Evaluation of some integrals over solid angles

Consider a vector $\vec{\boldsymbol{x}}$ that points from the origin to the location (x, y, z) in three dimensional Euclidean space. Denoted $r \equiv |\vec{\boldsymbol{x}}|$. Then, we shall write,

$$\vec{\boldsymbol{x}} = r\hat{\boldsymbol{n}}\,, \tag{1}$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ is a unit vector that points in the radial direction, where $\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$. In spherical coordinates,

$$\hat{n}_1 = \sin\theta\cos\phi\,, \qquad \hat{n}_2 = \sin\theta\sin\phi\,, \qquad \hat{n}_3 = \cos\theta\,, \tag{2}$$

where θ is the polar angle and ϕ is the azimuthal angle.

In this note, I will evaluate the following two integrals.

$$\int \hat{n}_i\hat{n}_j\,d\Omega = \frac{4\pi}{3}\,\delta_{ij}\,, \tag{3}$$

$$\int \hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_\ell\,d\Omega = \frac{4\pi}{15}\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big)\,. \tag{4}$$

These results could be obtained by employing eq. (2) and evaluating the integrals explicitly using $d\Omega = d\cos\theta\,d\phi$. However, I will provide a more elegant method based on tensor algebra.

Since $\hat{n}_i\hat{n}_j$ is a symmetric second rank Cartesian tensor, one can argue that after integrating over $d\Omega$, the end result must still be a symmetric second rank Cartesian tensor. What are the possibilities? There is only one such tensor available, namely δ_{ij}. For example, the tensor x_ix_j is not available since we have already integrated over all possible directions, so the integral cannot depend on the vector $\vec{\boldsymbol{x}}$. Thus, one can immediately write

$$\int \hat{n}_i\hat{n}_j\,d\Omega = a\delta_{ij}\,, \tag{5}$$

where the constant a is not yet fixed by the argument presented above. However, once we have established eq. (5), it is a simple matter to determine the constant a as follows. Multiply eq. (5) by δ_{ij} and sum over i and j. Since $\hat{\boldsymbol{n}}$ is a unit vector, it follows that $\delta_{ij}\hat{n}_i\hat{n}_j = 1$ (where the Einstein summation convention has been used by summing over the repeated indices). On the right hand side, we have $\delta_{ij}\delta_{ij} = 3$. Hence, it follows that

$$\int d\Omega = 3a\,. \tag{6}$$

Since the integral over solid angles yields 4π, it follows that $a = 4\pi/3$, which confirms the result stated in eq. (3).

The same technique can be used to derive eq. (4). Since $\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_\ell$ is a fourth rank symmetric tensor, it follows that it must have the form,

$$\int \hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_\ell \, d\Omega = a\delta_{ij}\delta_{k\ell} + b\delta_{ik}\delta_{j\ell} + c\delta_{i\ell}\delta_{jk} \,, \tag{7}$$

since there are only three possible fourth rank symmetric tensors that you can create using the Kronecker deltas. We now can determine the constants a, b and c as follows. First, multiply both sides of eq. (7) by δ_{ij} and sum over i and j. This yields,

$$\int \hat{n}_k\hat{n}_\ell \, d\Omega = (3a + b + c)\delta_{k\ell} \,. \tag{8}$$

Next multiply both sides of eq. (7) by δ_{ik} and sum over i and k. Finally, multiply both sides of eq. (7) by $\delta_{i\ell}$ and sum over i and ℓ. In each case, one obtains an integral of the form given by eq. (3). So, the end result yields three equations and three unknowns,

$$\frac{4\pi}{3} = 3a + b + c \,, \tag{9}$$

$$\frac{4\pi}{3} = a + 3b + c \,, \tag{10}$$

$$\frac{4\pi}{3} = a + b + 3c \,. \tag{11}$$

It is not too difficult to solve these equations, which yield the unique solution,[1]

$$a = b = c = \frac{4\pi}{15} \,. \tag{12}$$

Thus, we have recovered the result of eq. (4).

Note how much easier the above derivations are as compared to the explicit calculations that make use of eq. (2). For example, to employ this latter technique requires 9 separate integrations to derive eq. (3) [since all possible choices of $i = 1, 2, 3$ and $j = 1, 2, 3$ must be considered], and 81 separate integrations to derive eq. (4).

As a final exercise, see if you can argue that

$$\int \hat{n}_{i_1}\hat{n}_{i_2} \cdots \hat{n}_{i_{2m-1}} \, d\Omega = 0 \,, \qquad \text{for any positive integer } m. \tag{13}$$

Note that an odd number of factors appears in the integrand above. In particular, show that one cannot construct a $(2m-1)$-rank symmetric tensor using only Kronecker deltas other than the zero tensor. Thus, for example,

$$\int \hat{\boldsymbol{n}} \, d\Omega = 0 \,, \tag{14}$$

a result that can be trivially obtained by inspection using eq. (2). But for values of $m > 1$, the derivation of eq. (13) based on tensor algebra is certainly the preferred one.

[1] Subtracting eq. (10) from eq. (9) yields $a = b$. Subtracting eq. (11) from eq. (10) yields $b = c$. Thus, $a = b = c$. Substitute this result back into any of the three equations to obtain eq. (12).