The consistent electromagnetic radiation multipole expansion

In Sections 9.6–9.10 of Jackson, the multipole expansion of the radiation electromagnetic fields are developed. However, near the end of the development, approximations are made that are not internally consistent. In these notes, I will clarify this remark and provide a consistent expansion in the small parameter kd, where d measures the extent of the sources (charge and current densities) and $k = \omega/c$. For simplicity, I will set the magnetization to zero. In addition, I will employ gaussian units. The electromagnetic fields will be given in the far (radiation) zone, where $r \gg d$ and r is the distance from the origin (located inside the region of the sources) to the location of the observer.

We assume that the electromagnetic fields are of harmonic form. The leading $\mathcal{O}(r^{-1})$ contributions to the multipole expansions of the magnetic field, $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})e^{-i\omega t}$, and the electric field, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}$, in the far (radiation) zone are given by:

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{\sqrt{\ell(\ell+1)}}\left\{a_E(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{L}}\big[Y_{\ell m}(\theta,\phi)\big] - \frac{i}{k}a_M(\ell,m)\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\left[\frac{e^{ikr}}{kr}Y_{\ell m}(\theta,\phi)\right]\right\},$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{(-i)^{\ell+1}}{\sqrt{\ell(\ell+1)}}\left\{\frac{i}{k}a_E(\ell,m)\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\left[\frac{e^{ikr}}{kr}Y_{\ell m}(\theta,\phi)\right] + a_M(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{L}}\big[Y_{\ell m}(\theta,\phi)\big]\right\},$$

where $\vec{\boldsymbol{L}} \equiv -i\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}$ is a differential operator that acts on functions of $\vec{\boldsymbol{x}} = (r,\theta,\phi)$ and $r \equiv |\vec{\boldsymbol{x}}|$. The corresponding magnetic and electric multipole coefficients are given by:[1]

$$a_M(\ell,m) = -\frac{4\pi i k^2}{c\sqrt{\ell(\ell+1)}}\int d^3x'\, j_\ell(kr')Y^*_{\ell m}(\hat{\boldsymbol{n}}')\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\big(\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\big)\,, \tag{1}$$

$$a_E(\ell,m) = \frac{4\pi i k^2}{c\sqrt{\ell(\ell+1)}}\int d^3x'\, j_\ell(kr')Y^*_{\ell m}(\hat{\boldsymbol{n}}')\left[\frac{i}{k}\vec{\boldsymbol{\nabla}}'^2\big(\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\big) + \frac{c}{r'}\frac{\partial}{\partial r'}\big(r'^2\rho(\vec{\boldsymbol{x}}')\big)\right], \tag{2}$$

where $r' \equiv |\vec{\boldsymbol{x}}'|$ and $\hat{\boldsymbol{n}}' \equiv \vec{\boldsymbol{x}}'/r'$.

One can rewrite eq. (2) via an integration by parts. First, by using Green's theorem and dropping the surface terms (since the sources vanish at infinity), we can move the $\vec{\boldsymbol{\nabla}}'^2$ over to $j_\ell(kr)Y^*_{\ell m}(\hat{\boldsymbol{n}})$ and make use of the homogeneous Helmholtz equation,

$$\big(\vec{\boldsymbol{\nabla}}'^2 + k^2\big)\big[j_\ell(kr')Y^*_{\ell m}(\hat{\boldsymbol{n}}')\big] = 0\,. \tag{3}$$

Likewise, moving $\partial/\partial r'$ so that it acts on $r'^2 j_\ell(kr')$ [after writing $d^3x' = r'^2 dr'\,d\Omega$] and discarding the surface term at infinity, we end up with

$$a_E(\ell,m) = -\frac{4\pi i k^2}{c\sqrt{\ell(\ell+1)}}\int d^3x'\, Y^*_{\ell m}(\hat{\boldsymbol{n}}')\left[ik\big(\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\big)j_\ell(kr') + c\rho(\vec{\boldsymbol{x}}')\frac{\partial}{\partial r'}\Big(r' j_\ell(kr')\Big)\right]. \tag{4}$$

[1]In our notation, the primed coordinates indicate the location of the sources.

Jackson's next step is to invoke the small kd approximation. This has two effects. First, he approximates

$$j_\ell(kr') \simeq \frac{(kr')^\ell}{(2\ell+1)!!} + \mathcal{O}\big((kr')^{\ell+2}\big)\,. \tag{5}$$

In this approximation,

$$\frac{\partial}{\partial r'}\big(r' j_\ell(kr')\big) \simeq (\ell+1)\frac{(kr')^\ell}{(2\ell+1)!!} + \mathcal{O}\big((kr')^{\ell+2}\big)\,. \tag{6}$$

Furthermore, Jackson asserts that the term in eq. (4) proportional to $\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')$ is subdominant and can also be neglected. Under these assumptions, it then follows that

$$\begin{aligned} a_M(\ell,m) &= -\frac{4\pi i k^{\ell+2}}{c(2\ell+1)!!\sqrt{\ell(\ell+1)}}\int d^3x'\, r'^{\ell} Y^*_{\ell m}(\hat{\boldsymbol{n}}')\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\big(\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\big)\,, \\ &= \frac{4\pi i k^{\ell+2}}{(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}M_{\ell m}\,, \end{aligned} \tag{7}$$

$$a_E(\ell,m) = -\frac{4\pi i k^{\ell+2}}{(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}\int d^3x'\, r'^{\ell} Y^*_{\ell m}(\hat{\boldsymbol{n}}')\rho(\vec{\boldsymbol{x}}') = -\frac{4\pi i k^{\ell+2}}{(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}Q_{\ell m}\,, \tag{8}$$

where the magnetic and electric multipole spherical tensors are given by

$$\begin{aligned} M_{\ell m} &= -\frac{1}{c(\ell+1)}\int d^3x'\, r'^{\ell} Y^*_{\ell m}(\hat{\boldsymbol{n}}')\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\big(\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\big) \\ &= \frac{1}{c(\ell+1)}\int d^3x'\,\big(\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\big)\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}'\big(r'^{\ell} Y^*_{\ell m}(\hat{\boldsymbol{n}}')\big)\,, \end{aligned} \tag{9}$$

$$Q_{\ell m} = \int d^3x'\, r'^{\ell} Y^*_{\ell m}(\hat{\boldsymbol{n}}')\rho(\vec{\boldsymbol{x}}')\,. \tag{10}$$

However, inserting eqs. (7) and (8) back into the multipole expansions for $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ does not yield a consistent expansion. In the limit of $kd \ll 1$, the multipole spherical tensors are roughly of order $M_{\ell m} \sim (v'/c)d^\ell$ and $Q_{\ell m} \sim d^\ell$, where we noted that $\vec{\boldsymbol{J}} \sim \rho\vec{\boldsymbol{v}}'$. For harmonic sources $\vec{\boldsymbol{v}}' = d\vec{\boldsymbol{x}}'/dt = -i\omega\vec{\boldsymbol{x}}'$. Thus, $v'/c \sim \omega d/c \sim kd$. It follows that $M_{\ell m} \sim kd^{\ell+1}$. Hence, we conclude that

$$a_M(\ell,m)/k \sim (kd)^{\ell+1}\,, \qquad a_E(\ell,m)/k^2 \sim (kd)^\ell\,. \tag{11}$$

Inserting these estimates into the multipole expansions for $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ yields a series in which each successive term of the sum is suppressed by a power of kd. But, with the approximations made in eqs. (5) and (6) and dropping the term in eq. (4) proportional to $\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')$, the terms that are being neglected are of the same order as other terms that have been kept.

So, let us return to eqs. (1) and (4). We will keep the functions $j_\ell(kr')$ as they are without making any small argument approximation. Moreover, we will make use of the following relation,

$$\frac{\partial}{\partial r'}\Big(r' j_\ell(kr')\Big) = (\ell+1)j_\ell(kr') - kr' j_{\ell+1}(kr')\,. \qquad (12)$$

Next, we introduce:

$$M_{\ell m}(k^2) = -\frac{(2\ell+1)!!}{ck^\ell(\ell+1)}\int d^3x'\, j_\ell(kr')Y^*_{\ell m}(\boldsymbol{\hat n}')\boldsymbol{\vec\nabla}'\boldsymbol{\cdot}\big(\boldsymbol{\vec x}'\times\boldsymbol{\vec J}(\boldsymbol{\vec x}')\big)\,, \qquad (13)$$

$$Q_{\ell m}(k^2) = \frac{(2\ell+1)!!}{k^\ell}\int d^3x'\, j_\ell(kr')Y^*_{\ell m}(\boldsymbol{\hat n}')\rho(\boldsymbol{\vec x}')\,. \qquad (14)$$

Note that in the limit of $k\to 0$,

$$M_{\ell m}(0) = M_{\ell m}\,, \qquad\qquad Q_{\ell m}(0) = Q_{\ell m}\,, \qquad (15)$$

where $M_{\ell m}$ and $Q_{\ell m}$ are defined by eqs. (9) and (10).

In addition, we must introduce a third spherical tensor,

$$T_{\ell m}(k^2) = \frac{(2\ell+1)!!}{(\ell+1)ck^\ell}\int d^3x'\, Y^*_{\ell m}(\boldsymbol{\hat n}')\Big[j_\ell(kr')\boldsymbol{\vec x}'\boldsymbol{\cdot}\boldsymbol{\vec J}(\boldsymbol{\vec x}') + icr' j_{\ell+1}(kr')\rho(\boldsymbol{\vec x}')\Big]\,, \qquad (16)$$

where the last term on the right-hand side above is a consequence of the last term on the right hand side of eq. (12). Since the charge and current densities are harmonic, with $\boldsymbol{\vec J}(\boldsymbol{\vec x}',t) = \boldsymbol{\vec J}(\boldsymbol{\vec x}')e^{-i\omega t}$ and $\rho(\boldsymbol{\vec x}',t) = \rho(\boldsymbol{\vec x}')e^{-i\omega t}$, the continuity equation yields

$$\boldsymbol{\vec\nabla}'\boldsymbol{\cdot}\boldsymbol{\vec J}(\boldsymbol{\vec x}') = i\omega\rho(\boldsymbol{\vec x})\,. \qquad (17)$$

We will use this equation to rewrite eq. (16) as

$$T_{\ell m}(k^2) = \frac{(2\ell+1)!!}{(\ell+1)ck^\ell}\int d^3x'\, Y^*_{\ell m}(\boldsymbol{\hat n}')\Big[j_\ell(kr')\boldsymbol{\vec x}'\boldsymbol{\cdot}\boldsymbol{\vec J}(\boldsymbol{\vec x}') + \frac{r'}{k}\, j_{\ell+1}(kr')\boldsymbol{\vec\nabla}'\boldsymbol{\cdot}\boldsymbol{\vec J}(\boldsymbol{\vec x}')\Big]\,. \qquad (18)$$

We then obtain (without making any small argument approximation):

$$a_M(\ell,m) = \frac{4\pi i k^{\ell+2}}{(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}M_{\ell m}(k^2)\,, \qquad (19)$$

$$a_E(\ell,m) = -\frac{4\pi i k^{\ell+2}}{(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}\big[Q_{\ell m}(k^2) + ikT_{\ell m}(k^2)\big]\,. \qquad (20)$$

Inserting these results back into the multipole expansions for $\boldsymbol{\vec B}(\boldsymbol{\vec x})$ and $\boldsymbol{\vec E}(\boldsymbol{\vec x})$ yields consistent expansions to all orders in kd [in contrast to the results quoted in eqs. (7) and (8)].

Note that $T_{\ell m}(k^2)$ has a finite limit as $k\to 0$. We can estimate the size of $kT_{\ell m}(0)$ by using similar arguments to those employed in deriving eq. (11). In particular,

$$kT_{\ell m}(0) \sim (v/c)kd^{\ell+1} \sim (kd)^2 d^\ell\,. \qquad (21)$$

If we expand around $k=0$,

$$Q_{\ell m}(k^2)+ikT_{\ell m}(k^2)=Q_{\ell m}+k^2Q'_{\ell m}(0)+ikT_{\ell m}(0)+\cdots\,, \tag{22}$$

where $Q'_{\ell m}(0)\equiv\bigl(\partial Q_{\ell m}(k^2)/\partial k^2\bigr)_{k^2=0}$, then we see that $k^2Q'_{\ell m}(0)$ and $ikT_{\ell m}(0)$ are of the same order in the small argument expansion. This observation motivates the following definition of the *toroidal multipole moment*,

$$T_{\ell m}\equiv T_{\ell m}(0)-ik\,Q'_{\ell m}(0)\,. \tag{23}$$

To evaluate $Q'_{\ell m}(0)$, we need the second term in the small argument expansion of $j_\ell(kr')$,

$$j_\ell(kr')\simeq\frac{(kr')^\ell}{(2\ell+1)!!}-\frac{(kr')^{\ell+2}}{2(2\ell+3)!!}+\mathcal{O}\bigl((kr')^{\ell+4}\bigr)\,. \tag{24}$$

Hence, we obtain

$$Q'_{\ell m}(0)=-\frac{1}{2(2\ell+3)}\int d^3x'\,r'^{\,\ell+2}Y^*_{\ell m}(\hat{\boldsymbol{n}}')\rho(\vec{\boldsymbol{x}}')\,. \tag{25}$$

In light of eqs. (17) and (23),

$$T_{\ell m}=T_{\ell m}(0)+\frac{1}{2(2\ell+3)c}\int d^3x'\,r'^{\,\ell+2}Y^*_{\ell m}(\hat{\boldsymbol{n}}')\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\,, \tag{26}$$

after using $\omega=kc$.

Whereas the electric dipole moment enters at $a_E(1,m)/k^2\sim\mathcal{O}(kd)$ in the multipole expansion, the toroidal multipole moment makes its first appearance in the contribution to $a_E(1,m)/k^2\sim\mathcal{O}(kd)^3$. This is the reason that these effects were not seen in Sections 9.2 and 9.3 of Jackson, as the calculations presented there were sensitive only to the leading electric dipole contribution to $a_E(1,m)/k^2\sim\mathcal{O}(kd)$, and the leading electric quadrupole contribution to $a_E(1,m)/k^2\sim\mathcal{O}\bigl((kd)^2\bigr)$. Likewise, the leading magnetic dipole contributes to $a_M(\ell,m)/k\sim\mathcal{O}\bigl((kd)^2\bigr)$ and the leading magnetic quadrupole contributes to $a_M(\ell,m)/k\sim\mathcal{O}\bigl((kd)^3\bigr)$. Thus, had the methods of Sections 9.2 and 9.3 of Jackson been extended one additional order (corresponding to the third term in the series expansion of the exponential $e^{-ik\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\hat{\boldsymbol{n}}}$), one would have found contributions from the electric octupole, magnetic quadrupole and the toroidal dipole.

It is of interest to evaluate explicitly the leading contribution to the $\ell=1$ case of the toroidal multipole moment. Using eqs. (18) and (26),

$$T_{1m}=\frac{1}{2c}\int d^3x'\,r'\,Y^*_{1m}(\hat{\boldsymbol{n}}')\left[\vec{\boldsymbol{x}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')+\tfrac{2}{5}r'^2\,\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\right]\,. \tag{27}$$

With respect to the spherical basis, we shall normalize the components of the toroidal dipole moment vector $\vec{\boldsymbol{t}}$ with respect to the spherical basis as:

$$t_m=\sqrt{\frac{3}{4\pi}}T_{1m}\,. \tag{28}$$

Converting to the Cartesian basis and integrating by parts then yields,

$$\begin{aligned}\vec{\boldsymbol{t}} &= \frac{1}{2c}\int d^3x' \left[\vec{\boldsymbol{x}}\,'\Big(\vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\Big) - \tfrac{2}{5}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\,'\big(r'^2\vec{\boldsymbol{x}}\,'\big)\right] \\ &= \frac{1}{2c}\int d^3x' \left[\vec{\boldsymbol{x}}\,'\big(\vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big) - \tfrac{2}{5}\big[r'^2\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,') + 2\vec{\boldsymbol{x}}\,'\big(\vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big)\big]\right] \\ &= \frac{1}{10c}\int d^3x' \left[\vec{\boldsymbol{x}}\,'\big(\vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big) - 2r'^2\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\right]. \end{aligned} \tag{29}$$

This is the expression for the toroidal dipole moment vector that appears in the literature.

Note that the normalization of $\vec{\boldsymbol{t}}$ according to eq. (28) implies that the relation between the spherical and Cartesian components of $\vec{\boldsymbol{t}}$ are the same as the corresponding relations of the electric dipole moment $\vec{\boldsymbol{p}}$. Namely,

$$T_{11} = -\sqrt{\frac{3}{8\pi}}\big(t_x - it_y\big)\,, \qquad T_{10} = \sqrt{\frac{3}{4\pi}}\,t_z\,, \qquad T_{1,-1} = \sqrt{\frac{3}{8\pi}}\big(t_x + it_y\big)\,. \tag{30}$$

For further discussions on the significance of the toroidal multipole moments, see the references quoted below.

REFERENCES:

1. V.M. Dubovik and V.V. Tugushev, *Toroid Moments in Electrodynamics and Solid-State Physics*, Physics Reports **187** (1990) 145–202.

2. E.E. Radescu and G. Vaman, *Exact Calculation of the Angular Momentum Loss, Recoil Force, and Radiation Intensity for an Arbitrary Source in Terms of Electric, Magnetic, and Toroid Multipoles*, Physical Review E **65** (2002) 046609.

3. A. Góngora T. and E. Ley-Koo, *Complete electromagnetic multipole expansion including toroidal moments*, Revista Mexicana de Física, **52** (2006) 188–197.

4. Ivan Fernandez-Corbaton, Stefan Nanz, Rasoul Alaee, and Carsten Rockstuhl, *Exact Dipolar Moments of a Localized Electric Current Distribution*, Optics Express **23** (2015) 33044–33064.

5. N. Papasimakis, V.A. Fedotov, V. Savinov, T.A. Raybould, and N.I. Zheludev, *Electromagnetic toroidal Excitations in Matter and Free Space*, Nature Materials **15** (2016) 263–271.

6. Ivan Fernandez-Corbaton, Stefan Nanz, and Carsten Rockstuhl, *On the Dynamics Toroidal Multipoles from Localized Electric Circuit Distributions*, Nature Scientific Reports 7, Article number: 7527 (2017) [arXiv:1507.00755 [physics.optics]]; Supplementary information for "On the Dynamics Toroidal Multipoles from Localized Electric Circuit Distributions," op. cit.