Solution of the Inhomogeneous Wave Equation via the Green Function

For electromagnetic waves in vacuum (propagating at the speed of light c), the four-vector potential in the Lorenz gauge satisfies

$$\Box A^\mu(x) = \frac{4\pi}{c} J^\mu(x)\,, \tag{1}$$

where gaussian units are being employed, $A^\mu = (\Phi\,;\,\vec{\boldsymbol{A}})$, and $\Box \equiv \partial_\mu \partial^\mu$. This is an inhomogeneous wave equation with a source term of $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$. Naively, one might propose a solution of the form

$$A^\mu(x) = \frac{4\pi}{c} \Box^{-1} J^\mu(x)\,. \tag{2}$$

However, we know that $\Box^{-1}$ does not exist due to the fact that there exist eigenfunctions of the D'Alembertian operator $\Box$ with zero eigenvalues.[1] In particular, given the four vectors $x = (ct\,;\,\vec{\boldsymbol{x}})$ and $k = (\omega/c\,;\,\vec{\boldsymbol{k}})$, with corresponding dot product $k\cdot x = \omega t - \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}$, it follows that

$$\Box\, e^{ik\cdot x} = \Box\, e^{i\omega t - i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}} = \left(k^2 - \frac{\omega^2}{c^2}\right) e^{i\omega t - i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}} = 0\,, \tag{3}$$

after making use of the dispersion relation in vacuum, $|\vec{\boldsymbol{k}}| = \omega/c$. That is $e^{ik\cdot x}$ is an eigenfunction of $\Box$ with zero eigenvalue.

Consider the inhomogeneous wave equation,[2]

$$\Box\, \Psi(\vec{\boldsymbol{x}}, t) = 4\pi f(\vec{\boldsymbol{x}}, t)\,. \tag{4}$$

The most general solution to this equation is of the form

$$\Psi(\vec{\boldsymbol{x}}, t) = \Psi_0(\vec{\boldsymbol{x}}, t) + \Psi_1(\vec{\boldsymbol{x}}, t)\,, \tag{5}$$

where

$$\Box\, \Psi_0(\vec{\boldsymbol{x}}, t) = 0\,, \qquad\qquad \Box\, \Psi_1(\vec{\boldsymbol{x}}, t) = 4\pi f(\vec{\boldsymbol{x}}, t)\,. \tag{6}$$

The most general solution to the homogeneous wave equation is well known. It can be expanded in a Fourier series consisting of a linear combination of plane waves of mode $(\vec{\boldsymbol{k}}, \omega)$, where $\omega = |\vec{\boldsymbol{k}}|c$. The so-called particular solution, $\Psi_1(\vec{\boldsymbol{x}}, t)$, satisfies the inhomogeneous wave equation.

[1] It is well known that in finite dimension vector spaces, a linear transformation is noninvertible if it has a zero eigenvalue. This is equivalent to saying that the matrix that represents the linear transformation is noninvertible if its determinant vanishes, since the determinant of a matrix is equal to a product of its eigenvalues. This result generalizes to infinite dimensional function spaces.

[2] The factor of 4π in eq. (4) is conventional, as this matches the $\mu = 0$ component of eq. (1).

Since $\Box$ is not an invertible operator, it follows that either no solution exists or an infinite number of solutions to eq. (4) exist. It turns out that the latter is the case here. For the problem at hand, we wish to choose the unique solution that satisfies the physical requirements of our problem. In the case of electromagnetic wave propagation, the condition we shall impose is causality. In this context, causality implies that the waves are generated only after the source turns on and the signal reaches a distant observer only after the waves have propagated with a finite speed (equal to the speed of light c).

We will solve eq. (4) by employing the Green function technique. We define a Green function $G(x, x')$ (where x and x' are four vectors) by[3]

$$\Box_x G(x - x') = 4\pi\delta^4(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,') \,, \tag{7}$$

where $\Box_x$ indicates partial differentiation with respect to the four vector x (with x' held fixed), and the four-dimensional delta function is defined by

$$\delta^4(x - x') = \delta^3(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,')\delta(x - x_0) \,, \tag{8}$$

where $x_0 = ct$. By translational invariance, $G(x, x') = G(x - x')$. That is, the Green function can only depend on the difference of the two spacetime points.

As a check, assuming that $G(x - x')$ is known, the particular solution to eq. (4) is given by

$$\Psi_1(x) = \int d^4x'\, G(x - x') f(x') \,. \tag{9}$$

As a check, we apply $\Box_x$ to both sides eq. (9) to obtain

$$\Box\,\Psi_1(x) = 4\pi \int d^4x'\, f(x')\Box\, G(x - x') = 4\pi \int d^4x'\, f(x')\delta^4(x - x') = 4\pi f(x) \,, \tag{10}$$

as required.

Since $G(x, x') = G(x - x')$, we can set $x' = 0$ without loss of generality and solve the equation

$$\Box\, G(x) = 4\pi\delta^4(x) \,. \tag{11}$$

We shall solve eq. (11) using a Fourier transform method, as this will convert the differential equation into an algebraic equation that is easily solved. The Fourier representation of $G(x)$ is

$$G(x) = \frac{1}{(2\pi)^4} \int d^4k\, \widetilde{G}(k)\, e^{-ik\cdot x} \,, \tag{12}$$

where $k\cdot x = k_0 x_0 - \vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{x}}$. We shall also make use of the integral representation of the four-dimensional delta function,

$$\delta^4(x) = \frac{1}{(2\pi)^4} \int d^4x\, e^{-ik\cdot x} \,. \tag{13}$$

Plugging eqs. (12) and (13) into eq. (11), we obtain the algebraic equation,

$$-k^2 \widetilde{G}(k) = 4\pi \,, \tag{14}$$

[3]It is convenient to employ the factor of 4π in eq. (7) in light of the factor of 4π that appears in eq. (4).

which has the immediate solution

$$\widetilde{G}(k) = -\frac{4\pi}{k^2} = -\frac{4\pi}{k_0^2 - |\vec{\boldsymbol{k}}|^2} \,. \tag{15}$$

Plugging in eq. (15) back into eq. (11),

$$G(x) = -\frac{4\pi}{(2\pi)^4}\int \frac{d^4k}{k^2}\, e^{-ik\cdot x} = -\frac{1}{4\pi^3}\int d^3k\; e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}}\int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0x_0}}{k_0^2-\kappa^2}\,, \tag{16}$$

where $\kappa \equiv |\vec{\boldsymbol{k}}|$. We immediately see a problem. Namely, the integral over k_0 does not exist due to the singularities of the integrand at $k_0 = \pm\kappa$. Thus, $G(x)$ as derived above does not exist. Of course, this is equivalent to saying that $\Box^{-1}$ does not exist. If we had obtained a unique convergent expression for $G(x)$ we would have concluded that $G(x) = 4\pi\Box^{-1}\delta^4(x)$ is well-defined, implying the existence of $\Box^{-1}$.

However, as previously noted, eq. (11) does not possess a unique solution. Hence, we shall adopt a strategy to produce many solutions to eq. (11) and choose the unique solution among them that respects causality. This can be accomplished by examining four different expressions which are slightly perturbed versions eq. (16) that depend on a real positive infinitesimal parameter ϵ. Here are four possible choices:

$$G_R(x) = -\frac{1}{4\pi^3}\int d^3k\; e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}}\int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0x_0}}{(k_0-\kappa+i\epsilon)(k_0+\kappa+i\epsilon)}\,, \tag{17}$$

$$G_A(x) = -\frac{1}{4\pi^3}\int d^3k\; e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}}\int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0x_0}}{(k_0-\kappa-i\epsilon)(k_0+\kappa-i\epsilon)}\,, \tag{18}$$

$$G_F(x) = -\frac{1}{4\pi^3}\int d^3k\; e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}}\int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0x_0}}{(k_0-\kappa+i\epsilon)(k_0+\kappa-i\epsilon)}\,, \tag{19}$$

$$G_{AF}(x) = -\frac{1}{4\pi^3}\int d^3k\; e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}}\int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0x_0}}{(k_0-\kappa-i\epsilon)(k_0+\kappa+i\epsilon)}\,. \tag{20}$$

As long as $\epsilon \neq 0$, these integrals are all well-defined. Moreover, one can check that they are indeed solutions to eq. (11), where the limit of $\epsilon \to 0$ is taken after evaluating the corresponding integrals. Thus, we have actually found four possible solutions to eq. (11). Indeed, one can take arbitrary linear combinations of the above four solutions and then normalize the corresponding combination to find an infinite number of solutions to eq. (11).

The condition of causality is

$$G(x) = 0 \text{ for all } t < 0. \tag{21}$$

This condition arises since in eq. (11), we see that the source term (i.e., the delta function) is zero until $x = (ct\,;\,\vec{\boldsymbol{x}}) = 0$. Thus, no signal can be generated until $t \geq 0$. This condition uniquely selects the so-called retarded Green function,

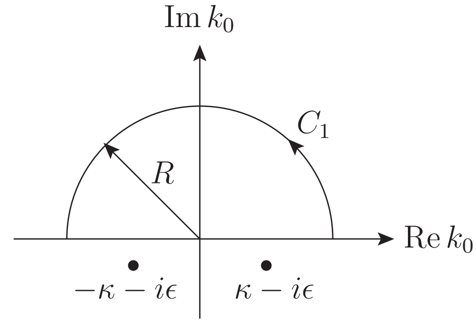
$$G(x) = \lim_{\epsilon\to 0} G_R(x) = \frac{1}{4\pi^3}\int d^3k\; e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}}\int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0x_0}}{(k_0-\kappa+i\epsilon)(k_0+\kappa+i\epsilon)}\,, \tag{22}$$

where the limit $\epsilon \to 0$ should be applied after evaluating the integral above. Let us verify this claim. Suppose that $x_0 = ct < 0$. Then $x_0 = -|x_0|$. We can perform the integral over x_0 by analyzing the integrand in the complex k_0 plane. Because $|x_0| < 0$, we can close the contour in the upper half complex lane with a semicircle of radius R, which we denote by C_1 in the figure below. By taking $R \to \infty$, we see that along the semicircular path C_1,

$$e^{-ik_0x_0} = e^{i(\operatorname{Re} k_0 + i \operatorname{Im} k_0)|x_0|} = e^{i|x_0| \operatorname{Re} k_0} e^{-|x_0| \operatorname{Im} k_0} \to 0 \text{ as } R \to \infty, \tag{23}$$

since $\operatorname{Im} k_0 > 0$ in the upper half complex plane. That is, in the limit of $R \to \infty$, there is no contribution to the integral along the semicircle C_1 as a result of eq. (23). In particular, the integral given in eq. (22) is equal to the integral over the closed integration path exhibited in the figure below.

$$I_1(R, \varepsilon) \equiv \int_{-R}^{R} + \int_{C_1} dk_0 \, \frac{e^{-ik_0x_0}}{(k_0 - \kappa + i\epsilon)(k_0 + \kappa + i\epsilon)}$$



Since there are no poles inside the closed contour, it follows (by Cauchy's theorem) that $\lim_{R\to\infty} I_1(R, \epsilon) = 0$. Hence, we conclude that

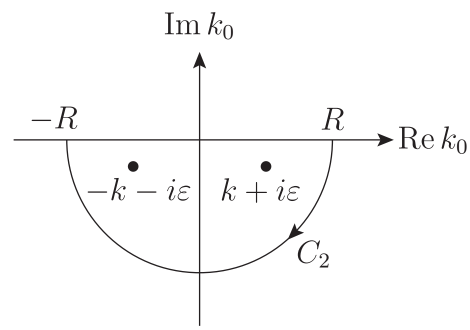
$$G_R(x) = \lim_{\epsilon\to 0} I_1(\infty, \epsilon) = 0\,, \text{ for } x_0 < 0. \tag{24}$$

Next, we consider the case of $x_0 = ct > 0$. Then $x_0 = |x_0|$. We can again perform the integral over x_0 by analyzing the integrand in the complex k_0 plane. Because $|x_0| > 0$, we can close the contour in the lower half complex plane with a semicircle of radius R, which we denote by C_2 in the figure below. By taking $R \to \infty$, we see that along the semicircular path C_2,

$$e^{-ik_0x_0} = e^{-i(\operatorname{Re} k_0 + i \operatorname{Im} k_0)|x_0|} = e^{-i|x_0| \operatorname{Re} k_0} e^{|x_0| \operatorname{Im} k_0} \to 0 \text{ as } R \to \infty, \tag{25}$$

since $\operatorname{Im} k_0 < 0$ in the lower half complex plane. That is, in the limit of $R \to \infty$, there is no contribution from the integration along the semicircle C_2 to the integral as a result of eq. (25). In particular, the integral given in eq. (22) is equal to the integral over the closed integration path exhibited in the figure below.

$$I_2(R, \varepsilon) \equiv \int_{-R}^{R} + \int_{C_2} dk_0 \, \frac{e^{-ik_0x_0}}{(k_0 - \kappa + i\epsilon)(k_0 + \kappa + i\epsilon)}$$



We now employ the theory of residues to evaluate the integral over the closed integration path exhibited above. Two poles are present inside the closed contour, and thus we pick up contribution from the two residues at the poles, respectively. Since the integration path shown above is clockwise, an extra minus sign appears in the residue theorem. Thus, in the limit of $\epsilon \to 0$,

$$\oint dk_0 \, \frac{e^{-ik_0 x_0}}{(k_0 - \kappa + i\epsilon)(k_0 + \kappa + i\epsilon)} = -2\pi i \, \frac{1}{2\kappa}\left[e^{-i\kappa x_0} - e^{i\kappa x_0}\right] = -\frac{2\pi}{\kappa}\sin\bigl(\kappa x_0\bigr)\,. \tag{26}$$

Returning to eq. (22), we have

$$G(x) = \lim_{\epsilon\to 0} G_R(x) = \frac{1}{2\pi^2}\Theta(x_0)\int d^3k \; e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}}\,\frac{\sin(\kappa x_0)}{\kappa}\,. \tag{27}$$

where we have introduced the step function,

$$\Theta(x_0) = \begin{cases} 1\,, & \text{for } x_0 > 0, \\ 0\,, & \text{for } x_0 < 0, \end{cases} \tag{28}$$

which automatically takes care of the two cases treated above. Writing $d^3k = \kappa^2\, d\kappa\, d\cos\theta\, d\phi$, the integration over ϕ is immediate and gives 2π. Hence,

$$\begin{aligned} G(x) = \lim_{\epsilon\to 0} G_R(x) &= \frac{1}{\pi}\Theta(x_0)\int_0^\infty \kappa\, d\kappa\, \sin(\kappa x_0)\int_{-1}^{1} e^{i\kappa r\cos\theta}\, d\cos\theta \\ &= \frac{1}{\pi}\Theta(x_0)\int_0^\infty \kappa\, d\kappa\, \sin(\kappa x_0)\,\frac{e^{i\kappa r} - e^{-i\kappa r}}{i\kappa r} \\ &= \frac{2}{\pi r}\Theta(x_0)\int_0^\infty d\kappa\, \sin(\kappa x_0)\,\sin(\kappa r) \\ &= \frac{\Theta(x_0)}{2\pi r}\int_{-\infty}^{\infty} d\kappa\, \left[e^{i(x_0 - r)\kappa} - e^{i(x_0 + r)\kappa}\right] \\ &= \frac{\Theta(x_0)}{r}\left[\delta(x_0 - r) + \delta(x_0 + r)\right. \\ &= \frac{\Theta(x_0)}{r}\,\delta(x_0 - r)\,, \end{aligned} \tag{29}$$

where we have introduced $r \equiv |\vec{\boldsymbol{x}}|$. Note that $\Theta(x_0)\delta(x_0 + r) = 0$, since the step function requires $x_0 > 0$ but the argument of the delta function never vanishes in this case. Finally, we note that $\Theta(x_0)\delta(x_0 - r) = \delta(x_0 - r)$, since the delta function is nonzero only in the case of $x_0 = r > 0$, in which case $\Theta(x_0) = 1$. We have therefore obtained our final result for the Green function of the D'Alembertian operator that is consistent with causality:

$$G(x) = \frac{\delta(x_0 - |\vec{\boldsymbol{x}}|)}{|\vec{\boldsymbol{x}}|}\,. \tag{30}$$

We can now insert eq. (30) back into eq. (9) to obtain the particular solution to the inhomogeneous wave equation,

$$\Psi_1(x) = \int d^3x' \int_{-\infty}^{\infty} dx_0' \, \frac{\delta(x_0 - x_0' - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|} \, f(\vec{\boldsymbol{x}}\,', t)\,. \tag{31}$$

The integration over x_0' is trivial due to the delta function. We end up with

$$\Psi_1(x) = \int \frac{d^3x'}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|} \, f\Big(x', t' = t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}{c}\Big)\,. \tag{32}$$

It is convenient to introduce the following notation:

$$\big[f(\vec{\boldsymbol{x}}\,', t')\big]_{\rm ret} \equiv f\Big(x', t' = t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}{c}\Big)\,. \tag{33}$$

Using this notation, the particular solution to the inhomogeneous wave equation is given by [cf. eq. (6.47) of Jackson]:

$$\Psi_1(\vec{\boldsymbol{x}}, t) = \int d^3x' \, \frac{\big[f(\vec{\boldsymbol{x}}\,', t')\big]_{\rm ret}}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}\,. \tag{34}$$

The time variable t' is called the retarded time. Its name derives from the fact that when the emission of radiation by charge and current density sources occurs, the radiated electromagnetic fields observed by an observer at the spacetime point $(ct\,;\vec{\boldsymbol{x}})$ is due to the sources located at $\vec{\boldsymbol{x}}\,'$ at and earlier (or retarded) time $t' = t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|/c$, allowing for the (causal) transmission of the radiated field from the source to the observer (which propagates at the speed of light c).

REFERENCE

For more details on Green function techniques, have a look at G. Barton, *Elements of Green's Functions and Propagation—Potentials, Diffusion and Waves* (Oxford Science Publications, Oxford, UK, 1989). The method discussed in these notes is treated in Appendix F.4 of this book. But there is much more valuable information to be found in Chapters 10–12 of this book which treats wave phenomena.