1. In the Drude dielectric model for metals, a conducting medium possesses a frequency-dependent electric permittivity (in gaussian units) of

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}\,, \qquad \text{where } \gamma > 0. \tag{1}$$

In class, we showed that under the assumption of causality and linearity,

$$\vec{\boldsymbol{D}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) + \int_{-\infty}^{\infty} d\tau\, f(\tau)\, \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t - \tau)\,, \qquad \text{where } f(\tau) = 0 \text{ for } \tau < 0. \tag{2}$$

We then showed that

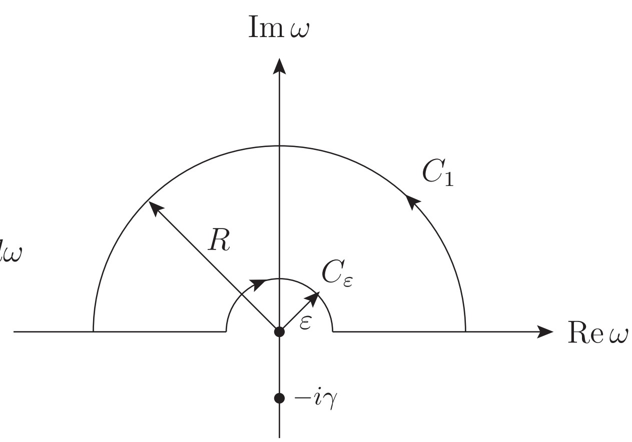
$$f(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \big[\epsilon(\omega) - 1\big] e^{-i\omega\tau} d\omega\,. \tag{3}$$

(a) Evaluate $f(\tau)$ given by eq. (3) with $\epsilon(\omega)$ as specified in eq. (1), and verify that $f(\tau) = 0$ for $\tau < 0$ if the integration path along the real ω axis is deformed near the origin by taking a semicircular path of radius ε in the complex ω plane from $\omega = -\varepsilon$ to $\omega = \varepsilon$. Using the same deformed integration path, evaluate $f(\tau)$ when $\tau > 0$.

Consider first the case of $\tau < 0$. By deforming the integration path as shown in the figure below, we note that the integrand is exponentially damped in the upper half complex plane as $R \to \infty$, so we are permitted to add C_1 to the integration path. The result is an integral over a closed curve with no singularities inside. Thus by Cauchy's theorem, the value of the integral

$$f(\tau) = -\frac{1}{2\pi} \lim_{R\to\infty} \lim_{\varepsilon\to 0} \oint \frac{\omega_p^2\, e^{-i\omega t}}{\omega(\omega + i\gamma)}\, d\omega = 0\,, \quad \text{for } \tau < 0. \tag{4}$$

$$I_1(R, \varepsilon) \equiv -\frac{1}{2\pi}\left[\int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_{\varepsilon}^{R} + \int_{C_1}\right] \frac{\omega_p^2\, e^{-i\omega t}}{\omega(\omega + i\gamma)}\, d\omega$$

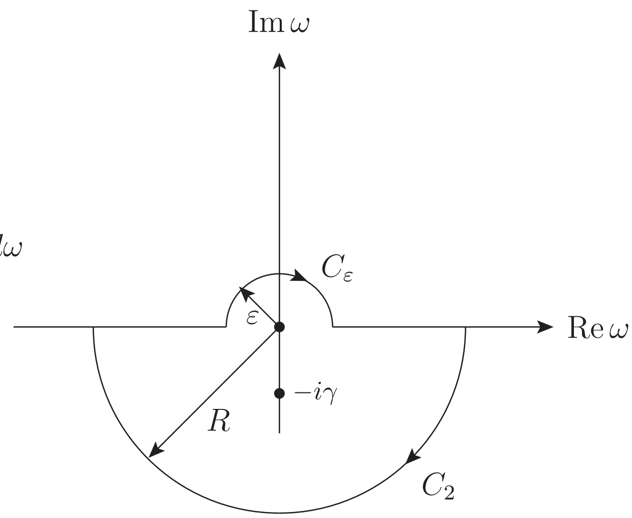


Next, we consider the case of $\tau > 0$. By deforming the integration path as shown in the second figure below, we note that the integrand is exponentially damped in the lower half complex plane as $R \to \infty$, so we are permitted to add C_2 to the integration path. The result is an integral over a closed curve with two poles inside. One pole resides at $\omega = 0$ and one pole resides at $\omega = -i\gamma$. Thus by the residue theorem, the value of the integral

$$f(\tau) = -\frac{1}{2\pi} \lim_{R\to\infty} \lim_{\varepsilon\to 0} \oint \frac{\omega_p^2\, e^{-i\omega t}}{\omega(\omega + i\gamma)}\, d\omega = \frac{\omega_p^2}{\gamma}\bigl(1 - e^{-\gamma t}\bigr)\,, \quad \text{for } \tau > 0. \tag{5}$$

after noting that the closed integration path is *clockwise*, which implies that the integral is given by $-2\pi i$ times the sum over the two residues.

$$I_2(R,\varepsilon) \equiv -\frac{1}{2\pi}\int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_{\varepsilon}^{R} + \int_{C_2} \frac{\omega_p^2\, e^{-i\omega t}}{\omega(\omega + i\gamma)}\, d\omega$$



In summary, we have found that

$$f(\tau) = \frac{\omega_p^2}{\gamma}\bigl(1 - e^{-\gamma t}\bigr)\Theta(\tau)\,, \tag{6}$$

where the step function $\Theta(\tau) = 1$ for $\tau > 0$ and $\Theta(\tau) = 0$ for $\tau < 0$. It is noteworthy that $f(\tau)$ does *not* vanish in the limit of $\tau \to \infty$, as one might expect from the Riemann-Lebesgue theorem. In this case, the theorem does not apply due to the pole in the integrand at $\omega = 0$ (which is the typical behavior of a conducting medium).

(b) One cannot use the Kramers-Kronig relation for $\epsilon(\omega)$ since it has a pole on the real axis at $\omega = 0$. However, note that the quantity

$$\tilde{\epsilon}(\omega) \equiv \epsilon(\omega) - \frac{i\omega_p^2}{\gamma\omega}\,, \tag{7}$$

is an analytic function in the upper half complex plane *including* the real axis. Thus, the Kramers-Kronig relation can be used for $\tilde{\epsilon}(\omega)$. Verify explicitly that the following result is satisfied:

$$\operatorname{Re}\tilde{\epsilon}(\omega) = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im}\tilde{\epsilon}(\omega')}{\omega' - \omega}\, d\omega'\,. \tag{8}$$

Using eq. (1),

$$\tilde{\epsilon}(\omega) = 1 - \frac{i\omega_p^2}{\gamma(\omega + i\gamma)}\,. \tag{9}$$

It follows that

$$\operatorname{Re}\tilde{\epsilon}(\omega) = 1 - \frac{\omega_p^2}{\omega^2+\gamma^2}\,, \qquad\qquad \operatorname{Im}\tilde{\epsilon}(\omega) = -\frac{\omega_p^2\omega}{\gamma(\omega^2+\gamma^2)}\,. \tag{10}$$

Thus, we are asked to evaluate the integral

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{Im}\tilde{\epsilon}(\omega')}{\omega'-\omega}\,d\omega'\,, \tag{11}$$

and show that eq. (8) is satisfied. Using eq. (10),

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{Im}\tilde{\epsilon}(\omega')}{\omega'-\omega}\,d\omega' = -\frac{\omega_p^2}{\pi\gamma}\,P\int_{-\infty}^{\infty}\frac{\omega'\,d\omega'}{(\omega'^2+\gamma^2)(\omega'-\omega)}\,. \tag{12}$$

The principal value prescription dictates how to treat the singularity at $\omega' = \omega$. The simplest approach is to use the Sokhotski-Plemelj formula,

$$P\,\frac{1}{\omega'-\omega} = \frac{1}{\omega'-\omega+i\varepsilon} + i\pi\delta(\omega'-\omega)\,, \qquad \text{for a real infinitesimal } \varepsilon > 0. \tag{13}$$

Hence, we can write:

$$\begin{aligned}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{Im}\tilde{\epsilon}(\omega')}{\omega'-\omega}\,d\omega' &= -\frac{\omega_p^2}{\pi\gamma}\int_{-\infty}^{\infty}\frac{\omega'\,d\omega'}{\omega'^2+\gamma^2}\left\{\frac{1}{\omega'-\omega+i\varepsilon} + i\pi\delta(\omega'-\omega)\right\} \\ &= -\frac{\omega_p^2}{\pi\gamma}\left\{\frac{i\pi\omega}{\omega^2+\gamma^2} + \int_{-\infty}^{\infty}\frac{\omega'\,d\omega'}{(\omega'^2+\gamma^2)(\omega'-\omega+i\varepsilon)}\right\}\,. \end{aligned} \tag{14}$$

The remaining integral in eq. (14) can be evaluating by closing the contour in the upper half complex ω' plane (since the integrand vanishes along the semicircle of radius R in the limit of $R \to \infty$). We can then evaluate the integral over the closed contour using the residue theorem. There is one pole inside the closed contour at $\omega' = i\gamma$. Thus, $2\pi i$ times the residue at the pole yields:

$$\oint\frac{\omega'\,d\omega'}{(\omega'^2+\gamma^2)(\omega'-\omega+i\varepsilon)} = 2\pi i\lim_{\omega'\to i\gamma}\frac{(\omega'-i\gamma)\omega'}{(\omega'-\omega)(\omega'^2+\gamma^2)} = \frac{\pi i}{i\gamma-\omega} = -\frac{\pi i(\omega+i\gamma)}{\omega^2+\gamma^2}\,. \tag{15}$$

Inserting this result back into eq. (14), we end up with

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{Im}\tilde{\epsilon}(\omega')}{\omega'-\omega}\,d\omega' = -\frac{\omega_p^2}{\omega^2+\gamma^2}\,. \tag{16}$$

Hence, in light of eq. (10),

$$1 + \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{Im}\tilde{\epsilon}(\omega')}{\omega'-\omega}\,d\omega' = 1 - \frac{\omega_p^2}{\omega^2+\gamma^2} = \operatorname{Re}\tilde{\epsilon}(\omega)\,, \tag{17}$$

which verifies the Kramers-Kronig relation for $\tilde{\epsilon}(\omega)$ given in eq. (8).

2. The theory of electromagnetism in $3+1$ spacetime dimensions can be generalized to $n+1$ spacetime dimensions as follows. The indices of the second-rank totally antisymmetric electromagnetic field strength tensor $F^{\mu\nu}$ now take on values $\mu,\ \nu \in \{0, 1, \ldots, n\}$. The dynamical Maxwell equations are given (in gaussian units) by:

$$\partial_\mu F^{\mu\nu} = \frac{S_{n-1}}{c} J^\nu \,, \tag{18}$$

where

$$S_{n-1} \equiv \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{19}$$

is the surface area of an n-dimension ball of unit radius. For example, $S_1 = 2\pi$, $S_2 = 4\pi$, etc. The dual electromagnetic field strength tensor is defined by employing the totally antisymmetric rank $(n+1)$ ϵ-tensor. The latter can be used to express the kinematical Maxwell equations,

$$\epsilon^{\mu\cdots\alpha\beta}\partial_\mu F_{\alpha\beta} = 0 \,, \tag{20}$$

where $\cdots$ in eq. (20) represents $n-2$ free indices that are not exhibited explicitly. By convention, we choose $\epsilon^{012\cdots n} = +1$.

(a) In $n+1$ spacetime dimensions, how many independent components are needed to describe $F^{\mu\nu}$? How many of these components represent the electric field and how many of these components represent the magnetic field?

In general, a rank 2 tensor in $n+1$ spacetime dimensions has $(n+1)^2$ components. But, an antisymmetric tensor satisfies $F^{00} = F^{11} = \cdots = 0$ and $F^{\mu\nu} = -F^{\nu\mu}$. Hence, the number of independent components is given by:

$$\frac{1}{2}\big[(n+1)^2 - (n+1)\big] = \frac{1}{2}n(n+1) \,. \tag{21}$$

As a check, in 4 spacetime dimensions (i.e., with $n = 3$), $F^{\mu\nu}$ has 6 components.

The components of the electric field are given by:

$$E^i = F^{i0} \,, \qquad \text{for } i \in \{1, 2, \ldots, n\} \,. \tag{22}$$

Since the electric field vector has n components, we can use eq. (21) to conclude that the number of components of the magnetic field is

$$\frac{1}{2}n(n+1) - n = \frac{1}{2}n(n-1) \,. \tag{23}$$

If you wish to have a more explicit result, recall that for $n = 3$, we showed in class that $B^k = -\frac{1}{2}\epsilon^{ijk}F^{ij}$, where there is an implicit sum over the two pairs of repeated indices. In n spatial dimensions, the Levi-Civita tensor has n components. Thus, for $n \geq 4$,

$$B^{k\ell\cdots} = -\frac{1}{2}\epsilon^{ijk\ell\cdots}F^{ij} \,, \tag{24}$$

where $\cdots$ represents the remaining $n-4$ space indices. That is, B is a totally antisymmetric rank $n-2$ tensor, which (with a little help from combinatorial mathematics) has

$$\frac{n!}{(n-2)!\,2!} = \frac{1}{2}n(n-1)\,, \tag{25}$$

components, in agreement with eq. (23).

(b) Consider the theory of electromagnetism in $2+1$ spacetime dimensions, where $F^{\mu\nu}$ can be constructed by deleting the fourth row and fourth column of the $3+1$ dimensional version of $F^{\mu\nu}$. Note that the electric field vector is now of the form $\vec{\boldsymbol{E}} = \hat{\boldsymbol{x}}E_x + \hat{\boldsymbol{y}}E_y$, as expected, but magnetic field consists of a single "component" which you can denote by B. Define a "dual" electromagnetic field strength tensor and show that it is a Lorentz three-vector of the $2+1$ dimensional spacetime. Determine its components in terms of the electric and magnetic fields. In light of the $2+1$ dimensional version of eq. (20), show that

$$\frac{d}{dt}\int B(\vec{\boldsymbol{x}}, t)\, d^2x = 0\,, \tag{26}$$

assuming that the electric and magnetic fields vanish sufficiently fast at spatial infinity.

Starting with $F^{\mu\nu}$ in $3+1$ dimensional spacetime and deleting the four row and column, we obtain:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & -B \\ E_y & B & 0 \end{pmatrix} . \tag{27}$$

Note that what was B_z in $3+1$ dimensions is now denoted by B. More explicitly,

$$E^i = F^{i0}\,, \qquad B = -\frac{1}{2}\epsilon^{ij}F^{ij}\,, \qquad \text{for } i, j \in \{1, 2\}, \tag{28}$$

where ϵ^{ij} is the Levi-Civita tensor in 2 spatial dimensions ($\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 0$), and there is an implicit sum over the pair of repeated indices. In particular, note that

$$E_x \equiv E^1 \quad \text{and} \quad E_y \equiv E^2\,. \tag{29}$$

In $3+1$ dimensional spacetime, the dual of the electromagnetic field strength tensor is an antisymmetric rank 2 tensor that is defined as:

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,, \tag{30}$$

where $F_{\alpha\beta} = g_{\alpha\rho}g_{\beta\sigma}F^{\rho\sigma}$ where $g = \text{diag}(1\,,\,-1\,,\,-1\,,\,-1)$ is the Minkowski metric in $3+1$ spacetime dimensions. In $2+1$ spacetime dimensions, the Levi-Civita tensor has only three indices. Hence, in this case the dual of the electromagnetic field strength tensor is a Lorentz three-vector that is given by

$$\widetilde{F}^{\mu} \equiv -\frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}\,, \tag{31}$$

where the minus sign has been inserted for convenience. In eq. (31), $F_{\alpha\beta} = g_{\alpha\rho}g_{\beta\sigma}F^{\rho\sigma}$, where $g = \text{diag}(1\,,\,-1\,,\,-1)$ is the Minkowski metric in $2+1$ spacetime dimensions. Using eqs. (27) and (31), the components of $\widetilde{F}^{\mu}$ are:

$$\widetilde{F}^0 = -\frac{1}{2}\big(\epsilon^{012}F_{12} + \epsilon^{021}F_{21}\big) = -F^{12} = B\,, \tag{32}$$

$$\widetilde{F}^1 = -\frac{1}{2}\big(\epsilon^{102}F_{02} + \epsilon^{120}F_{20}\big) = -F^{02} = E^2 = E_y\,, \tag{33}$$

$$\widetilde{F}^2 = -\frac{1}{2}\big(\epsilon^{201}F_{01} + \epsilon^{210}F_{10}\big) = F^{01} = -E^1 = -E_x\,, \tag{34}$$

after making use of eq. (28). Note that eqs. (33) and (34) can be rewritten as

$$\widetilde{F}^i = \epsilon^{ij}E^j\,, \qquad \text{for } i \in \{1,2\}, \tag{35}$$

with an implicit sum over the repeated index j.

In $2+1$ spacetime dimensions, eq. (20) yields

$$\partial_\mu \widetilde{F}^{\mu} = 0\,, \tag{36}$$

which is equivalent to

$$\frac{1}{c}\frac{\partial F^0}{\partial t} + \partial_i \widetilde{F}^i = 0\,. \tag{37}$$

Using the results of eqs. (32) and (35),

$$\frac{1}{c}\frac{\partial B}{\partial t} + \epsilon^{ij}\partial_i E^j = 0\,. \tag{38}$$

Integrating this result over two-dimensional space,

$$\frac{1}{c}\frac{d}{dt}\int B(\vec{\boldsymbol{x}},t)\,d^2x + \epsilon^{ij}\int \partial_i E^j(\vec{\boldsymbol{x}},t)\,d^2x = 0\,. \tag{39}$$

The second integral in eq. (39) can be converted into a line integral at infinity by using the two-dimensional version of Stokes's theorem. Under the assumption that the electric fields vanish at spatial infinity, the end result is

$$\frac{d}{dt}\int B(\vec{\boldsymbol{x}},t)\,d^2x = 0\,. \tag{40}$$

(c) Consider a reference frame K' that moves at a constant velocity $c\beta\hat{\boldsymbol{x}}$ with respect to reference frame K. Using the behavior of a Lorentz three-vector under a boost, obtain expressions for the electric and magnetic fields, E_x, E_y, and B, in reference frame K' in terms of the corresponding fields in reference frame K. Check that your results coincide with the expected result in $3+1$ dimensional spacetime.

A Lorentz three-vector in $2+1$ spacetime dimensions transforms under a boost similarly to the transformation of a Lorentz four-vector in $3+1$ spacetime dimensions. Writing $w^\mu = (w^0\,;\, w^1\,,\, w^2)$, the corresponding Lorentz transformation is given by

$$w'^0 = \gamma(w^0 - \beta w^1)\,, \tag{41}$$

$$w'^1 = \gamma(w^1 - \beta w^0)\,, \tag{42}$$

$$w'^2 = w^2\,. \tag{43}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$. The only difference between 2 and 3 spatial dimensions is that in 2 dimensions there is only one spatial direction transverse to $\vec{\boldsymbol{\beta}}$ whereas in 3 dimensions there are two spatial directions transverse to $\vec{\boldsymbol{\beta}}$.

Using eqs. (32)–(34), it follows that

$$B' = \gamma(B - \beta E_y)\,, \tag{44}$$

$$E'_y = \gamma(E_y - \beta B)\,, \tag{45}$$

$$E'_x = E_x\,. \tag{46}$$

To check that eqs. (44)–(46) are correct, note that in $3+1$ spacetime, one can employ eq. (11.148) of Jackson, where the third component of the electric field and the first and second components of the magnetic field are discarded whereas B_3 is identified as B.

(d) In 2 spatial dimensions, there are two different vector differential operators,

$$\vec{\boldsymbol{\nabla}} \equiv \boldsymbol{\hat{x}}\frac{\partial}{\partial x} + \boldsymbol{\hat{y}}\frac{\partial}{\partial y}\,, \tag{47}$$

$$\vec{\boldsymbol{\nabla}}_\perp \equiv \boldsymbol{\hat{x}}\frac{\partial}{\partial y} - \boldsymbol{\hat{y}}\frac{\partial}{\partial x}\,, \tag{48}$$

where $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}_\perp = 0$ (which justifies the notation). Using eqs. (18) and (20), write out Maxwell equations explicitly in terms of the electric field vector, the magnetic field, the charge density and current density vector, and the differential operators defined above. Show that in 2 spatial dimensions, there are only three Maxwell equations (in contrast to the four equations obtained in three spatial dimensions).

In $2+1$ dimensional spacetime, Maxwell's equations are given by eqs. (18) and (36),

$$\partial_\mu F^{\mu\nu} = \frac{2\pi}{c} J^\nu\,, \tag{49}$$

$$\partial_\mu \widetilde{F}^\mu = 0\,. \tag{50}$$

Note that eq. (28) implies that

$$F^{i0} = -F^{0i} = E^i\,, \qquad\qquad F^{ij} = -F^{ji} = -\epsilon^{ij} B\,. \tag{51}$$

Moreover, in terms of components, eqs. (47) and (48) are equivalent to

$$\boldsymbol{\nabla}_i = \partial_i \,, \qquad \left(\boldsymbol{\nabla}_\perp\right)_i = \epsilon^{ij}\partial_j \,. \tag{52}$$

Using eqs. (27), (32), and (35) and $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$, eq. (49) yields

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 2\pi\rho\,, \tag{53}$$

$$\vec{\boldsymbol{\nabla}}_\perp B - \frac{1}{c}\frac{\partial\vec{\boldsymbol{E}}}{\partial t} = \frac{2\pi}{c}\vec{\boldsymbol{J}}\,. \tag{54}$$

Finally, we have already shown that eq. (50) yields eq. (38) which can be rewritten as

$$\frac{1}{c}\frac{\partial B}{\partial t} = \vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,. \tag{55}$$

It is noteworthy that there is no 2 dimensional analog of the 3 dimensional Maxwell equation, $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = 0$. A further exploration of electrodynamics in 2 spatial dimensions can be found in a paper by Kirk T. McDonald, *Electrodynamics in 1 and 2 Spatial Dimensions*, which is available at `http://kirkmcd.princeton.edu/examples/2dem.pdf`.

3. A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency ω and angle ϑ_0 with respect to the z-axis as shown in Fig. 1. That is, the time-dependence of the azimuthal angle is $\varphi_0(t) = \varphi_0 - \omega t$. Electromagnetic radiation is emitted by the precessing dipole.

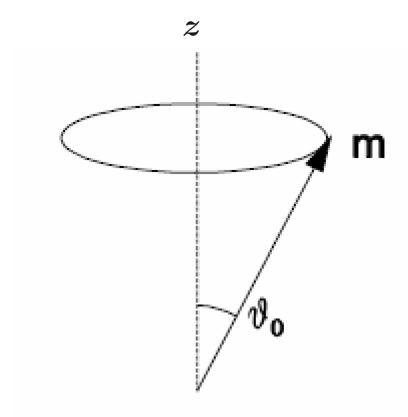


Figure 1: A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency ω and angle ϑ_0 with respect to the z-axis.

(a) Write out an explicit expression for the time-dependent magnetic dipole vector $\vec{\boldsymbol{m}}$ in terms of its magnitude m_0, the angles ϑ_0 and φ_0 and the time t. Show that $\vec{\boldsymbol{m}}$ consists of the sum of a time-dependent term and a time-independent term. Verify that the time-dependent term can be written as $\mathrm{Re}(\vec{\boldsymbol{\mu}}\,e^{-i\omega t})$, for some suitably chosen complex vector $\vec{\boldsymbol{\mu}}$.

In light of Fig. 1, the magnetic dipole moment vector is given by:

$$\vec{\boldsymbol{m}} = m_0\Big[\hat{\boldsymbol{x}}\sin\vartheta_0\cos(\varphi_0-\omega t)+\hat{\boldsymbol{y}}\sin\vartheta_0\sin(\varphi_0-\omega t)+\hat{\boldsymbol{z}}\cos\vartheta_0\Big] \tag{56}$$

$$= \mathrm{Re}\Big[m_0\sin\vartheta_0\, e^{i\varphi_0}(\hat{\boldsymbol{x}}-i\hat{\boldsymbol{y}})e^{-i\omega t}\Big]+m_0\cos\vartheta_0\,\hat{\boldsymbol{z}}\,.$$

Thus, we can write the time-dependent term of $\vec{\boldsymbol{m}}$ as $\mathrm{Re}(\vec{\boldsymbol{\mu}}\, e^{-i\omega t})$, where

$$\vec{\boldsymbol{\mu}} = m_0\sin\varphi_0\, e^{i\varphi_0}(\hat{\boldsymbol{x}}-i\hat{\boldsymbol{y}})\,. \tag{57}$$

(b) Compute the angular distribution of the time-averaged radiated power, with respect to the z-axis defined in the above figure.

The angular distribution of the time-averaged power is given by eq. (9.21) of Jackson in SI units,

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\big[r^2\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^{\,*}\big]\,.$$

The magnetic and electric fields of the magnetic dipole are given by eqs. (9.35) and (9.36) of Jackson. Keeping only the leading terms of $\mathcal{O}(1/r)$, we see that

$$\vec{\boldsymbol{H}} = -\frac{1}{Z_0}\vec{\boldsymbol{E}}\times\hat{\boldsymbol{n}}\,,$$

where $Z_0=\sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. It follows that

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^{\,*} = -\frac{1}{Z_0}\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{E}}^{\,*}\times\hat{\boldsymbol{n}}) = \frac{1}{Z_0}\Big[|\vec{\boldsymbol{E}}|^2-|\vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}|^2\Big] = \frac{1}{Z_0}|\vec{\boldsymbol{E}}|^2\,,$$

since $\vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}=0$ (due to the transverse nature of electromagnetic radiation). Hence,

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|\vec{\boldsymbol{E}}|^2\,, \tag{58}$$

where the leading $\mathcal{O}(1/r)$ term of eq. (9.36) of Jackson, applied to the complex magnetic moment vector $\vec{\boldsymbol{\mu}}$, yields

$$\vec{\boldsymbol{E}} = -\frac{Z_0}{4\pi}k^2(\vec{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}})\frac{e^{ikr}}{r}\,. \tag{59}$$

Inserting this result into eq. (58), we end up with

$$\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2}\,k^4|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}|^2\,. \tag{60}$$

The squared magnitude of the cross product above is easily computed,

$$|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}|^2 = (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}})\cdot(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}^{\,*}) = |\vec{\boldsymbol{\mu}}|^2-|\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\mu}}|^2\,,$$

since $\hat{\boldsymbol{n}}$ is a unit vector. Explicitly, $\vec{\boldsymbol{\mu}}$ is given by eq. (57) and

$$\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}}+\sin\theta\sin\phi\,\hat{\boldsymbol{y}}+\cos\theta\,\hat{\boldsymbol{z}}\,.$$

Hence, it follows that

$$|\vec{\boldsymbol{\mu}}|^2 = 2m_0^2\sin^2\vartheta_0\,, \qquad\qquad |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\mu}}| = m_0\sin\vartheta_0\sin\theta\,,$$

and

$$|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}|^2 = m_0^2\sin^2\vartheta_0(2-\sin^2\theta) = m_0^2\sin^2\vartheta_0(1+\cos^2\theta)\,.$$

Thus, the angular distribution of the time-averaged radiated power is given by[1]

$$\frac{dP}{d\Omega} = \frac{Z_0 m_0^2\sin^2\vartheta_0}{32\pi^2}\,k^4(1+\cos^2\theta)\,. \tag{61}$$

An alternative technique for computing the time-averaged radiated power

Instead of evaluating eq. (60), which requires the complex magnetic moment $\vec{\boldsymbol{\mu}}$ given in eq. (57), one can instead employ the result of problem 9.7(a) of Jackson,

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2c^4}|\ddot{\vec{\boldsymbol{m}}}\times\hat{\boldsymbol{n}}|^2\,, \tag{62}$$

where $\ddot{\vec{\boldsymbol{m}}} \equiv d^2\vec{\boldsymbol{m}}/dt^2$, and $\vec{\boldsymbol{m}}$ is the time-dependent magnetic dipole moment given in eq. (56). Note that eq. (62) yields the time dependent power distribution, so to recover the results obtained in problem 1(b), we must time-average over one cycle.

For convenience, we rewrite eq. (56) here:

$$\vec{\boldsymbol{m}} = m_0\Big[\hat{\boldsymbol{x}}\sin\vartheta_0\cos(\varphi_0-\omega t)+\hat{\boldsymbol{y}}\sin\vartheta_0\sin(\varphi_0-\omega t)+\hat{\boldsymbol{z}}\cos\vartheta_0\Big]$$

Taking two time derivatives, we obtain:

$$\ddot{\vec{\boldsymbol{m}}} = -m_0\omega^2\Big[\hat{\boldsymbol{x}}\sin\vartheta_0\cos(\varphi_0-\omega t)+\hat{\boldsymbol{y}}\sin\vartheta_0\sin(\varphi_0-\omega t)\Big]\,. \tag{63}$$

Next, we compute the square of the cross product,

$$|\ddot{\vec{\boldsymbol{m}}}\times\hat{\boldsymbol{n}}|^2 = \ddot{\vec{\boldsymbol{m}}}\cdot\ddot{\vec{\boldsymbol{m}}} - (\hat{\boldsymbol{n}}\cdot\ddot{\vec{\boldsymbol{m}}})^2\,,$$

after using the fact that $\hat{\boldsymbol{n}}$ is a unit vector,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi+\hat{\boldsymbol{y}}\sin\theta\sin\phi+\hat{\boldsymbol{z}}\cos\theta\,. \tag{64}$$

Using eqs. (63) and (64), it follows that

$$\ddot{\vec{\boldsymbol{m}}}\cdot\ddot{\vec{\boldsymbol{m}}} = m_0^2\omega^4\sin^2\vartheta_0\,,$$

and

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\ddot{\vec{\boldsymbol{m}}} &= -m_0\omega^2\sin\vartheta_0\sin\theta\Big[\cos\phi\cos(\varphi-\omega t)+\sin\phi\sin(\varphi_0-\omega t)\Big]\\ &= -m_0\omega^2\sin\vartheta_0\sin\theta\cos(\omega t-\varphi_0+\phi)\,.\end{aligned}$$

[1]To obtain the angular distribution of the time-averaged radiated power in gaussian units, one must replace $Z_0\to 4\pi/c$ and $m_0\to m_0c$ in eq. (61).

Hence,

$$|\dddot{\boldsymbol{m}} \times \hat{\boldsymbol{n}}|^2 = m_0^2\omega^4 \sin^2\vartheta\Big[1 - \sin^2\theta\cos^2(\omega t - \varphi_0 + \phi)\Big]\,.$$

Inserting the above result into eq. (62) and using $\omega = kc$, we end up with

$$\frac{dP(t)}{d\Omega} = \frac{Z_0 m_0^2 \sin^2\vartheta}{16\pi^2}\, k^4\Big[1 - \sin^2\theta\cos^2(\omega t - \varphi_0 + \phi)\Big]\,. \tag{65}$$

Time-averaging over one cycle, $\langle\cos^2(\omega t - \varphi_0 + \phi)\rangle = \frac{1}{2}$. Since $1 - \frac{1}{2}\sin^2\theta = \frac{1}{2}(1 + \cos^2\theta)$, we recover eq. (61). One can also check that the total power obtained by integrating eq. (65) over solid angles is time-independent and coincides with eq. (67).

(c) Compute the total power radiated.

Integrating eq. (61) over solid angles,

$$\int d\Omega(1 + \cos^2\theta) = 2\pi\int_{-1}^{1}(1 + \cos^2\theta)\, d\cos\theta = \frac{16\pi}{3}\,. \tag{66}$$

Hence,

$$P = \frac{Z_0 m_0^2 k^4 \sin^2\vartheta_0^2}{6\pi}\,. \tag{67}$$

(d) What is the polarization of the radiation measured by an observer located along the positive z-axis far from the precessing dipole? How would your answer change if the observer were located in the x–y plane?

The polarization is determined from the electric field given in eq. (59). Thus, we must evaluate $\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}}$,

$$\begin{aligned}\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}} &= \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ m_0\sin\vartheta_0 e^{i\varphi_0} & -im_0\sin\vartheta_0 e^{i\varphi_0} & 0 \end{pmatrix} \\ &= im_0 e^{i\varphi_0}\sin\vartheta_0\cos\theta(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}) - im_0\sin\vartheta_0\sin\theta\, e^{i(\varphi_0 - \phi)}\,\hat{\boldsymbol{z}}\,. \end{aligned} \tag{68}$$

The polarization depends on the location of the observer. If the observer is located on the positive z-axis then $\theta = 0$. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}$, which corresponds to right-circularly polarized light [cf. p. 300 of Jackson]. If the observer is located in the x–y plane, the $\theta = \frac{1}{2}\pi$. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{z}}$, which corresponds to linearly polarized light in the z-direction.

<u>*REMARK*</u>: If $\theta = \pi$, then $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}$, which corresponds to left-circularly polarized light. For any other value of $\theta \neq 0, \frac{1}{2}\pi$ or π, the radiation is elliptically polarized.