Evaluating $f(\tau)$ using complex integration

In class, we argued that the $\vec{\boldsymbol{D}}$ and $\vec{\boldsymbol{E}}$ fields in a linear, homogeneous and isotropic medium are related by,

$$\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)+\int_{-\infty}^{\infty} f(\tau)\,\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t-\tau)\,d\tau\,, \tag{1}$$

where $f(\tau)=0$ for $\tau<0$ (since causality requires that $\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}},t)$ can only depend on the values of $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t')$ for $t'<t$). We then showed that

$$f(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bigl[\epsilon(\omega)-1\bigr]e^{-i\omega\tau}\,d\omega\,, \tag{2}$$

where the index of refraction, $n(\omega)=\sqrt{\epsilon(\omega)}$, depends on the frequency of the Fourier mode of the electric field (under the assumption that $\mu=1$).

Consider a simple model in which the electrons in the medium are bound by a harmonic force (with natural frequency ω_0) with damping factor $\gamma>0$ acted on by an oscillating electric field of angular frequency ω. In this model, we showed in class that

$$\epsilon(\omega)=1+\frac{\omega_p^2}{\omega_0^2-\omega^2-i\gamma\omega}\,, \tag{3}$$

where the so-called plasma frequency ω_p depends on the properties of the medium. In the model under consideration,

$$\omega_p^2\equiv\frac{4\pi Ne^2}{m}\,, \tag{4}$$

for a medium with N electrons per unit volume (where m is the electron mass and e is the electron charge). Plugging eq. (3) into eq. (2),

$$f(\tau)=\frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}\,d\omega}{\omega_0^2-\omega^2-i\gamma\omega}\,,\qquad\text{where }\gamma>0. \tag{5}$$

The goal of this note is to evaluate $f(\tau)$.

The first step is to factor the denominator of the integrand,

$$\omega_0^2-\omega^2-i\gamma\omega=-(\omega-\omega_1)(\omega-\omega_2)\,, \tag{6}$$

where

$$\omega_{1,2}=-\tfrac{1}{2}i\gamma\pm\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}\,, \tag{7}$$

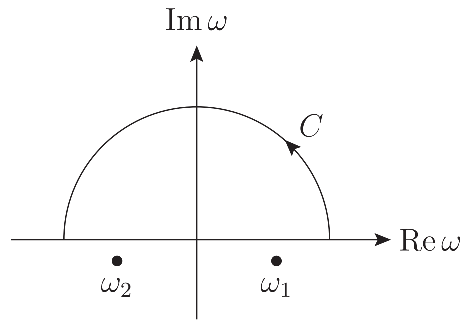
with $\operatorname{Re}\omega_1>0$ and $\operatorname{Re}\omega_2<0$. Note that since $\gamma>0$ it follows that $\operatorname{Im}\omega_{1,2}<0$. Thus,

$$f(\tau)=-\frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}\,d\omega}{(\omega-\omega_1)(\omega-\omega_2)}\,,\qquad\text{where }\operatorname{Im}\omega_{1,2}<0. \tag{8}$$

To evaluate $f(\tau)$, we consider a semicircular contour in the complex ω plane. Two cases will now be treated.

Case 1: $\tau < 0$. Then it follows that

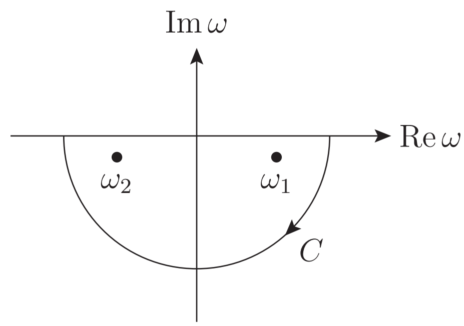
$$f(\tau) = -\frac{\omega_p^2}{2\pi}\int_C \frac{e^{-i\omega\tau}\,d\omega}{(\omega-\omega_1)(\omega-\omega_2)}$$



where C is the closed contour shown above, and the radius of the contour is taken to infinity. Note that because $\tau < 0$, the integrand is exponentially damped along the semicircular part of the contour C and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity. Note that there are no singularities inside the closed contour C (since the two poles at $\omega = \omega_{1,2}$ lie outside the closed contour. Hence, by Cauchy's Theorem of complex analysis, it follows that $f(\tau) = 0$ for $\tau < 0$.

Case 2: $\tau > 0$. Then it follows that

$$f(\tau) = -\frac{\omega_p^2}{2\pi}\int_C \frac{e^{-i\omega\tau}\,d\omega}{(\omega-\omega_1)(\omega-\omega_2)}$$



where the contour C is now closed in the lower half plane. Since in this case $\tau > 0$, the integrand is again exponentially damped along the semicircular part of the contour C and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity. Two simple poles reside inside the clockwise contour C. Thus, by the residue theorem of complex analysis applied to a closed clockwise contour,

$$\int_C \frac{e^{-i\omega\tau}\,d\omega}{(\omega-\omega_1)(\omega-\omega_2)} = -2\pi i\left[\frac{1}{\omega_1-\omega_2}\text{Res}\left(\frac{e^{-i\omega\tau}}{\omega-\omega_1}\right) + \frac{1}{\omega_2-\omega_1}\text{Res}\left(\frac{e^{-i\omega\tau}}{\omega-\omega_2}\right)\right], \quad (9)$$

where $\text{Res} f(z) = \lim_{z\to z_0}(z-z_0)f(z)$ is the residue due to a simple pole at $z = z_0$. Note that the minus sign in front of $2\pi i$ appears because the closed contour is a *clockwise* path (for a counterclockwise path no minus sign would appear). Hence, after evaluating the residues,

$$\int_C \frac{e^{-i\omega\tau}\,d\omega}{(\omega-\omega_1)(\omega-\omega_2)} = -\frac{2\pi i}{\omega_1-\omega_2}\left[e^{-i\omega_1\tau} - e^{-i\omega_2\tau}\right], \qquad \text{for } \tau > 0. \quad (10)$$

It is convenient to introduce the notation,

$$\nu_0 \equiv \sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}\,. \quad (11)$$

Then, eq. (7) reads

$$\omega_1 = \nu_0 - \tfrac{1}{2}i\gamma\,, \qquad \omega_2 = -\nu_0 - \tfrac{1}{2}i\gamma\,. \quad (12)$$

Plugging the results of eq. (12) into eq. (10), we end up with

$$\int_C \frac{e^{-i\omega\tau}\,d\omega}{(\omega-\omega_1)(\omega-\omega_2)} = -\frac{\pi i}{\nu_0}e^{-\gamma\tau/2}\big[e^{-i\nu_0\tau}-e^{i\nu_0\tau}\big] = -\frac{2\pi}{\nu_0}e^{-\gamma\tau/2}\sin\nu_0\tau\,. \tag{13}$$

Combining the two cases treated above, it follows that

$$f(\tau) = \begin{cases} \omega_p^2\, e^{-\gamma\tau/2}\,\dfrac{\sin\nu_0\tau}{\nu_0}\,, & \text{for } \tau>0, \\[6pt] 0\,, & \text{for } \tau<0. \end{cases} \tag{14}$$

We can combine the two cases more neatly by introducing the Heavyside step function,

$$\Theta(\tau) = \begin{cases} 1\,, & \text{if } \tau>0\,, \\ 0\,, & \text{if } \tau<0\,. \end{cases} \tag{15}$$

We then conclude that,

$$\boxed{f(\tau) = \Theta(\tau)\,\omega_p^2\, e^{-\gamma\tau/2}\,\frac{\sin\nu_0\tau}{\nu_0}.} \tag{16}$$

Note that the step function ensures that our result respects causality.

The requirement of causality actually imposes an interesting constraint on the analytic properties of $\epsilon(\omega)$. First, we note that inverting the Fourier transform given in eq. (2) yields,

$$\epsilon(\omega) = 1 + \int_{-\infty}^{\infty} f(\tau)\, e^{i\omega\tau}\, d\tau\,. \tag{17}$$

Hence, regarding $\epsilon(\omega)$ as a function of a *complex* variable ω, it immediately follows that

$$\epsilon(-\omega) = \epsilon^*(\omega^*)\,, \tag{18}$$

since $f(\tau)$ defined in eq. (1) must be a real function as it relates the real physical electric displacement field and electric field, respectively.

Furthermore, writing $\omega \equiv \omega_R + i\omega_I$ (where $\omega_R \equiv \mathrm{Re}\,\omega$ and $\omega_I = \mathrm{Im}\,\omega$), it follows that

$$\epsilon(\omega_R + i\omega_I) = 1 + \int_0^{\infty} e^{i\omega_R\tau} f(\tau)\, e^{-\omega_I\tau}\, d\tau\,, \tag{19}$$

$$\frac{d^n}{d\omega^n}\,\epsilon(\omega)\bigg|_{\omega=\omega_R+i\omega_I} = i^n \int_0^{\infty} \tau^n\, e^{i\omega_R\tau} f(\tau)\, e^{-\omega_I\tau}\, d\tau\,, \quad \text{for } n=1,2,3,\ldots, \tag{20}$$

where we have invoked causality in setting the lower limit of integration to zero, since $f(\tau)=0$ for $\tau<0$. Assuming that $f(\tau)$ is finite on the real axis [an assumption that is physically sensible in light of eq. (1)], then eqs. (19) and (20) imply that:

1. $\epsilon(\omega)$ is non-singular when $\mathrm{Im}\,\omega > 0$,

2. $d\epsilon/d\omega$ is non-singular when $\mathrm{Im}\,\omega > 0$,

since $\mathrm{Im}\,\omega > 0$ implies that the integrals of eqs. (19) and (20) converge due to the damping factor $e^{-\omega_I\tau}$. Consequently, $\epsilon(\omega)$ can be expanded in a Taylor series around any point in the upper half complex ω plane (i.e., it is an analytic function), as a result of causality. Indeed, the simple model employed above that yielded eq. (3) for $\epsilon(\omega)$ satisfies this latter requirement [as well as satisfying eq. (18)], as expected.