3$ denote a displacement field which describes the displacement of each point within the material. With reference to the figure 2.3-14 let P and Q denote two neighboring points within the material while it is in an unstrained state. These points move to the points P' and Q' when the material is in a state of strain. We let $y_i, i = 1, 2, 3$ represent the position vector to the general point P in the material, which is in an unstrained state, and denote by $y_i + u_i, i = 1, 2, 3$ the position vector of the point P' when the material is in a state of strain.

For Q a neighboring point of P which moves to Q' when the material is in a state of strain, we have from the figure 2.3-14 the following vectors:

$$\begin{aligned}
\text{position of } P: &\quad y_i, \quad i=1,2,3\\
\text{position of } P': &\quad y_i+u_i(y_1,y_2,y_3), \quad i=1,2,3\\
\text{position of } Q: &\quad y_i+\Delta y_i, \quad i=1,2,3\\
\text{position of } Q': &\quad y_i+\Delta y_i+u_i(y_1+\Delta y_1,y_2+\Delta y_2,y_3+\Delta y_3), \quad i=1,2,3
\end{aligned} \tag{2.3.52}$$

Employing our earlier one dimensional definition of strain, we define the strain associated with the point P in the direction $\overline{PQ}$ as $e=\dfrac{L-L_0}{L_0}$, where $L_0=\overline{PQ}$ and $L=\overline{P'Q'}$. To calculate the strain we need to first calculate the distances L_0 and L. The quantities L_0^2 and L^2 are easily calculated by considering dot products of vectors. For example, we have $L_0^2=\Delta y_i\Delta y_i$, and the distance $L=\overline{P'Q'}$ is the magnitude of the vector

$$y_i+\Delta y_i+u_i(y_1+\Delta y_1,y_2+\Delta y_2,y_3+\Delta y_3)-(y_i+u_i(y_1,y_2,y_3)), \quad i=1,2,3.$$

Expanding the quantity $u_i(y_1+\Delta y_1,y_2+\Delta y_2,y_3+\Delta y_3)$ in a Taylor series about the point P and neglecting higher order terms of the expansion we find that

$$L^2=(\Delta y_i+\frac{\partial u_i}{\partial y_m}\Delta y_m)(\Delta y_i+\frac{\partial u_i}{\partial y_n}\Delta y_n).$$

Expanding the terms in this expression produces the equation

$$L^2=\Delta y_i\Delta y_i+\frac{\partial u_i}{\partial y_n}\Delta y_i\Delta y_n+\frac{\partial u_i}{\partial y_m}\Delta y_m\Delta y_i+\frac{\partial u_i}{\partial y_m}\frac{\partial u_i}{\partial y_n}\Delta y_m\Delta y_n.$$

Note that L and L_0 are very small and so we express the difference $L^2-L_0^2$ in terms of the strain e. We can write

$$L^2-L_0^2=(L+L_0)(L-L_0)=(L-L_0+2L_0)(L-L_0)=(e+2)eL_0^2.$$

Now for e very small, and e^2 negligible, the above equation produces the approximation

$$eL_0^2\approx\frac{L^2-L_0^2}{2}=\frac{1}{2}\left[\frac{\partial u_m}{\partial y_n}+\frac{\partial u_n}{\partial y_m}+\frac{\partial u_r}{\partial y_m}\frac{\partial u_r}{\partial y_n}\right]\Delta y_m\Delta y_n.$$

The quantities

$$e_{mn}=\frac{1}{2}\left[\frac{\partial u_m}{\partial y_n}+\frac{\partial u_n}{\partial y_m}+\frac{\partial u_r}{\partial y_m}\frac{\partial u_r}{\partial y_n}\right] \tag{2.3.53}$$

is called the Green strain tensor or Lagrangian strain tensor. To show that e_{ij} is indeed a tensor, we consider the transformation $y_i=\ell_{ij}\overline{y}_j+b_i$, where $\ell_{ji}\ell_{ki}=\delta_{jk}=\ell_{ij}\ell_{ik}$. Note that from the derivative relation $\frac{\partial y_i}{\partial \overline{y}_j}=\ell_{ij}$ and the transformation equations $\overline{u}_i=\ell_{ij}u_j, i=1,2,3$ we can express the strain in the barred system of coordinates. Performing the necessary calculations produces

$$\begin{aligned}
\overline{e}_{ij}&=\frac{1}{2}\left[\frac{\partial \overline{u}_i}{\partial \overline{y}_j}+\frac{\partial \overline{u}_j}{\partial \overline{y}_i}+\frac{\partial \overline{u}_r}{\partial \overline{y}_i}\frac{\partial \overline{u}_r}{\partial \overline{y}_j}\right]\\
&=\frac{1}{2}\left[\frac{\partial}{\partial y_n}(\ell_{ik}u_k)\frac{\partial y_n}{\partial \overline{y}_j}+\frac{\partial}{\partial y_m}(\ell_{jk}u_k)\frac{\partial y_m}{\partial \overline{y}_i}+\frac{\partial}{\partial y_k}(\ell_{rs}u_s)\frac{\partial y_k}{\partial \overline{y}_i}\frac{\partial}{\partial y_t}(\ell_{rm}u_m)\frac{\partial y_t}{\partial \overline{y}_j}\right]\\
&=\frac{1}{2}\left[\ell_{im}\ell_{nj}\frac{\partial u_m}{\partial y_n}+\ell_{jk}\ell_{mi}\frac{\partial u_k}{\partial y_m}+\ell_{rs}\ell_{rp}\ell_{ki}\ell_{tj}\frac{\partial u_s}{\partial y_k}\frac{\partial u_p}{\partial y_t}\right]\\
&=\frac{1}{2}\left[\frac{\partial u_m}{\partial y_n}+\frac{\partial u_n}{\partial y_m}+\frac{\partial u_s}{\partial y_m}\frac{\partial u_s}{\partial y_n}\right]\ell_{im}\ell_{nj}
\end{aligned}$$

or $\quad \overline{e}_{ij}=e_{mn}\ell_{im}\ell_{nj}$.Consequently, the strain e_{ij} transforms like a second order Cartesian tensor.

Lagrangian and Eulerian Systems

Let $\overline{x}^i$ denote the initial position of a material particle in a continuum. Assume that at a later time the particle has moved to another point whose coordinates are x^i. Both sets of coordinates are referred to the same coordinate system. When the final position can be expressed as a function of the initial position and time we can write $x^i = x^i(\overline{x}^1, \overline{x}^2, \overline{x}^3, t)$. Whenever the changes of any physical quantity is represented in terms of its initial position and time, the representation is referred to as a Lagrangian or material representation of the quantity. This can be thought of as a transformation of the coordinates. When the Jacobian $J(\frac{x}{\overline{x}})$ of this transformation is different from zero, the above set of equations have a unique inverse $\overline{x}^i = \overline{x}^i(x^1, x^2, x^3, t)$, where the position of the particle is now expressed in terms of its instantaneous position and time. Such a representation is referred to as an Eulerian or spatial description of the motion.

Let $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ denote the initial position of a particle whose motion is described by $x_i = x_i(\overline{x}_1, \overline{x}_2, \overline{x}_3, t)$, then $u_i = x_i - \overline{x}_i$ denotes the displacement vector which can by represented in a Lagrangian or Eulerian form. For example, if

$$
\begin{aligned}
x_1 &= 2(\overline{x}_1 - \overline{x}_2)(e^t - 1) + (\overline{x}_2 - \overline{x}_1)(e^{-t} - 1) + \overline{x}_1 \\
x_2 &= (\overline{x}_1 - \overline{x}_2)(e^t - 1) + (\overline{x}_2 - \overline{x}_1)(e^{-t} - 1) + \overline{x}_2 \\
x_3 &= \overline{x}_3
\end{aligned}
$$

then the displacement vector can be represented in the Lagrangian form

$$
\begin{aligned}
u_1 &= 2(\overline{x}_1 - \overline{x}_2)(e^t - 1) + (\overline{x}_2 - \overline{x}_1)(e^{-t} - 1) \\
u_2 &= (\overline{x}_1 - \overline{x}_2)(e^t - 1) + (\overline{x}_2 - \overline{x}_1)(e^{-t} - 1) \\
u_3 &= 0
\end{aligned}
$$

or the Eulerian form

$$
\begin{aligned}
u_1 &= x_1 - (2x_2 - x_1)(1 - e^{-t}) - (x_1 - x_2)(e^{-2t} - e^{-t}) - x_1 e^{-t} \\
u_2 &= x_2 - (2x_2 - x_1)(1 - e^{-t}) - (x_2 - x_1)(e^{-2t} - e^{-t}) - x_2 e^{-t} \\
u_3 &= 0.
\end{aligned}
$$

Note that in the Lagrangian system the displacements are expressed in terms of the initial position and time, while in the Eulerian system the independent variables are the position coordinates and time. Euler equations describe, as a function of time, how such things as density, pressure, and fluid velocity change at a fixed point in the medium. In contrast, the Lagrangian viewpoint follows the time history of a moving individual fluid particle as it moves through the medium.

General Tensor Derivation of Strain.

With reference to the figure 2.3-15 consider the deformation of a point P within a continuum. Let (y^1, y^2, y^3) denote a Cartesian coordinate system which is fixed in space. We can introduce a coordinate transformation $y^i = y^i(x^1, x^2, x^3), \quad i = 1, 2, 3$ and represent all points within the continuum with respect to a set of generalized coordinates (x^1, x^2, x^3). Let P denote a general point in the continuum while it is in an unstrained state and assume that this point gets transformed to a point P' when the continuum experiences external forces. If P moves to P', then all points Q which are near P will move to points Q' near P'. We can imagine that in the unstrained state all the points of the continuum are referenced with respect to the set of generalized coordinates (x^1, x^2, x^3). After the strain occurs, we can imagine that it will be convenient to represent all points of the continuum with respect to a new barred system of coordinates $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$. We call the original set of coordinates the Lagrangian system of coordinates and the new set of barred coordinates the Eulerian coordinates. The Eulerian coordinates are assumed to be described by a set of coordinate transformation equations $\overline{x}^i = \overline{x}^i(x^1, x^2, x^3), \quad i = 1, 2, 3$ with inverse transformations $x^i = x^i(\overline{x}^1, \overline{x}^2, \overline{x}^3), \quad i = 1, 2, 3$, which are assumed to exist. The barred and unbarred coordinates can be related to a fixed set of Cartesian coordinates $y^i, i = 1, 2, 3$, and we may assume that there exists transformation equations

$$y^i = y^i(x^1, x^2, x^3), \quad i = 1, 2, 3 \quad \text{and} \quad y^i = y^i(\overline{x}^1, \overline{x}^2, \overline{x}^3), \quad i = 1, 2, 3$$

which relate the barred and unbarred coordinates to the Cartesian axes. In the discussion that follows be sure to note whether there is a bar over a symbol, as we will be jumping back and forth between the Lagrangian and Eulerian reference frames.

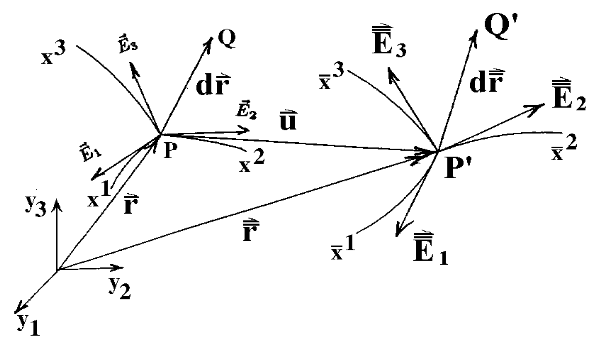

Figure 2.3-15. Strain in generalized coordinates

In the Lagrangian system of unbarred coordinates we have the basis vectors $\vec{E}_i = \dfrac{\partial \vec{r}}{\partial x^i}$ which produce the metrices $g_{ij} = \vec{E}_i \cdot \vec{E}_j$. Similarly, in the Eulerian system of barred coordinates we have the basis vectors $\vec{\overline{E}}_i = \dfrac{\partial \vec{\overline{r}}}{\partial \overline{x}^i}$ which produces the metrices $\overline{G}_{ij} = \vec{\overline{E}}_i \cdot \vec{\overline{E}}_j$. These basis vectors are illustrated in the figure 2.3-15.

We assume that an element of arc length squared ds^2 in the unstrained state is deformed to the element of arc length squared $d\overline{s}^2$ in the strained state. An element of arc length squared can be expressed in terms of the barred or unbarred coordinates. For example, in the Lagrangian system, let $d\vec{r} = \overline{PQ}$ so that

$$L_0^2 = d\vec{r} \cdot d\vec{r} = ds^2 = g_{ij} dx^i dx^j, \tag{2.3.54}$$

where g_{ij} are the metrices in the Lagrangian coordinate system. This same element of arc length squared can be expressed in the barred system by

$$L_0^2 = ds^2 = \overline{g}_{ij} d\overline{x}^i d\overline{x}^j, \quad \text{where} \quad \overline{g}_{ij} = g_{mn} \frac{\partial x^m}{\partial \overline{x}^i} \frac{\partial x^n}{\partial \overline{x}^j}. \tag{2.3.55}$$

Similarly, in the Eulerian system of coordinates the deformed arc length squared is

$$L^2 = d\vec{\overline{r}} \cdot d\vec{\overline{r}} = d\overline{s}^2 = \overline{G}_{ij} d\overline{x}^i d\overline{x}^j, \tag{2.3.56}$$

where $\overline{G}_{ij}$ are the metrices in the Eulerian system of coordinates. This same element of arc length squared can be expressed in the Lagrangian system by the relation

$$L^2 = d\overline{s}^2 = G_{ij} dx^i dx^j, \quad \text{where} \quad G_{ij} = \overline{G}_{mn} \frac{\partial \overline{x}^m}{\partial x^i} \frac{\partial \overline{x}^n}{\partial x^j}. \tag{2.3.57}$$

In the Lagrangian system we have

$$d\overline{s}^2 - ds^2 = (G_{ij} - g_{ij}) dx^i dx^j = 2e_{ij} dx^i dx^j$$

where

$$e_{ij} = \frac{1}{2} \left(G_{ij} - g_{ij}\right) \tag{2.3.58}$$

is called the Green strain tensor or Lagrangian strain tensor. Alternatively, in the Eulerian system of coordinates we may write

$$d\overline{s}^2 - ds^2 = \left(\overline{G}_{ij} - \overline{g}_{ij}\right) d\overline{x}^i d\overline{x}^j = 2\overline{e}_{ij} d\overline{x}^i d\overline{x}^j$$

where

$$\overline{e}_{ij} = \frac{1}{2} \left(\overline{G}_{ij} - \overline{g}_{ij}\right) \tag{2.3.59}$$

is called the Almansi strain tensor or Eulerian strain tensor.

Note also in the figure 2.3-15 there is the displacement vector $\vec{u}$. This vector can be represented in any of the following forms:

$$\begin{aligned}
&\vec{u} = u^i \vec{E}_i \quad \text{contravariant, Lagrangian basis}\\
&\vec{u} = u_i \vec{E}^i \quad \text{covariant, Lagrangian reciprocal basis}\\
&\vec{u} = \overline{u}^i \vec{\overline{E}}_i \quad \text{contravariant, Eulerian basis}\\
&\vec{u} = \overline{u}_i \vec{\overline{E}}^i \quad \text{covariant, Eulerian reciprocal basis.}
\end{aligned}$$

By vector addition we have $\vec{r} + \vec{u} = \vec{\overline{r}}$ and consequently $d\vec{r} + d\vec{u} = d\vec{\overline{r}}$. In the Lagrangian frame of reference at the point P we represent $\vec{u}$ in the contravariant form $\vec{u} = u^i \vec{E}_i$ and write $d\vec{r}$ in the form $d\vec{r} = dx^i \vec{E}_i$. By use of the equation (1.4.48) we can express $d\vec{u}$ in the form $d\vec{u} = u^i_{\ ,k} dx^k \vec{E}_i$. These substitutions produce the representation $d\vec{\overline{r}} = (dx^i + u^i_{\ ,k} dx^k)\vec{E}_i$ in the Lagrangian coordinate system. We can then express $d\overline{s}^2$ in the Lagrangian system. We find

$$\begin{aligned}
d\vec{\overline{r}} \cdot d\vec{\overline{r}} = d\overline{s}^2 &= (dx^i + u^i_{\ ,k} dx^k)\vec{E}_i \cdot (dx^j + u^j_{\ ,m} dx^m)\vec{E}_j\\
&= (dx^i dx^j + u^j_{\ ,m} dx^m dx^i + u^i_{\ ,k} dx^k dx^j + u^i_{\ ,k} u^j_{\ ,m} dx^k dx^m) g_{ij}
\end{aligned}$$

and consequently from the relation (2.3.58) we derive the representation

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{m,i} u^m_{\ ,j}\right). \tag{2.3.60}$$

This is the representation of the Lagrangian strain tensor in any system of coordinates. The strain tensor e_{ij} is symmetric. We will restrict our study to small deformations and neglect the product terms in equation (2.3.60). Under these conditions the equation (2.3.60) reduces to $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

If instead, we chose to represent the displacement $\vec{u}$ with respect to the Eulerian basis, then we can write

$$\vec{u} = \overline{u}^i \vec{\overline{E}}_i \quad \text{with} \quad d\vec{u} = \overline{u}^i_{\ ,k} d\overline{x}^k \vec{\overline{E}}_i.$$

These relations imply that

$$d\vec{r} = d\vec{\overline{r}} - d\vec{u} = (d\overline{x}^i - \overline{u}^i_{\ ,k} d\overline{x}^k)\vec{\overline{E}}_i.$$

This representation of $d\vec{r}$ in the Eulerian frame of reference can be used to calculate the strain $\overline{e}_{ij}$ from the relation $d\overline{s}^2 - ds^2$. It is left as an exercise to show that there results

$$\overline{e}_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} + \overline{u}_{j,i} - \overline{u}_{m,i} \overline{u}^m_{\ ,j}\right). \tag{2.3.61}$$

The equation (2.3.61) is the representation of the Eulerian strain tensor in any system of coordinates. Under conditions of small deformations both the equations (2.3.60) and (2.3.61) reduce to the linearized Lagrangian and Eulerian strain tensor $\overline{e}_{ij} = \frac{1}{2}(\overline{u}_{i,j} + \overline{u}_{j,i})$. In the case of large deformations the equations (2.3.60) and (2.3.61) describe the strains. In the case of linear elasticity, where the deformations are very small, the product terms in equations (2.3.60) and (2.3.61) are neglected and the Lagrangian and Eulerian strains reduce to their linearized forms

$$e_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right] \qquad \overline{e}_{ij} = \frac{1}{2}\left[\overline{u}_{i,j} + \overline{u}_{j,i}\right]. \tag{2.3.62}$$

■

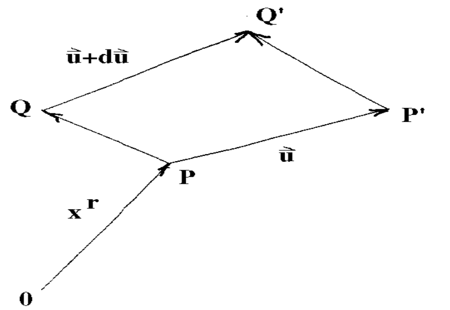

Figure 2.3-16. Displacement due to strain

Compressible and Incompressible Material With reference to figure 2.3-16, let x^i, $i = 1, 2, 3$ denote the position vector of an arbitrary point P in a continuum before there is a state of strain. Let Q be a neighboring point of P with position vector $x^i + dx^i$, $i = 1, 2, 3$. Also in the figure 2.3-16 there is the displacement vector $\vec{u}$. Here it is assumed that $\vec{u} = \vec{u}(x^1, x^2, x^3)$ denotes the displacement field when the continuum is in a state of strain. The figure 2.3-16 illustrates that in a state of strain P moves to P' and Q moves to Q'. Let us find a relationship between the distance $\overline{PQ}$ before the strain and the distance $\overline{P'Q'}$ when the continuum is in a state of strain. For $\vec{E}_1, \vec{E}_2, \vec{E}_3$ basis functions constructed at P we have previously shown that if

$$\vec{u}(x^1, x^2, x^3) = u^i \vec{E}_i \qquad \text{then} \qquad d\vec{u} = u^i_{\ ,j} dx^j \, \vec{E}_i.$$

Now for $\vec{u} + d\vec{u}$ the displacement of the point Q we may use vector addition and write

$$\overline{PQ} + \vec{u} + d\vec{u} = \vec{u} + \overline{P'Q'}. \tag{2.3.63}$$

Let $\overline{PQ} = dx^i \vec{E}_i = a^i \vec{E}_i$ denote an arbitrary small change in the continuum. This arbitrary displacement gets deformed to $\overline{P'Q'} = A^i \vec{E}_i$ due to the state of strain in the continuum. Employing the equation (2.3.63) we write

$$dx^i + u^i_{\ ,j} dx^j = a^i + u^i_{\ ,j} a^j = A^i$$

which can be written in the form

$$\delta a^i = A^i - a^i = u^i_{\ ,j} a^j \quad \text{where} \quad dx^i = a^i, i = 1, 2, 3 \tag{2.3.64}$$

denotes an arbitrary small change. The tensor $u^i_{\ ,j}$ and the associated tensor $u_{i,j} = g_{it} u^t_{\ ,j}$ are in general not symmetric tensors. However, we know we can express $u_{i,j}$ as the sum of a symmetric (e_{ij}) and skew-symmetric(ω_{ij}) tensor. We therefore write

$$u_{i,j} = e_{ij} + \omega_{ij} \quad \text{or} \quad u^i_{\ ,j} = e^i_{\ j} + \omega^i_{\ j},$$

where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}(g_{im} u^m_{\ \ ,j} + g_{jm} u^m_{\ \ ,i}) \qquad \text{and} \qquad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \frac{1}{2}(g_{im} u^m_{\ \ ,j} - g_{jm} u^m_{\ \ ,i}).$$

The deformation of a small quantity a^i can therefore be represented by a pure strain $A^i - a^i = e^i_s a^s$ followed by a rotation $A^i - a^i = \omega^i_s a^s$.

Consider now a small element of volume inside a material medium. With reference to the figure 2.3-17(a) we let $\vec{a}, \vec{b}, \vec{c}$ denote three small arbitrary independent vectors constructed at a general point P within the material before any external forces are applied. We imagine $\vec{a}, \vec{b}, \vec{c}$ as representing the sides of a small parallelepiped before any deformation has occurred. When the material is placed in a state of strain the point P will move to P' and the vectors $\vec{a}, \vec{b}, \vec{c}$ will become deformed to the vectors $\vec{A}, \vec{B}, \vec{C}$ as illustrated in the figure 2.3-17(b). The vectors $\vec{A}, \vec{B}, \vec{C}$ represent the sides of the parallelepiped after the deformation.

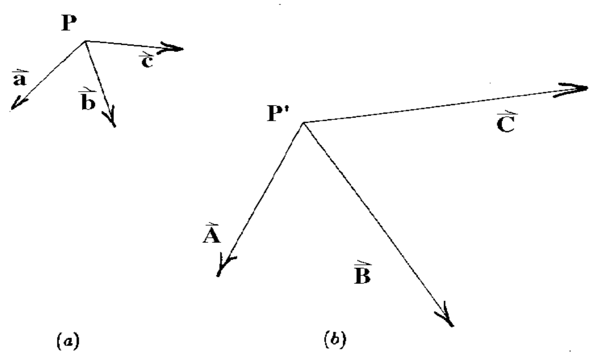

Figure 2.3-17. Deformation of a parallelepiped

Let ΔV denote the volume of the parallelepiped with sides $\vec{a}, \vec{b}, \vec{c}$ at P before the strain and let $\Delta V'$ denote the volume of the deformed parallelepiped after the strain, when it then has sides $\vec{A}, \vec{B}, \vec{C}$ at the point P'. We define the ratio of the change in volume due to the strain divided by the original volume as the dilatation at the point P. The dilatation is thus expressed as

$$\Theta = \frac{\Delta V' - \Delta V}{\Delta V} = dilatation. \tag{2.3.65}$$

Since $u^i, i = 1, 2, 3$ represents the displacement field due to the strain, we use the result from equation (2.3.64) and represent the displaced vectors $\vec{A}, \vec{B}, \vec{C}$ in the form

$$\begin{aligned} A^i &= a^i + u^i_{\ ,j} a^j \\ B^i &= b^i + u^i_{\ ,j} b^j \\ C^i &= c^i + u^i_{\ ,j} c^j \end{aligned} \tag{2.3.66}$$

where $\vec{a}, \vec{b}, \vec{c}$ are arbitrary small vectors emanating from the point P in the unstrained state. The element of volume ΔV, before the strain, is calculated from the triple scalar product relation

$$\Delta V = \vec{a} \cdot (\vec{b} \times \vec{c}) = e_{ijk} a^i b^j c^k.$$

The element of volume $\Delta V'$, which occurs due to the strain, is calculated from the triple scalar product

$$\Delta V' = \vec{A} \cdot (\vec{B} \times \vec{C}) = e_{ijk} A^i B^j C^k.$$

Substituting the relations from the equations (2.3.66) into the triple scalar product gives

$$\Delta V' = e_{ijk}(a^i + u^i_{,m}a^m)(b^j + u^j_{,n}b^n)(c^k + u^k_{,p}c^p).$$

Expanding the triple scalar product and employing the result from Exercise 1.4, problem 34, we find the simplified result gives us the dilatation

$$\Theta = \frac{\Delta V' - \Delta V}{\Delta V} = u^r_{,r} = \text{div }(\vec{u}). \tag{2.3.67}$$

That is, the dilatation is the divergence of the displacement field. If the divergence of the displacement field is zero, there is no volume change and the material is said to be incompressible. If the divergence of the displacement field is different from zero, the material is said to be compressible.

Note that the strain e_{ij} is expressible in terms of the displacement field by the relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{and consequently} \quad g^{mn}e_{mn} = u^r_{,r}. \tag{2.3.68}$$

Hence, for an orthogonal system of coordinates the dilatation can be expressed in terms of the strain elements along the main diagonal.

Conservation of Mass

Consider the material in an arbitrary region R of a continuum. Let $\varrho = \varrho(x, y, z, t)$ denote the density of the material within the region. Assume that the dimension of the density ϱ is gm/cm^3 in the cgs system of units. We shall assume that the region R is bounded by a closed surface S with exterior unit normal $\vec{n}$ defined everywhere on the surface. Further, we let $\vec{v} = \vec{v}(x, y, z, t)$ denote a velocity field associated with all points within the continuum. The velocity field has units of cm/sec in the cgs system of units. Neglecting sources and sinks, the law of conservation of mass examines all the material entering and leaving a region R. Enclosed within R is the material mass m where $m = \iiint_R \varrho\, d\tau$ with dimensions of gm in the cgs system of units. Here $d\tau$ denotes an element of volume inside the region R. The change of mass with time is obtained by differentiating the above relation. Differentiating the mass produces the equation

$$\frac{\partial m}{\partial t} = \iiint_R \frac{\partial \varrho}{\partial t}\, d\tau \tag{2.3.69}$$

and has the dimensions of gm/sec.

Consider also the surface integral

$$I = \iint_S \varrho \vec{v} \cdot \hat{n}\, d\sigma \tag{2.3.70}$$

where $d\sigma$ is an element of surface area on the surface S which encloses R and $\hat{n}$ is the exterior unit normal vector to the surface S. The dimensions of the integral I is determined by examining the dimensions of each term in the integrand of I. We find that

$$[I] = \frac{gm}{cm^3} \cdot \frac{cm}{sec} \cdot cm^2 = \frac{gm}{sec}$$

and so the dimension of I is the same as the dimensions for the change of mass within the region R. The surface integral I is the flux rate of material crossing the surface of R and represents the change of mass entering the region if $\vec{v}\cdot\hat{n}$ is negative and the change of mass leaving the region if $\vec{v}\cdot\hat{n}$ is positive, as $\hat{n}$ is always an exterior unit normal vector. Equating the relations from equations (2.3.69) and (2.3.70) we obtain a mathematical statement for mass conservation

$$\frac{\partial m}{\partial t} = \iiint_R \frac{\partial \varrho}{\partial t}\, d\tau = -\iint_S \varrho\vec{v}\cdot\vec{n}\, d\sigma. \tag{2.3.71}$$

The equation (2.3.71) implies that the rate at which the mass contained in R increases must equal the rate at which the mass flows into R through the surface S. The negative sign changes the direction of the exterior normal so that we consider flow of material into the region. Employing the Gauss divergence theorem, the surface integral in equation (2.3.71) can be replaced by a volume integral and the law of conservation of mass is then expressible in the form

$$\iiint_R \left[\frac{\partial \varrho}{\partial t} + \operatorname{div}\,(\varrho\vec{v})\right] d\tau = 0. \tag{2.3.72}$$

Since the region R is an arbitrary volume we conclude that the term inside the brackets must equal zero. This gives us the continuity equation

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}\,(\varrho\vec{v}) = 0 \tag{2.3.73}$$

which represents the mass conservation law in terms of velocity components. This is the Eulerian representation of continuity of mass flow.

Equivalent forms of the continuity equation are:

$$\begin{aligned}\frac{\partial \varrho}{\partial t} + \vec{v}\cdot\operatorname{grad}\,\varrho + \varrho\operatorname{div}\,\vec{v} &= 0\\ \frac{\partial \varrho}{\partial t} + v_i\frac{\partial \varrho}{\partial x^i} + \varrho\frac{\partial v_i}{\partial x^i} &= 0\\ \frac{D\varrho}{Dt} + \varrho\frac{\partial v_i}{\partial x^i} &= 0\end{aligned}$$

where $\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x^i}v_i$ is called the material derivative of the density ϱ. Note that the material derivative contains the expression $\frac{\partial\varrho}{\partial x^i}v_i$ which is known as the convective or advection term. If the density $\varrho = \varrho(x,y,z,t)$ is a constant we have

$$\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x}\frac{dx}{dt} + \frac{\partial \varrho}{\partial y}\frac{dy}{dt} + \frac{\partial \varrho}{\partial z}\frac{dz}{dt} = \frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x^i}\frac{dx^i}{dt} = 0 \tag{2.3.74}$$

and hence the continuity equation reduces to $\operatorname{div}\,(\vec{v}) = 0$. Thus, if $\operatorname{div}\,(\vec{v})$ is zero, then the material is incompressible.

EXAMPLE 2.3-2. (Continuity Equation) Find the Lagrangian representation of mass conservation.

Solution: Let (X,Y,Z) denote the initial position of a fluid particle and denote the density of the fluid by $\varrho(X,Y,Z,t)$ so that $\varrho(X,Y,Z,0)$ denotes the density at the time $t=0$. Consider a simple closed region in our continuum and denote this region by $R(0)$ at time $t=0$ and by $R(t)$ at some later time t. That is, all the points in $R(0)$ move in a one-to-one fashion to points in $R(t)$. Initially the mass of material in $R(0)$ is $m(0) = \iiint_{R(0)} \varrho(X,Y,Z,0)\, d\tau(0)$ where $d\tau(0) = dX\,dY\,dZ$ is an element of volume in $R(0)$. We have after a time t has elapsed the mass of material in the region $R(t)$ given by $m(t) = \iiint_{R(t)} \varrho(X,Y,Z,t)\, d\tau(t)$ where $d\tau(t) = dxdydz$ is a deformed element of volume related to the $d\tau(0)$ by $d\tau(t) = J\left(\frac{x,y,z}{X,Y,Z}\right) d\tau(0)$ where J is the Jacobian of the Eulerian (x,y,z) variables with respect to the Lagrangian (X,Y,Z) representation. For mass conservation we require that $m(t) = m(0)$ for all t. This implies that

$$\varrho(X,Y,Z,t)J = \varrho(X,Y,Z,0) \tag{2.3.75}$$

for all time, since the initial region $R(0)$ is arbitrary. The right hand side of equation (2.3.75) is independent of time and so

$$\frac{d}{dt}\left(\varrho(X,Y,Z,t)J\right) = 0. \tag{2.3.76}$$

This is the Lagrangian form of the continuity equation which expresses mass conservation. Using the result that $\frac{dJ}{dt} = J\mathrm{div}\,\vec{V}$, (see problem 28, Exercise 2.3), the equation (2.3.76) can be expanded and written in the form

$$\frac{D\varrho}{Dt} + \varrho\,\mathrm{div}\,\vec{V} = 0 \tag{2.3.77}$$

where $\frac{D\varrho}{Dt}$ is from equation (2.3.74). The form of the continuity equation (2.3.77) is one of the Eulerian forms previously developed.

■

In the Eulerian coordinates the continuity equation is written $\frac{\partial\varrho}{\partial t} + \mathrm{div}\,(\varrho\vec{v}) = 0$, while in the Lagrangian system the continuity equation is written $\frac{d(\varrho\, J)}{dt} = 0$. Note that the velocity carries the Lagrangian axes and the density change $\mathrm{grad}\,\varrho$. This is reflective of the advection term $\vec{v}\cdot\mathrm{grad}\,\varrho$. Thus, in order for mass to be conserved it need not remain stationary. The mass can flow and the density can change. The material derivative is a transport rule depicting the relation between the Eulerian and Lagrangian viewpoints.

In general, from a Lagrangian viewpoint, any quantity $Q(x,y,z,t)$ which is a function of both position and time is seen as being transported by the fluid velocity (v_1,v_2,v_3) to $Q(x+v_1dt, y+v_2dt, z+v_3dt, t+dt)$. Then the time derivative of Q contains both $\frac{\partial Q}{\partial t}$ and the advection term $\vec{v}\cdot\nabla Q$. In terms of mass flow, the Eulerian viewpoint sees flow into and out of a fixed volume in space, as depicted by the equation (2.3.71), In contrast, the Lagrangian viewpoint sees the same volume moving with the fluid and consequently

$$\frac{D}{Dt}\int\int\int_{R(t)} \rho\, d\tau = 0,$$

where $R(t)$ represents the volume moving with the fluid. Both viewpoints produce the same continuity equation reflecting the conservation of mass.

Summary of Basic Equations

Let us summarize the basic equations which are valid for all types of a continuum. We have derived:

- Conservation of mass (continuity equation)

$$\frac{\partial\varrho}{\partial t} + (\varrho v^i)_{,i} = 0$$

- Conservation of linear momentum sometimes called the Cauchy equation of motion.

$$\sigma^{ij}{}_{,i} + \varrho b^j = \varrho f^j, \quad j = 1, 2, 3.$$

- Conservation of angular momentum

$$\sigma_{ij} = \sigma_{ji}$$

- Strain tensor for linear elasticity

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

If we assume that the continuum is in equilibrium, and there is no motion, then the velocity and acceleration terms above will be zero. The continuity equation then implies that the density is a constant. The conservation of angular momentum equation requires that the stress tensor be symmetric and we need find only six stresses. The remaining equations reduce to a set of nine equations in the fifteen unknowns:

$$\begin{array}{ll} \text{3 displacements} & u_1, u_2, u_3 \\ \text{6 strains} & e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33} \\ \text{6 stresses} & \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33} \end{array}$$

Consequently, we still need additional information if we desire to determine these unknowns.

Note that the above equations do not involve any equations describing the material properties of the continuum. We would expect solid materials to act differently from liquid material when subjected to external forces. An equation or equations which describe the material properties are called constitutive equations. In the following sections we will investigate constitutive equations for solids and liquids. We will restrict our study to linear elastic materials over a range where there is a linear relationship between the stress and strain. We will not consider plastic or viscoelastic materials. Viscoelastic materials have the property that the stress is not only a function of strain but also a function of the rates of change of the stresses and strains and consequently properties of these materials are time dependent.

EXERCISE 2.3

▶ **1.** Assume an orthogonal coordinate system with metric tensor $g_{ij} = 0$ for $i \neq j$ and $g_{(i)(i)} = h_i^2$ (no summation on i). Use the definition of strain

$$e_{rs} = \frac{1}{2}(u_{r,s} + u_{s,r}) = \frac{1}{2}\left(g_{rt}u^t_{\ ,s} + g_{st}u^t_{\ ,r}\right)$$

and show that in terms of the physical components

$$e(ij) = \frac{e_{ij}}{h_i h_j} \quad \text{no summation on } i \text{ or } j$$

$$u(i) = h_i u^i \quad \text{no summation on } i$$

there results the equations:

$$e_{ii} = g_{it}\left[\frac{\partial u^t}{\partial x^i} + \left\{ {t \atop m\, i} \right\} u^m\right] \quad \text{no summation on } i$$

$$2e_{ij} = g_{it}\frac{\partial u^t}{\partial x^j} + g_{jt}\frac{\partial u^t}{\partial x^i}, \qquad i \neq j$$

$$e(ii) = \frac{\partial}{\partial x^i}\left(\frac{u(i)}{h_i}\right) + \frac{1}{2h_i^2}\sum_{m=1}^{3}\frac{u(m)}{h_m}\frac{\partial}{\partial x^m}\left(h_i^2\right) \quad \text{no summation on } i$$

$$2e(ij) = \frac{h_i}{h_j}\frac{\partial}{\partial x^j}\left(\frac{u(i)}{h_i}\right) + \frac{h_j}{h_i}\frac{\partial}{\partial x^i}\left(\frac{u(j)}{h_j}\right), \quad \text{no summation on } i \text{ or } j,\ i \neq j.$$

▶ **2.** Use the results from problem 1 to write out all components of the strain tensor in Cartesian coordinates. Use the notation $u(1) = u, u(2) = v, u(3) = w$ and

$$e(11) = e_{xx}, \quad e(22) = e_{yy}, \quad e(33) = e_{zz}, \quad e(12) = e_{xy}, \quad e(13) = e_{xz}, \quad e(23) = e_{yz}$$

to verify the relations:

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$e_{yy} = \frac{\partial v}{\partial y} \qquad e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$e_{zz} = \frac{\partial w}{\partial z} \qquad e_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

▶ **3.** Use the results from problem 1 to write out all components of the strain tensor in cylindrical coordinates. Use the notation $u(1) = u_r$, $u(2) = u_\theta$, $u(3) = u_z$ and

$$e(11) = e_{rr}, \quad e(22) = e_{\theta\theta}, \quad e(33) = e_{zz}, \quad e(12) = e_{r\theta}, \quad e(13) = e_{rz}, \quad e(23) = e_{\theta z}$$

to verify the relations:

$$e_{rr} = \frac{\partial u_r}{\partial r} \qquad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad e_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

$$e_{zz} = \frac{\partial u_z}{\partial z} \qquad e_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

▶ **4.** Use the results from problem 1 to write out all components of the strain tensor in spherical coordinates. Use the notation $u(1) = u_\rho, u(2) = u_\theta, u(3) = u_\phi$ and

$$e(11) = e_{\rho\rho}, \quad e(22) = e_{\theta\theta}, \quad e(33) = e_{\phi\phi}, \quad e(12) = e_{\rho\theta}, \quad e(13) = e_{\rho\phi}, \quad e(23) = e_{\theta\phi}$$

to verify the relations

$$\begin{aligned} e_{\rho\rho} &= \frac{\partial u_\rho}{\partial \rho} & e_{\rho\theta} &= \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_\rho}{\partial \theta} - \frac{u_\theta}{\rho} + \frac{\partial u_\theta}{\partial \rho}\right) \\ e_{\theta\theta} &= \frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\rho}{\rho} & e_{\rho\phi} &= \frac{1}{2}\left(\frac{1}{\rho\sin\theta}\frac{\partial u_\rho}{\partial \phi} - \frac{u_\phi}{\rho} + \frac{\partial u_\phi}{\partial \rho}\right) \\ e_{\phi\phi} &= \frac{1}{\rho\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\rho}{\rho} + \frac{u_\theta}{\rho}\cot\theta & e_{\theta\phi} &= \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{\rho}\cot\theta + \frac{1}{\rho\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right) \end{aligned}$$

▶ **5.** Expand equation (2.3.67) and find the dilatation in terms of the physical components of an orthogonal system and verify that

$$\Theta = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3u(1))}{\partial x^1} + \frac{\partial(h_1h_3u(2))}{\partial x^2} + \frac{\partial(h_1h_2u(3))}{\partial x^3}\right]$$

▶ **6.** Verify that the dilatation in Cartesian coordinates is

$$\Theta = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

▶ **7.** Verify that the dilatation in cylindrical coordinates is

$$\Theta = e_{rr} + e_{\theta\theta} + e_{zz} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z}.$$

▶ **8.** Verify that the dilatation in spherical coordinates is

$$\Theta = e_{\rho\rho} + e_{\theta\theta} + e_{\phi\phi} = \frac{\partial u_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} + \frac{2}{\rho}u_\rho + \frac{1}{\rho\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta\cot\theta}{\rho}.$$

▶ **9.** Show that in an orthogonal set of coordinates the rotation tensor ω_{ij} can be written in terms of physical components in the form

$$\omega(ij) = \frac{1}{2h_ih_j}\left[\frac{\partial(h_iu(i))}{\partial x^j} - \frac{\partial(h_ju(j))}{\partial x^i}\right], \quad \text{no summations}$$

Hint: See problem 1.

▶ **10.** Use the result from problem 9 to verify that in Cartesian coordinates

$$\begin{aligned} \omega_{yx} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ \omega_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_{zy} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \end{aligned}$$

▶ **11.** Use the results from problem 9 to verify that in cylindrical coordinates

$$\omega_{\theta r} = \frac{1}{2r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right]$$
$$\omega_{rz} = \frac{1}{2}\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right]$$
$$\omega_{z\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right]$$

▶ **12.** Use the results from problem 9 to verify that in spherical coordinates

$$\omega_{\theta\rho} = \frac{1}{2\rho}\left[\frac{\partial(\rho u_\theta)}{\partial \rho} - \frac{\partial u_\rho}{\partial \theta}\right]$$
$$\omega_{\rho\phi} = \frac{1}{2\rho}\left[\frac{1}{\sin\theta}\frac{\partial u_\rho}{\partial \phi} - \frac{\partial(\rho u_\phi)}{\partial \rho}\right]$$
$$\omega_{\phi\theta} = \frac{1}{2\rho\sin\theta}\left[\frac{\partial(u_\phi \sin\theta)}{\partial \theta} - \frac{\partial u_\theta}{\partial \phi}\right]$$

▶ **13.** The conditions for static equilibrium in a linear elastic material are determined from the conservation law

$$\sigma_{i\ ,j}^{j} + \varrho b_i = 0, \quad i,j = 1,2,3,$$

where σ_j^i are the stress tensor components, b_i are the external body forces per unit mass and ϱ is the density of the material. Assume an orthogonal coordinate system and verify the following results.

(a) Show that

$$\sigma_{i\ ,j}^{j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\sigma_i^j) - [ij,m]\sigma^{mj}$$

(b) Use the substitutions

$$\sigma(ij) = \sigma_i^j \frac{h_j}{h_i} \quad \text{no summation on } i \text{ or } j$$
$$b(i) = \frac{b_i}{h_i} \quad \text{no summation on } i$$
$$\sigma(ij) = \sigma^{ij} h_i h_j \quad \text{no summation on } i \text{ or } j$$

and express the equilibrium equations in terms of physical components and verify the relations

$$\sum_{j=1}^{3} \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}h_i\sigma(ij)}{h_j}\right) - \frac{1}{2}\sum_{j=1}^{3}\frac{\sigma(jj)}{h_j^2}\frac{\partial(h_j^2)}{\partial x^i} + h_i\varrho b(i) = 0,$$

where there is no summation on i.

▶ **14.** Use the results from problem 13 and verify that the equilibrium equations in Cartesian coordinates can be expressed

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \varrho b_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \varrho b_y = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \varrho b_z = 0$$

▶ **15.** Use the results from problem 13 and verify that the equilibrium equations in cylindrical coordinates can be expressed

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \varrho b_r = 0$$
$$\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{\theta r} + \varrho b_\theta = 0$$
$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{zr} + \varrho b_z = 0$$

▶ **16.** Use the results from problem 13 and verify that the equilibrium equations in spherical coordinates can be expressed

$$\frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\rho\theta}}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial \sigma_{\rho\phi}}{\partial \phi} + \frac{1}{\rho}\left(2\sigma_{\rho\rho} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{\rho\theta}\cot\theta\right) + \varrho b_\rho = 0$$
$$\frac{\partial \sigma_{\theta\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{\rho}\left(3\sigma_{\rho\theta} + [\sigma_{\theta\theta} - \sigma_{\phi\phi}]\cot\theta\right) + \varrho b_\theta = 0$$
$$\frac{\partial \sigma_{\phi\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\phi\theta}}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{\rho}\left(3\sigma_{\rho\phi} + 2\sigma_{\theta\phi}\cot\theta\right) + \varrho b_\phi = 0$$

▶ **17.** Derive the result for the Lagrangian strain defined by the equation (2.3.60).

▶ **18.** Derive the result for the Eulerian strain defined by equation (2.3.61).

▶ **19.** The equation $\delta a^i = u^i{}_{,j} a^j$, describes the deformation in an elastic solid subjected to forces. The quantity δa^i denotes the difference vector $A^i - a^i$ between the undeformed and deformed states.

(a) Let $|a|$ denote the magnitude of the vector a^i and show that the strain e in the direction a^i can be represented

$$e = \frac{\delta |a|}{|a|} = e_{ij}\left(\frac{a^i}{|a|}\right)\left(\frac{a^j}{|a|}\right) = e_{ij}\lambda^i\lambda^j,$$

where λ^i is a unit vector in the direction a^i.

(b) Show that for $\lambda^1 = 1, \lambda^2 = 0, \lambda^3 = 0$ there results $e = e_{11}$, with similar results applying to vectors λ^i in the y and z directions.

Hint: Consider the magnitude squared $|a|^2 = g_{ij}a^i a^j$.

▶ **20.** At the point $(1, 2, 3)$ of an elastic solid construct the small vector $\vec{a} = \epsilon(\frac{2}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{1}{3}\,\hat{\mathbf{e}}_3)$, where $\epsilon > 0$ is a small positive quantity. The solid is subjected to forces such that the following displacement field results.

$$\vec{u} = (xy\,\hat{\mathbf{e}}_1 + yz\,\hat{\mathbf{e}}_2 + xz\,\hat{\mathbf{e}}_3) \times 10^{-2}$$

Calculate the deformed vector $\vec{A}$ after the displacement field has been imposed.

▶ **21.** For the displacement field

$$\vec{u} = (x^2 + yz)\,\hat{\mathbf{e}}_1 + (xy + z^2)\,\hat{\mathbf{e}}_2 + xyz\,\hat{\mathbf{e}}_3$$

(a) Calculate the strain matrix at the point $(1, 2, 3)$.

(b) Calculate the rotation matrix at the point $(1, 2, 3)$.

▶ **22.** Show that for an orthogonal coordinate system the ith component of the convective operator can be written

$$[(\vec{V} \cdot \nabla)\,\vec{A}]_i = \sum_{m=1}^{3} \frac{V(m)}{h_m} \frac{\partial A(i)}{\partial x^m} + \sum_{\substack{m=1 \\ m \neq i}}^{3} \frac{A(m)}{h_m h_i} \left(V(i) \frac{\partial h_i}{\partial x^m} - V(m) \frac{\partial h_m}{\partial x^i} \right)$$

▶ **23.** Consider a parallelepiped with dimensions ℓ, w, h which has a uniform pressure P applied to each face. Show that the volume strain can be expressed as

$$\frac{\Delta V}{V} = \frac{\Delta \ell}{\ell} + \frac{\Delta w}{w} + \frac{\Delta h}{h} = \frac{-3P(1-2\nu)}{E}.$$

The quantity $k = E/3(1-2\nu)$ is called the bulk modulus of elasticity.

▶ **24.** Show in Cartesian coordinates the continuity equation is

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} + \frac{\partial(\varrho w)}{\partial z} = 0,$$

where (u, v, w) are the velocity components.

▶ **25.** Show in cylindrical coordinates the continuity equation is

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r} \frac{\partial(r \varrho V_r)}{\partial r} + \frac{1}{r} \frac{\partial(\varrho V_\theta)}{\partial \theta} + \frac{\partial(\varrho V_z)}{\partial z} = 0$$

where V_r, V_θ, V_z are the velocity components.

▶ **26.** Show in spherical coordinates the continuity equation is

$$\frac{\partial \varrho}{\partial t} + \frac{1}{\rho^2} \frac{\partial(\rho^2 \varrho V_\rho)}{\partial \rho} + \frac{1}{\rho \sin\theta} \frac{\partial(\varrho V_\theta \sin\theta)}{\partial \theta} + \frac{1}{\rho \sin\theta} \frac{\partial(\varrho V_\phi)}{\partial \phi} = 0$$

where V_ρ, V_θ, V_ϕ are the velocity components.

▶ **27.** (a) Apply a stress σ_{yy} to both ends of a square element in a x, y continuum. Illustrate and label all changes that occur due to this stress. (b) Apply a stress σ_{xx} to both ends of a square element in a x, y continuum. Illustrate and label all changes that occur due to this stress. (c) Use superposition of your results in parts (a) and (b) and explain each term in the relations

$$e_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E} \qquad \text{and} \qquad e_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{xx}}{E}.$$

▶ **28.** Show that the time derivative of the Jacobian $J = J\left(\frac{x, y, z}{X, Y, Z}\right)$ satisfies $\frac{dJ}{dt} = J \operatorname{div} \vec{V}$ where

$$\operatorname{div} \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \quad \text{and} \quad V_1 = \frac{dx}{dt}, \quad V_2 = \frac{dy}{dt}, \quad V_3 = \frac{dz}{dt}.$$

Hint: Let $(x, y, z) = (x_1, x_2, x_3)$ and $(X, Y, Z) = (X_1, X_2, X_3)$, then note that

$$e_{ijk} \frac{\partial V_1}{\partial X_i} \frac{\partial x_2}{\partial X_j} \frac{\partial x_3}{\partial X_k} = e_{ijk} \frac{\partial V_1}{\partial x_m} \frac{\partial x_m}{\partial X_i} \frac{\partial x_2}{\partial X_j} \frac{\partial x_3}{\partial X_k} = e_{ijk} \frac{\partial x_1}{\partial X_i} \frac{\partial x_2}{\partial X_j} \frac{\partial x_3}{\partial X_k} \frac{\partial V_1}{\partial x_1}, \quad \text{etc.}$$