The velocity and momentum four-vectors

1. The four-velocity vector

The velocity four-vector of a particle is defined by:

$$U^\mu = \frac{dx^\mu}{d\tau} = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}})\,, \tag{1}$$

where $x^\mu = (ct\,;\,\vec{\boldsymbol{x}})$ is the four-position vector and $d\tau$ is the differential proper time. To derive eq. (1), we must express $d\tau$ in terms of dt, where t is the time coordinate. Consider the infinitesimal invariant spacetime separation,

$$ds^2 = -c^2 d\tau^2 = \eta_{\mu\nu}dx^\mu dx^\nu\,, \tag{2}$$

in a convention where

$$\eta_{\mu\nu} = \text{diag}(-1\,,\,1\,,\,1\,,\,1)\,.$$

In eq. (2), there is an implicit sum over the repeated indices as dictated by the Einstein summation convention. Dividing by $-c^2$ yields

$$d\tau^2 = \frac{1}{c^2}\left(c^2dt^2 - \sum_{i=1}^3 dx^i dx^i\right) = \frac{c^2-v^2}{c^2}dt^2 = \left(1-\frac{v^2}{c^2}\right)dt^2 = \gamma^{-2}dt^2\,,$$

where we have employed the three-velocity $v^i = dx^i/dt$ and $v^2 \equiv \sum_i v^i v^i$. In the last step we have introduced $\gamma \equiv (1-v^2/c^2)^{-1/2}$. It follows that

$$d\tau = \gamma^{-1}\,dt\,. \tag{3}$$

Using eq. (3) and the definition of the three-velocity, $\vec{\boldsymbol{v}} = d\vec{\boldsymbol{x}}/dt$, we easily obtain eq. (1).

Note that the squared magnitude of the four-velocity vector,

$$U^2 \equiv \eta_{\mu\nu}U^\mu U^\nu = -c^2 \tag{4}$$

is a Lorentz invariant, which is most easily evaluated in the rest frame of the particle where $\vec{\boldsymbol{v}} = 0$, in which case $U^\mu = c\,(1\,;\,\vec{\boldsymbol{0}})$.

2. The relativistic law of addition of velocities

Let us now consider the following question. Suppose that in an inertial frame K' is moving with relative velocity $\vec{\boldsymbol{w}}$ with respect to an inertial frame K. Given a particle with three-velocity $\vec{\boldsymbol{v}}\boldsymbol{'}$ as seen in inertial frame K', what is the three-velocity $\vec{\boldsymbol{v}}$ as seen by an observer in inertial frame K? To answer this question, we consider the corresponding four-velocity vectors in inertial frames K and K',

$$U^\mu = (U^0\,;\,\vec{\boldsymbol{U}}) = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}}) \quad\text{and}\quad U'^{\,\mu} = (U'^{\,0}\,;\,\vec{\boldsymbol{U}}\boldsymbol{'}) = (\gamma' c\,;\,\gamma'\vec{\boldsymbol{v}}\boldsymbol{'})\,,$$

where $\gamma \equiv (1-v^2/c^2)^{-1/2}$ and $\gamma' \equiv (1-v'^2/c^2)^{-1/2}$, respectively.

The two four-vectors U^μ and U'^μ are related by a Lorentz boost,

$$U'^\mu = \Lambda^\mu{}_\nu U^\nu \tag{5}$$

with the boost matrix $\Lambda^\mu{}_\nu$ given by[1]

$$\Lambda = \begin{pmatrix} \gamma_w & -\gamma_w \vec{\boldsymbol{\beta}} \\ -\gamma_w \vec{\boldsymbol{\beta}} & \delta^{ij} + (\gamma_w - 1)\dfrac{\beta^i \beta^j}{\beta^2} \end{pmatrix}, \tag{6}$$

where $\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{w}}/c$, $\beta \equiv |\vec{\boldsymbol{\beta}}|$ and $\gamma_w \equiv (1-\beta^2)^{-1/2}$. Then, eqs. (5) and (6) imply that:

$$U'^0 = \gamma_w (U^0 - \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{U}}\,), \tag{7}$$

$$\vec{\boldsymbol{U}}' = \vec{\boldsymbol{U}} + \frac{(\gamma_w - 1)}{\beta^2} (\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{U}})\, \vec{\boldsymbol{\beta}} - \gamma_w \vec{\boldsymbol{\beta}} U^0 . \tag{8}$$

Actually, we need the inverse transformation, which can be obtained by interchanging the primed and unprimed variables and replacing $\vec{\boldsymbol{\beta}}$ with $-\vec{\boldsymbol{\beta}}$,

$$U^0 = \gamma_w (U'^0 + \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{U}}'\,), \tag{9}$$

$$\vec{\boldsymbol{U}} = \vec{\boldsymbol{U}}' + \frac{(\gamma_w - 1)}{\beta^2} (\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{U}}')\, \vec{\boldsymbol{\beta}} + \gamma_w \vec{\boldsymbol{\beta}} U'^0 . \tag{10}$$

Dividing these two equations yields:

$$\frac{\vec{\boldsymbol{U}}}{U^0} = \frac{1}{U'^0 + \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{U}}'} \left[\frac{\vec{\boldsymbol{U}}'}{\gamma_w} + \frac{(\gamma_w - 1)}{\gamma_w \beta^2} (\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{U}}')\, \vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}} U'^0 \right] . \tag{11}$$

Substituting $U'^0 = \gamma' c$, $\vec{\boldsymbol{U}}' = \gamma' \vec{\boldsymbol{v}}'$, and $\vec{\boldsymbol{U}}/U^0 = \vec{\boldsymbol{v}}/c$ in eq. (11) and using $\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{w}}/c$, we arrive at:

$$\vec{\boldsymbol{v}} = \frac{1}{1 + \dfrac{\vec{\boldsymbol{v}}' \cdot \vec{\boldsymbol{w}}}{c^2}} \left[\frac{\vec{\boldsymbol{v}}'}{\gamma_w} + \frac{(\gamma_w - 1)}{|\vec{\boldsymbol{w}}|^2 \gamma_w} (\vec{\boldsymbol{v}}' \cdot \vec{\boldsymbol{w}})\, \vec{\boldsymbol{w}} + \vec{\boldsymbol{w}} \right] . \tag{12}$$

This result can be rewritten as:

$$\boxed{\vec{\boldsymbol{v}} = \frac{1}{1 + \dfrac{\vec{\boldsymbol{v}}' \cdot \vec{\boldsymbol{w}}}{c^2}} \left[\frac{1}{\gamma_w} \left(\vec{\boldsymbol{v}}' - \frac{\vec{\boldsymbol{v}}' \cdot \vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2} \vec{\boldsymbol{w}} \right) + \left(1 + \frac{\vec{\boldsymbol{v}}' \cdot \vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2} \right) \vec{\boldsymbol{w}} \right] .} \tag{13}$$

This is the relativistic law of addition of velocities.

[1] A derivation of the form of the most general Lorentz boost matrix is given in Appendix A. For consistency, I should really define $\vec{\boldsymbol{\beta}}_w \equiv \vec{\boldsymbol{w}}/c$. However, there should be no confusion in the present discussion, so I will omit the subscript w to reduce the clutter in the typography.

In the simple case where $\vec{\boldsymbol{v}}'$ and $\vec{\boldsymbol{w}}$ are parallel, it follows that these two vectors are proportional. More explicitly,[2]

$$\vec{\boldsymbol{v}}' = \left(\frac{\vec{\boldsymbol{v}}' \cdot \vec{\boldsymbol{w}}}{|\vec{\boldsymbol{w}}|^2}\right) \vec{\boldsymbol{w}} \,. \tag{14}$$

In this case, eq. (13) simplifies immediately to:

$$\vec{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}}' + \vec{\boldsymbol{w}}}{1 + \vec{\boldsymbol{v}}' \cdot \vec{\boldsymbol{w}}/c^2} \,. \tag{15}$$

In the non-relativistic limit, $\gamma_w \simeq 1$ and $\beta \ll 1$. Thus, eqs. (13) and (15) both reduce to the expected form: $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}' + \vec{\boldsymbol{w}}$.

EXERCISE: Compare the result of eq. (15) with eq. (2.22) of our textbook. Explain why minus signs appear in the numerator and denominator in eq. (2.22) of our textbook in contrast to the plus signs that appear in eq. (15).

3. The four-momentum vector

The four-momentum vector is related in a simple way to the velocity four-vector:

$$P^\mu = mU^\mu = (E/c\,;\,\vec{\boldsymbol{p}}\,)\,, \tag{16}$$

where [using eq. (1)]

$$\vec{\boldsymbol{p}} = \gamma m \vec{\boldsymbol{v}}\,, \tag{17}$$

$$E = \gamma m c^2\,. \tag{18}$$

Note that by dividing these two equations, one deduces an expression for the particle velocity:

$$\vec{\boldsymbol{v}} = \frac{\vec{\boldsymbol{p}}\, c^2}{E}\,. \tag{19}$$

Here, $\vec{\boldsymbol{v}}$ [which also appears implicitly in the factors of γ in eqs. (17) and (18)] corresponds to the velocity of the particle. Thus, in the rest frame of the particle, $\vec{\boldsymbol{v}} = 0$ and $\gamma = 1$, which implies that $P^\mu = mc(1\,;\,\vec{\boldsymbol{0}}\,)$.

Furthermore, the mass m is a scalar quantity (which is Lorentz invariant); it corresponds to the rest energy of the particle divided by c^2. This also follows from the observation[3] that the Lorentz invariant scalar $P_\mu P^\mu = -m^2c^2$. Finally, by noting that

$$\eta_{\mu\nu} P^\mu P^\nu = |\vec{\boldsymbol{P}}|^2 - (P^0)^2 = -m^2 c^2\,, \tag{20}$$

[2]One can check the correctness of eq. (14) by taking the dot product of both sides of the equation with $\vec{\boldsymbol{w}}$.

[3]Since Lorentz scalars do not depend on the reference frame, I may compute it in any reference frame. By choosing the rest frame of the particle, the computation is trivial.

and inserting $P^0 = E/c$ and $\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}}$, one obtains an expression for the relativistic energy:

$$E^2 = c^2|\vec{\boldsymbol{p}}|^2 + m^2c^4 \,. \tag{21}$$

Taking the square root, and expanding out the resulting expression in the limit of $|\vec{\boldsymbol{v}}| \ll c$ yields:

$$E \simeq mc^2 + \frac{|\vec{\boldsymbol{p}}|^2}{2m} \,, \tag{22}$$

which we recognize as the sum of the rest energy and the non-relativistic kinetic energy. More generally, the relativistic energy can be written as $E = mc^2 + T$, which defines the relativistic kinetic energy as:

$$T = \sqrt{c^2|\vec{\boldsymbol{p}}|^2 + m^2c^4} - mc^2 \,. \tag{23}$$

Appendix A: The most general Lorentz boost matrix

Consider two reference frames K and K', where K' is moving with velocity $\vec{\boldsymbol{v}}$ with respect to K. Four-vectors in K' are related to four-vectors in K by a Lorentz transformation that is called a *boost*.[4] Eq. (3.16) of our textbook exhibits a Lorentz boost along the x^1-direction,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

where

$$\beta \equiv v/c \,, \qquad \gamma = (1 - v^2/c^2)^{-1/2} \,.$$

Here, v is the velocity along the x-direction. The four-position vector in reference frames K and K' are related by

$$x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu \,. \tag{24}$$

In this appendix, I shall derive the most general boost matrix, where the velocity of K' relative to K points in an arbitrary direction. To do this, we first write out eq. (24) explicitly,

$$x'^{\,0} = \gamma(x^0 - \beta x^1) \,, \tag{25}$$
$$x'^{\,1} = \gamma(x^1 - \beta x^0) \,, \tag{26}$$
$$x'^{\,2} = x^2 \,, \tag{27}$$
$$x'^{\,3} = x^3 \,. \tag{28}$$

[4]The most general Lorentz transformation consists of a combination of a three-dimensional rotation and a boost. The Lorentz transformations considered in these notes and in Chapters 2 and 3 of our textbook are pure boosts, since no extra three-dimensional rotation of the inertial frame K' relative to K is preformed.

We now define $\vec{\boldsymbol{x}}_\parallel$ as the component of $\vec{\boldsymbol{x}}$ that is parallel to $\vec{\boldsymbol{\beta}} = \vec{\boldsymbol{v}}/c$. Likewise, we define $\vec{\boldsymbol{x}}_\perp$ as the component of $\vec{\boldsymbol{x}}$ that is perpendicular to $\vec{\boldsymbol{\beta}}$. In equations, we have

$$\vec{\boldsymbol{x}}_\parallel = \left(\frac{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\beta}}}{\beta^2}\right)\vec{\boldsymbol{\beta}}\,, \qquad\qquad \vec{\boldsymbol{x}}_\perp = \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_\parallel\,, \tag{29}$$

where $\beta^2 \equiv \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}$ is the squared magnitude of the three-vector $\vec{\boldsymbol{\beta}}$. Note that $\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}} = \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}_\parallel$ and $\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}_\perp = 0$. The magnitude of $\vec{\boldsymbol{x}}_\parallel$, denoted by $x_\parallel$, is given by

$$x_\parallel = \frac{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\beta}}}{\beta}\,.$$

As an example, for $\vec{\boldsymbol{\beta}} = (v/c, 0, 0)$, we have $\vec{\boldsymbol{x}}_\parallel = (x^1, 0, 0)$ and $\vec{\boldsymbol{x}}_\perp = (0, x^2, x^3)$. It immediately follows that $x_\parallel = x^1$. Thus, we can rewrite eqs. (25)–(28) as

$$x'^0 = \gamma(x^0 - \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}})\,, \tag{30}$$
$$\vec{\boldsymbol{x}}'_\parallel = \gamma(\vec{\boldsymbol{x}}_\parallel - \vec{\boldsymbol{\beta}}\,x^0)\,, \tag{31}$$
$$\vec{\boldsymbol{x}}'_\perp = \vec{\boldsymbol{x}}_\perp\,. \tag{32}$$

In particular, for $\vec{\boldsymbol{\beta}} = (v/c, 0, 0)$, we have $\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}} = \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}_\parallel = vx^1/c$. Note that eq. (31) can also be rewritten as

$$x'_\parallel = \gamma(x_\parallel - \beta x^0)\,,$$

where β is the magnitude of the vector $\vec{\boldsymbol{\beta}}$.

I now claim that eqs. (30)–(32) provides the correct Lorentz transformation for an arbitrary boost in the direction of $\vec{\boldsymbol{\beta}} = \vec{\boldsymbol{v}}/c$. This should be clear since I can always rotate my coordinate system to redefine what is meant by the components (x^1, x^2, x^3) and (v^1, v^2, v^3). However, dot products of two three-vectors are invariant under such a rotation. Thus, the definitions of $\vec{\boldsymbol{x}}_\parallel$ and $\vec{\boldsymbol{x}}_\perp$ given in eq. (29) are unchanged. Thus, eqs. (30)–(32) must be valid for an arbitrary boost.

We can rewrite eqs. (30)–(32) in matrix form,

$$\begin{pmatrix} x'^0 \\ x'^i \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta^j \\ -\gamma\beta^i & \delta^{ij} + (\gamma-1)\beta^i\beta^j/\beta^2 \end{pmatrix} \begin{pmatrix} x^0 \\ x^j \end{pmatrix}\,, \tag{33}$$

where $\beta^2 \equiv \sum_i \beta^i\beta^i$, and there is an implicit sum over repeated indices. To validate eq. (33), we perform the matrix multiplication to obtain

$$x'^0 = \gamma(x^0 - \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}})\,, \tag{34}$$
$$\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{x}} - \gamma\vec{\boldsymbol{\beta}}x^0 + (\gamma-1)\frac{(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}}{\beta^2}\,. \tag{35}$$

Eq. (34) has reproduced eq. (30). We next show that eq. (35) is equivalent to eqs. (31) and (32). First, we take the dot product of eq. (35) with $\vec{\boldsymbol{\beta}}$ and then multiply by $\vec{\boldsymbol{\beta}}/\beta^2$ to obtain

$$\frac{(\vec{\boldsymbol{x}}\,'\cdot\vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}}{\beta^2} = \gamma\left(\frac{(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}}{\beta^2} - \vec{\boldsymbol{\beta}}x^0\right)\,, \tag{36}$$

which reproduces eq. (31). Subtracting eqs. (35) and (36) then yields

$$\vec{\boldsymbol{x}}' - \frac{(\vec{\boldsymbol{x}}' \cdot \vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}}{\beta^2} = \vec{\boldsymbol{x}} - \frac{(\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\beta}})\vec{\boldsymbol{\beta}}}{\beta^2}\,,$$

which reproduces eq. (32). Thus, we have proven that

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\vec{\boldsymbol{\beta}} \\ -\gamma\vec{\boldsymbol{\beta}} & \delta^{ij} + (\gamma-1)\beta^i\beta^j/\beta^2 \end{pmatrix}\,, \tag{37}$$

represents the most general Lorentz boost in the direction of $\vec{\boldsymbol{\beta}} = \vec{\boldsymbol{v}}/c$.

Finally, it is useful to compute the inverse matrix Λ^{-1}, where Λ is the general boost matrix given in eq. (37). Physically, the inverse of a Lorentz boost in the direction of $\vec{\boldsymbol{\beta}}$ is equal to a Lorentz boost in the direction of $-\vec{\boldsymbol{\beta}}$. That is,

$$(\Lambda^{-1})^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\vec{\boldsymbol{\beta}} \\ \gamma\vec{\boldsymbol{\beta}} & \delta^{ij} + (\gamma-1)\beta^i\beta^j/\beta^2 \end{pmatrix}\,.$$

Indeed, it is easy to verify by explicit calculation that

$$\begin{pmatrix} \gamma & -\gamma\beta^j \\ -\gamma\beta^i & \delta^{ij} + (\gamma-1)\beta^i\beta^j/\beta^2 \end{pmatrix}\begin{pmatrix} \gamma & \gamma\beta^k \\ \gamma\beta^j & \delta^{jk} + (\gamma-1)\beta^j\beta^k/\beta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta^{ik} \end{pmatrix}\,,$$

after noting that $\gamma^2 = (1-\beta^2)^{-1}$ and $\gamma^2\beta^2 = \gamma^2 - 1$.