This is a take-home exam. You may refer to the textbook by Jackson, any material linked to the class website, and any second classical electromagnetism textbook of your choosing. (If you do consult a second text, please indicate which one you used.) Reference to integrals or other mathematical facts, and any personal handwritten notes are also OK. However, you should *not* collaborate with anyone else (this includes AI help such as ChatGPT).

The point value of each problem is indicated in the square brackets below (each part being worth 10 points). You should use these values as a guide to manage your time during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.*

There is no need to rederive results that have been previously obtained in the textbook, the class notes or the class handouts. But if you make use of any previously derived result, please cite the source of the result.

Completed exams should be delivered in class on Thursday February 19, 2026.

---

1. [20] In the Drude dielectric model for metals, a conducting medium possesses a frequency-dependent electric permittivity (in gaussian units) of

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}\,, \qquad \text{where } \gamma > 0. \tag{1}$$

In class, we showed that under the assumption of causality and linearity,

$$\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) + \int_{-\infty}^{\infty} d\tau\, f(\tau)\, \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t-\tau)\,, \qquad \text{where } f(\tau) = 0 \text{ for } \tau < 0. \tag{2}$$

We then showed that

$$f(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \big[\epsilon(\omega) - 1\big] e^{-i\omega\tau} d\omega\,. \tag{3}$$

(a) Evaluate $f(\tau)$ given by eq. (3) with $\epsilon(\omega)$ as specified in eq. (1), and verify that $f(\tau) = 0$ for $\tau < 0$ if the integration path along the real ω axis is deformed near the origin by taking a semicircular path of radius ε in the complex ω plane from $\omega = -\varepsilon$ to $\omega = \varepsilon$. Using the same deformed integration path, evaluate $f(\tau)$ when $\tau > 0$.

(b) One cannot use the Kramers-Kronig relation for $\epsilon(\omega)$ since it has a pole on the real axis at $\omega = 0$. However, note that the quantity

$$\tilde{\epsilon}(\omega) \equiv \epsilon(\omega) - \frac{i\omega_p^2}{\gamma\omega}\,, \tag{4}$$

is an analytic function in the upper half complex plane *including* the real axis. Thus, the Kramers-Kronig relation can be used for $\tilde{\epsilon}(\omega)$. Verify explicitly that the following result is satisfied:

$$\operatorname{Re}\tilde{\epsilon}(\omega) = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\operatorname{Im}\tilde{\epsilon}(\omega')}{\omega' - \omega}\, d\omega'\,. \tag{5}$$

HINT for part (b) of Problem 1: The integral in eq. (5) is most easily obtained with the help of the Sokhotski-Plemelj formula,

$$P\,\frac{1}{\omega'-\omega}=\frac{1}{\omega'-\omega+i\varepsilon}+i\pi\delta(\omega'-\omega)\,,\qquad \text{for a real infinitesimal } \varepsilon>0. \tag{6}$$

2. [40] The theory of electromagnetism in $3+1$ spacetime dimensions can be generalized to $n+1$ spacetime dimensions as follows. The indices of the second-rank totally antisymmetric electromagnetic field strength tensor $F^{\mu\nu}$ now take on values $\mu,\ \nu\in\{0,1,\ldots,n\}$. The dynamical Maxwell equations are given (in gaussian units) by:

$$\partial_\mu F^{\mu\nu}=\frac{S_{n-1}}{c}J^\nu\,, \tag{7}$$

where

$$S_{n-1}\equiv\int d\Omega_{n-1}=\frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{8}$$

is the surface area of an n dimension ball of unit radius. For example, $S_1=2\pi$, $S_2=4\pi$, etc. The dual electromagnetic field strength tensor is defined by employing the totally antisymmetric rank $(n+1)$ ϵ-tensor. The latter can be used to express the kinematical Maxwell equations,

$$\epsilon^{\mu\cdots\alpha\beta}\partial_\mu F_{\alpha\beta}=0\,, \tag{9}$$

where $\cdots$ in eq. (9) represents $n-2$ free indices that are not exhibited explicitly. By convention, we choose $\epsilon^{012\cdots n}=+1$.

(a) In $n+1$ spacetime dimensions, how many independent components are needed to describe $F^{\mu\nu}$? How many of these components represent the electric field and how many of these components represent the magnetic field?

(b) Consider the theory of electromagnetism in $2+1$ spacetime dimensions, where $F^{\mu\nu}$ can be constructed by deleting the fourth row and fourth column of the $3+1$ dimensional version of $F^{\mu\nu}$. Note that the electric field vector is now of the form $\vec{\boldsymbol{E}}=\hat{\boldsymbol{x}}E_x+\hat{\boldsymbol{y}}E_y$, as expected, but magnetic field consists of a single "component" which you can denote by B. Define a "dual" electromagnetic field strength tensor and show that it is a Lorentz three-vector of the $2+1$ dimensional spacetime. Determine its components in terms of the electric and magnetic fields. In light of the $2+1$ dimensional version of eq. (9), show that

$$\frac{d}{dt}\int B(\vec{\boldsymbol{x}},t)\,d^2x=0\,, \tag{10}$$

assuming that the electric and magnetic fields vanish sufficiently fast at spatial infinity.

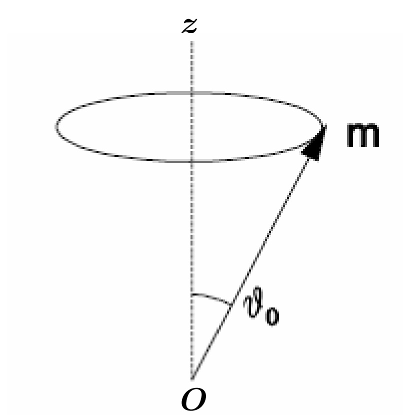
(c) Consider a reference frame K' that moves at a constant velocity $c\beta\hat{\boldsymbol{x}}$ with respect to reference frame K. Using the behavior of a Lorentz three-vector under a boost, obtain expressions for the electric and magnetic fields, E_x, E_y, and B, in reference frame K' in terms of the corresponding fields in reference frame K. Check that your results coincide with the expected result in $3+1$ dimensional spacetime.

(d) In 2 spatial dimensions, there are two different vector differential operators,

$$\vec{\boldsymbol{\nabla}} \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\,, \qquad\qquad \vec{\boldsymbol{\nabla}}_\perp \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial y} - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\,, \tag{11}$$

where $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}}_\perp = 0$. Using eqs. (7) and (9), write out Maxwell equations explicitly in terms of the electric field vector, the magnetic field, the charge density and current density vector, and the differential operators defined above. Show that in 2 spatial dimensions, there are only three Maxwell equations (in contrast to the four equations obtained in three spatial dimensions).

3. [40] A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency ω and angle ϑ_0 with respect to the z-axis as shown below. (The origin of the coordinate system is labeled by $\boldsymbol{O}$.) That is, the time-dependence of the azimuthal angle is $\varphi_0(t) = \varphi_0 - \omega t$.



Electromagnetic radiation is emitted by the precessing dipole.[1]

(a) Write out an explicit expression for the time-dependent magnetic dipole vector $\vec{\boldsymbol{m}}$ in terms of its magnitude m_0, the angles ϑ_0 and φ_0 and the time t. Show that $\vec{\boldsymbol{m}}$ consists of the sum of a time-dependent term and a time-independent term. Verify that the time-dependent term can be written as $\mathrm{Re}(\vec{\boldsymbol{m}}\, e^{-i\omega t})$, for some suitably chosen complex vector $\vec{\boldsymbol{m}}$.

(b) Compute the angular distribution of the time-averaged radiated power measured by an observer located at the point $\vec{\boldsymbol{x}}$ with respect to the coordinate system defined in the above figure.

(c) Compute the total power radiated.

(d) What is the polarization of the radiation measured by an observer located along the positive z-axis far from the precessing dipole? How would your answer change if the observer were located in the x–y plane?

[1] Radiation from pulsars is believed to be due to this mechanism.