The electromagnetic fields of a uniformly moving charge

1. Relativistic transformation laws

Consider a charge q moving at constant velocity $\vec{\boldsymbol{v}}$ with respect to the laboratory frame K. The rest frame of the charge will be denoted by K'. In particular, we define the origin of K' to be the location of the charge. A laboratory observer is located at the point $\vec{\boldsymbol{x}} = (x, y, z)$, which denotes the vector that points from the origin of the laboratory frame to the observer. As seen in the rest frame of the charge, the observer is located at the point $\vec{\boldsymbol{x}}\,' = (x', y', z')$, which denotes the vector that points from the origin of K' to the observer.

At time $t = 0$, the charge is located at the origin of the laboratory frame. After a time t has elapsed (as measured in frame K), the charge is located at the point $\vec{\boldsymbol{v}}t$ in the laboratory frame. It is convenient to define the axes of the K' coordinate system such that the K and K' coordinate systems (and their origins) coincide at $t = t' = 0$. As usual we define $x_0 \equiv ct$ and $x'_0 \equiv ct'$. The relation between $(x_0\,; \vec{\boldsymbol{x}})$ and $(x'_0\,; \vec{\boldsymbol{x}}\,')$ is given by

$$x'_0 = \gamma(x_0 - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\,, \tag{1}$$

$$\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{x}} + \frac{(\gamma-1)}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}x_0\,, \tag{2}$$

where

$$\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{v}}/c\,, \qquad \beta \equiv |\vec{\boldsymbol{\beta}}|\,, \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}\,.$$

It is convenient to resolve all vectors into components parallel and perpendicular to the direction of the velocity $\hat{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{\beta}}/\beta$. We shall write

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_\parallel + \vec{\boldsymbol{x}}_\perp\,, \tag{3}$$

where $\vec{\boldsymbol{x}}_\parallel \times \vec{\boldsymbol{\beta}} = \vec{\boldsymbol{x}}_\perp\boldsymbol{\cdot}\vec{\boldsymbol{\beta}} = 0$. Note that

$$\vec{\boldsymbol{x}}_\parallel = \frac{\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})}{\beta^2}\,, \qquad \vec{\boldsymbol{x}}_\perp = \frac{\vec{\boldsymbol{\beta}}\times(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\beta}})}{\beta^2}. \tag{4}$$

To verify these results, first note that $\vec{\boldsymbol{x}}_\parallel$ is parallel to $\vec{\boldsymbol{\beta}}$, which implies that there exists a constant κ such that $\vec{\boldsymbol{x}}_\parallel = \kappa\vec{\boldsymbol{\beta}}$. Taking the dot product of this result with $\vec{\boldsymbol{\beta}}$ and solving for κ yields

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}} = \beta^2\kappa\,. \tag{5}$$

Hence, $\kappa = \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}/\beta^2$. Plugging this result back into $\vec{\boldsymbol{x}}_\parallel = \kappa\vec{\boldsymbol{\beta}}$ yields the first result quoted in eq. (4). To verify the second result quoted in eq. (4), we evaluate the triple vector product using the identity $\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \vec{\boldsymbol{b}}(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}) - \vec{\boldsymbol{c}}(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}})$. Hence, using the results of eq. (4),

$$\vec{\boldsymbol{x}}_\parallel + \vec{\boldsymbol{x}}_\perp = \frac{\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{x}})}{\beta^2} + \frac{\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\beta}})}{\beta^2} = \frac{\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{x}})}{\beta^2} + \frac{\beta^2 \vec{\boldsymbol{x}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{x}})}{\beta^2} = \vec{\boldsymbol{x}}\,, \tag{6}$$

as required by eq. (3). Hence, eqs. (1) and (2) can be rewritten as:

$$x_0' = \gamma(x_0 - \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{x}}_\parallel)\,, \tag{7}$$

$$\vec{\boldsymbol{x}}_\parallel' = \gamma(\vec{\boldsymbol{x}}_\parallel - \vec{\boldsymbol{\beta}}\, x_0)\,, \tag{8}$$

$$\vec{\boldsymbol{x}}_\perp' = \vec{\boldsymbol{x}}_\perp\,. \tag{9}$$

The transformation laws for the electromagnetic fields are given by:

$$\vec{\boldsymbol{E}}\,' = \gamma(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}) - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}})\,, \tag{10}$$

$$\vec{\boldsymbol{B}}\,' = \gamma(\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}) - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{B}})\,. \tag{11}$$

Again, we resolve the vectors into components parallel and perpendicular to the velocity,

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_\perp + \vec{\boldsymbol{E}}_\parallel\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_\perp + \vec{\boldsymbol{B}}_\parallel\,, \tag{12}$$

where in light of eq. (4), we have

$$\vec{\boldsymbol{E}}_\parallel = \frac{\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}})}{\beta^2}\,, \qquad\qquad \vec{\boldsymbol{E}}_\perp = \frac{\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{\beta}})}{\beta^2}\,, \tag{13}$$

and similarly for $\vec{\boldsymbol{B}}_\parallel$ and $\vec{\boldsymbol{B}}_\perp$. Hence eq. (10) yields

$$\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}}\,' = \gamma \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}} - \frac{\gamma^2 \beta^2}{\gamma+1}\,\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}}\,. \tag{14}$$

Multiplying both sides of eq. (14) by $\vec{\boldsymbol{\beta}}/\beta^2$ and using eq. (13), it follows that

$$\vec{\boldsymbol{E}}_\parallel' = \left(\gamma - \frac{\gamma^2\beta^2}{\gamma+1}\right)\vec{\boldsymbol{E}}_\parallel = \vec{\boldsymbol{E}}_\parallel\,, \tag{15}$$

after noting that $\gamma^2 = 1/(1-\beta^2)$ yields $\gamma^2\beta^2 = \gamma^2 - 1$. Similarly, eqs. (10) and (13) yield

$$\begin{aligned}
\vec{\boldsymbol{E}}_\perp' = \frac{\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{\beta}})}{\beta^2} &= \frac{\vec{\boldsymbol{\beta}}}{\beta^2} \times \left[\gamma \vec{\boldsymbol{E}} \times \vec{\boldsymbol{\beta}} + \gamma(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}) \times \vec{\boldsymbol{\beta}}\right] \\
&= \frac{\gamma\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{\beta}})}{\beta^2} + \frac{\gamma\vec{\boldsymbol{\beta}} \times \left[\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{\beta}})\right]}{\beta^2} \\
&= \gamma\big(\vec{\boldsymbol{E}}_\perp + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}_\perp\big)\,.
\end{aligned} \tag{16}$$

Repeating the analogous calculations for the magnetic fields, the end result is:

$$\vec{\boldsymbol{E}}'_{\parallel} = \vec{\boldsymbol{E}}_{\parallel}\,, \qquad\qquad \vec{\boldsymbol{E}}'_{\perp} = \gamma(\vec{\boldsymbol{E}}_{\perp} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}_{\perp})\,, \tag{17}$$

$$\vec{\boldsymbol{B}}'_{\parallel} = \vec{\boldsymbol{B}}_{\parallel}\,, \qquad\qquad \vec{\boldsymbol{B}}'_{\perp} = \gamma(\vec{\boldsymbol{B}}_{\perp} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}_{\perp})\,. \tag{18}$$

In analyzing the uniformly moving charge, $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ are known, so we have to invert eqs. (17) and (18) to obtain the electromagnetic fields in frame K. This is easily done by interchanging the primed and unprimed fields while reversing the sign of $\vec{\boldsymbol{\beta}}$. That is,

$$\vec{\boldsymbol{E}}_{\perp} = \gamma(\vec{\boldsymbol{E}}'_{\perp} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}'_{\perp})\,, \qquad\qquad \vec{\boldsymbol{B}}_{\perp} = \gamma(\vec{\boldsymbol{B}}'_{\perp} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'_{\perp})\,. \tag{19}$$

2. Electromagnetic fields of a uniformly moving charge[1]

Let us compare the views from reference frames K and K'. The moving charge as seen from the laboratory frame K is shown in Figure 1.[2] In addition, we define $\vec{\boldsymbol{R}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}t$ to be the vector in frame K that points from the location of the charge at time t to the location of the observer.

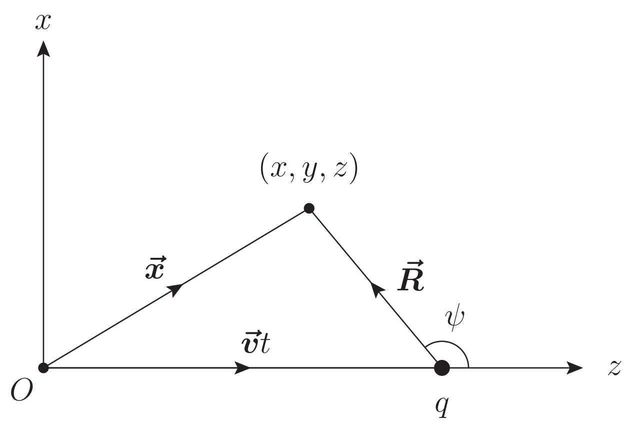

Figure 1: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z-direction as seen from reference frame K. The origin of the laboratory frame K is denoted by O, and $\vec{\boldsymbol{R}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}t$. The angle ψ is defined so that $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{R}} = \cos\psi$. By convention, we take $0 \leq \psi \leq \pi$.

The rest frame K' of the moving charge is depicted in Figure 2. In this frame, the vector that points from the origin of frame K to the location of the charge is $\vec{\boldsymbol{v}}t'$, where t' is the time elapsed as measured in frame K' (where $t = t' = 0$ marks the time when the frames K and K' coincided). In particular, eq. (2) can be rewritten in the following equivalent form,

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{R}} + \frac{(\gamma - 1)}{\beta^2}(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}}\,, \tag{20}$$

after noting that $\vec{\boldsymbol{\beta}}x_0 = \vec{\boldsymbol{v}}t$.

[1]The derivation presented in these notes is inspired by Section 22.6.4 on pp. 844–845 of Andrew Zangwill, *Modern Electrodynamics* (Cambridge University Press, Cambridge, UK, 2013).

[2]In this figure, we have defined the z-axis to point in the direction of the velocity $\vec{\boldsymbol{v}}$, although we do not make us of this fact in the derivation presented in these notes.

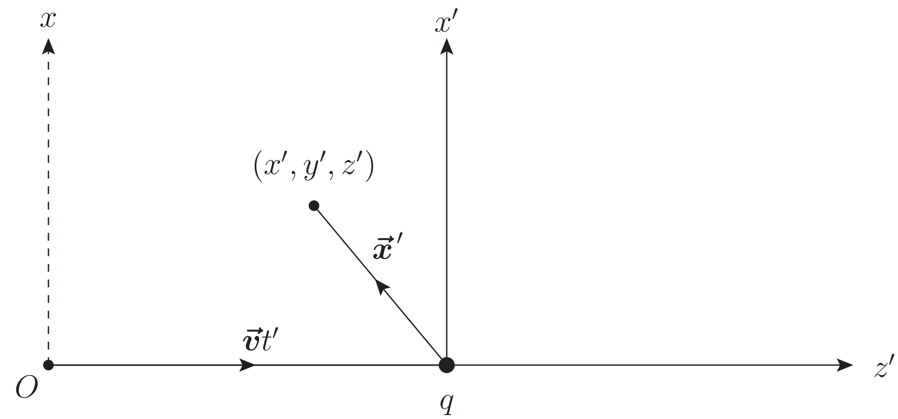

Figure 2: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z'-direction as seen from reference frame K' in which the charge is at rest. The origin of the laboratory frame K is denoted by O. The x-axis of frame K is indicated by a dashed line. The origin O and the location of the charge coincide at $t = t' = 0$.

The goal of our calculation is to compute the electromagnetic fields, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ in the laboratory frame K. First, we note that the corresponding electromagnetic fields in the rest frame K' of the charge are given (in gaussian units) by:

$$\vec{\boldsymbol{E}}\,' = \frac{q\vec{\boldsymbol{x}}\,'}{r'^3}\,, \qquad\qquad \vec{\boldsymbol{B}}\,' = 0\,, \tag{21}$$

where $r' \equiv |\vec{\boldsymbol{x}}\,'|$. We resolve the vectors above into components parallel and perpendicular to the velocity vector. In particular, we can identify the longitudinal and transverse electric fields in frame K',

$$\vec{\boldsymbol{E}}'_{\parallel} = \frac{q\vec{\boldsymbol{x}}'_{\parallel}}{r'^3}\,, \qquad\qquad \vec{\boldsymbol{E}}'_{\perp} = \frac{q\vec{\boldsymbol{x}}'_{\perp}}{r'^3}\,. \tag{22}$$

We are now ready to evaluate the electromagnetic fields in frame K. First, we employ eqs. (17)–(19) and (21) to obtain

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_{\parallel} + \vec{\boldsymbol{E}}_{\perp} = \vec{\boldsymbol{E}}'_{\parallel} + \gamma\vec{\boldsymbol{E}}'_{\perp}\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_{\parallel} + \vec{\boldsymbol{B}}_{\perp} = \gamma\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'_{\perp}\,.$$

Using eq. (22), it follows that

$$\vec{\boldsymbol{E}} = \frac{q}{r'^3}\left(\vec{\boldsymbol{x}}'_{\parallel} + \gamma\vec{\boldsymbol{x}}'_{\perp}\right)\,, \qquad\qquad \vec{\boldsymbol{B}} = \frac{\gamma q}{r'^3}\,\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{x}}'_{\perp}\,.$$

Next, we need to convert the primed coordinate into the unprimed coordinates. In light of eq. (20),

$$\vec{\boldsymbol{x}}'_{\parallel} = \gamma\vec{\boldsymbol{R}}_{\parallel}\,, \qquad\qquad \vec{\boldsymbol{x}}'_{\perp} = \vec{\boldsymbol{R}}_{\perp}\,, \tag{23}$$

after noting that $\vec{\boldsymbol{R}}_{\parallel} = (\boldsymbol{\beta} \boldsymbol{\cdot} \vec{\boldsymbol{R}})\vec{\boldsymbol{\beta}}/\beta^2$. Hence,

$$\vec{\boldsymbol{x}}'_{\parallel} + \gamma\vec{\boldsymbol{x}}'_{\perp} = \gamma\vec{\boldsymbol{R}}\,. \tag{24}$$

Noting that $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{R}}_\perp = \vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}_\parallel) = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{R}}$, it then follows that

$$\vec{\boldsymbol{E}} = \frac{\gamma q \vec{\boldsymbol{R}}}{r'^{\,3}}\,, \qquad\qquad \vec{\boldsymbol{B}} = \frac{\gamma q}{r'^{\,3}}\,\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{R}},. \tag{25}$$

Finally, in light of the identity $\gamma^2 - 1 = \beta^2\gamma^2$, eq. (20) yields

$$r'^2 = \vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,' = R^2 + \frac{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2}{\beta^2}\bigl[(\gamma-1)^2 + 2(\gamma-1)\bigr] = R^2 + \frac{\gamma^2-1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2 = R^2 + \gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\,. \tag{26}$$

Inserting this result back into eq. (25), we end up with

$$\vec{\boldsymbol{E}} = \frac{\gamma q \vec{\boldsymbol{R}}}{\bigl[R^2 + \gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\bigr]^{3/2}}\,, \tag{27}$$

$$\vec{\boldsymbol{B}} = \frac{\gamma q \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{R}}}{\bigl[R^2 + \gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2\bigr]^{3/2}}\,. \tag{28}$$

As exhibited in Figure 1, the angle between the vectors $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ and $\vec{\boldsymbol{R}}$ is denoted by ψ. In particular, $R_\perp = R\sin\psi$. By convention, we take $0 \leq \psi \leq \pi$ (that is, ψ is a polar angle of $\vec{\boldsymbol{R}}$ with respect to the vector $\vec{\boldsymbol{\beta}}$). It follows that

$$R^2 + \gamma^2(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}})^2 = \gamma^2 R^2\bigl[1 - \beta^2 + \beta^2\cos^2\psi\bigr] = \gamma^2 R^2(1 - \beta^2\sin^2\psi)\,, \tag{29}$$

after writing $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{R}} = \beta\cos\psi$ and noting that $\gamma^{-2} = 1 - \beta^2$. Inserting the result of eq. (29) into eqs. (27) and (28) and putting $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}$, we arrive at our final result:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \frac{q\vec{\boldsymbol{R}}}{\gamma^2 R^3(1 - \beta^2\sin^2\psi)^{3/2}}\,, \tag{30}$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t) = \frac{q\,\vec{\boldsymbol{v}} \times \vec{\boldsymbol{R}}}{c\gamma^2 R^3(1 - \beta^2\sin^2\psi)^{3/2}}\,, \tag{31}$$

where $\vec{\boldsymbol{R}} \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}t$. The expression for the electric field reproduces eq. (11.154) on p. 560 of Jackson.[3] However, our derivation is more general than the one given in Jackson, as Figure 11.8 of Jackson assumes that the observer is located on the x-axis of Figure 1; whereas in the derivation presented here the observer is located at an arbitrary point $\vec{\boldsymbol{x}} = (x, y, z)$.

[3]Jackson employs the symbol $\vec{\boldsymbol{r}}$ for what we call $\vec{\boldsymbol{R}}$. Our choice is motivated by the desire to avoid possible confusion between the meaning of the vector $\vec{\boldsymbol{x}}$ (with length $r \equiv |\vec{\boldsymbol{x}}|$) and the vector $\vec{\boldsymbol{R}}$ (with length R). In particular, it is important to note that although $\vec{\boldsymbol{R}}$ points from the charge to the observer in frame K and $\vec{\boldsymbol{x}}\,'$ points from the charge to the observer in frame K' [cf. Figs. 1 and 2], the vectors $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{x}}\,'$ are *not* related by a Lorentz transformation. Of course, the coordinate vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}\,'$ are related by a Lorentz transformation; namely a Lorentz boost along the direction of $\vec{\boldsymbol{v}}$ as indicated by eq. (2).

3. Electromagnetic fields of a uniformly moving charge revisited

The electromagnetic fields given in eqs. (30) and (31) are functions of $\vec{\boldsymbol{x}}$ and t that are expressed in terms of the variables R and ψ. In particular, the definitions of R and ψ are based on the location of the charge at the same time t [cf. Figure 1]. When we study the electromagnetic fields of a charge in general motion as in Chapter 14 of Jackson, the electromagnetic fields are expressed in terms of quantities whose definitions are based on the location of the charge at the *retarded time*, henceforth denoted by t'. By definition,[4]

$$t' = t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')|}{c}\,, \tag{32}$$

where $\vec{\boldsymbol{r}}(t') \equiv \vec{\boldsymbol{v}}t'$ is the location of the charge at time t'. The retarded time has the following interpretation: if a light signal originated from the location of the moving charge at time t', then the light signal would reach a fixed observer (located at the point $\vec{\boldsymbol{x}}$) at time t. It is instructive to rewrite eqs. (30) and (31) in terms of quantities whose definitions are based on the location of the charge at the retarded time t'. The relevant quantities are defined in Figure 3.

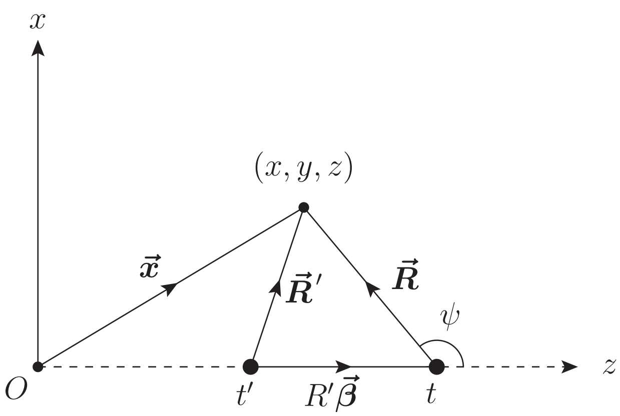

Figure 3: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z-direction as seen from reference frame K. The origin of the laboratory frame K is denoted by O. The charge q is located at the origin at time $t = 0$. At the retarded time t', the charge q is located at $\vec{\boldsymbol{v}}t'$ (labeled by t'), and at time t, the charge is located at $\vec{\boldsymbol{v}}t$ (labeled by t). A fixed observer is located at the point $\vec{\boldsymbol{x}}$.

In particular, we have defined the vector $\vec{\boldsymbol{R}}' = \vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')$ to be the vector that points from the location of the charge at time t' to the location of the fixed observer. Thus, the retarded time given in eq. (32) can be rewritten as

$$t' = t - \frac{R'}{c}\,,$$

where $R' \equiv |\vec{\boldsymbol{R}}|$. Consequently, the vector that points from t' to t in Figure 3 is simply given by

$$\vec{\boldsymbol{v}}t - \vec{\boldsymbol{v}}t' = \frac{R'\vec{\boldsymbol{v}}}{c} = R'\vec{\boldsymbol{\beta}}\,.$$

[4]In Section 3, a primed variable indicate a variable that depends on the retarded time t'. In particular, primed variables in this section are *not* associated with the reference frame K' used in Sections 1 and 2. All computations in this section are performed in the laboratory frame K.

It follows that

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{R}}' - R'\vec{\boldsymbol{\beta}} = R'(\hat{\boldsymbol{n}}' - \vec{\boldsymbol{\beta}})\,, \tag{33}$$

where $\hat{\boldsymbol{n}}' \equiv \vec{\boldsymbol{R}}'/R'$ is the unit vector that points in the direction of $\vec{\boldsymbol{R}}'$.

The angle ψ in Figure 3 can be defined via $\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}} = R\beta\cos\psi$. It follows that

$$R^3(1-\beta^2\sin^2\psi)^{3/2} = R^3(1-\beta^2+\beta^2\cos^2\psi)^{3/2} = \left[R^2(1-\beta)^2 + (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2\right]^{3/2}\,, \tag{34}$$

which was obtained previously in eq. (29). Using eq. (33), we can take the dot product with $\vec{\boldsymbol{\beta}}$ to obtain

$$\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}} = R'(\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}} - \beta^2)\,,$$

after using $\vec{\boldsymbol{R}}' = R'\,\hat{\boldsymbol{n}}'$. Squaring eq. (33) yields

$$R^2 = R'^2\left(1+\beta^2 - 2\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}}\right)\,.$$

It follows that

$$\begin{aligned}
R^2(1-\beta)^2 + (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{R}})^2 &= R'^2\left[(1-\beta^2)(1+\beta^2-2\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}}) + (\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}}-\beta^2)^2\right] \\
&= R'^2\left[1-\beta^4 - 2\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}} + 2\beta^2\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}} + (\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^2 - 2\beta^2\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}} + \beta^4\right] \\
&= R'^2\left[1-2\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}} + (\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^2\right] = \left[R'(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})\right]^2\,.
\end{aligned} \tag{35}$$

Note that for massive particles, we have $0 \leq \beta < 1$ so that $1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}} > 0$. Hence, it follows that $[(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^2]^{1/2} = 1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}}$. That is, when we take the 3/2 power of eq. (35), we end up with

$$R^3(1-\beta^2\sin^2\psi)^{3/2} = R'^3(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^3\,. \tag{36}$$

Applying eqs. (33) and (36) to eqs. (30) and (31), we obtain

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = \frac{q(\hat{\boldsymbol{n}}'-\vec{\boldsymbol{\beta}})}{\gamma^2 R'^2(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^3}\,, \qquad\qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}) = \frac{q\,\vec{\boldsymbol{\beta}}\times\hat{\boldsymbol{n}}'}{\gamma^2 R'^2(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^3} = \vec{\boldsymbol{n}}'\times\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})\,.$$

Note that all primed variables are functions of the retarded time t', since $\vec{\boldsymbol{R}}' \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')$ and $\hat{\boldsymbol{n}}' \equiv \vec{\boldsymbol{R}}'/R'$. Thus, we have confirmed that the velocity fields obtained in eqs. (14.13) and (14.14) of Jackson are equivalent to the results obtained in eqs. (30) and (31) by the Lorentz transformation technique of Section 2.