The equations of Jefimenko, Panofsky, and Phillips

1. Solution to the inhomogeneous wave equation

For electromagnetic waves in vacuum (propagating at the speed of light c), the four-vector potential in the Lorenz gauge satisfies

$$\Box A^\mu(x) = \frac{4\pi}{c} J^\mu(x)\,, \tag{1}$$

where gaussian units are being employed, $A^\mu = (\Phi\,;\,\vec{\boldsymbol{A}})$, and $\Box \equiv \partial_\mu \partial^\mu$. This is an inhomogeneous wave equation with a source term of $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$. The solution to the inhomogeneous wave equation is derived in the class handout entitled *Solution of the Inhomogeneous Wave Equation via the Green Function* and is given by:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t) = \frac{1}{c}\int d^3x'\,\frac{\big[\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')\big]_{\rm ret}}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}\,, \tag{2}$$

$$\Phi(\vec{\boldsymbol{x}}, t) = \frac{1}{c}\int d^3x'\,\frac{\big[\rho(\vec{\boldsymbol{x}}\,', t')\big]_{\rm ret}}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}\,, \tag{3}$$

where we have employed the notation

$$\big[f(\vec{\boldsymbol{x}}\,', t')\big]_{\rm ret} \equiv f\big(\vec{\boldsymbol{x}}\,', t' = t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|/c\big)\,, \tag{4}$$

and $t' \equiv t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|/c$ is the retarded time.

Using the vector and scalar potentials, one can obtain the electric and magnetic fields,

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi - \frac{1}{c}\frac{\partial \vec{\boldsymbol{A}}}{\partial t}\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}\,. \tag{5}$$

Using these relations, it follows from eq. (1) that the electric and magnetic fields also satisfy inhomogeneous wave equations:

$$\Box\vec{\boldsymbol{E}} = -4\pi\left(\vec{\boldsymbol{\nabla}}\rho + \frac{1}{c^2}\frac{\partial \vec{\boldsymbol{J}}}{\partial t}\right), \tag{6}$$

$$\Box\vec{\boldsymbol{B}} = \frac{4\pi}{c}\,\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{J}}\,. \tag{7}$$

The solutions to eqs. (6) and (7) are then given by

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = -\int \frac{d^3x'}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}\left[\vec{\boldsymbol{\nabla}}'\rho(\vec{\boldsymbol{x}}\,', t') + \frac{1}{c^2}\frac{\partial J(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\right]_{\rm ret}, \tag{8}$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t) = \frac{1}{c}\int \frac{d^3x'}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|}\left[\vec{\boldsymbol{\nabla}}' \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')\right]_{\rm ret}, \tag{9}$$

where $\vec{\boldsymbol{\nabla}}'$ corresponds to differentiation with respect to $\vec{\boldsymbol{x}}\,'$.

2. Applications of the chain rule

In order to analyze the consequences of eqs. (8) and (9), it is critical to note that

$$\vec{\boldsymbol{\nabla}}'\big[f(x',t')\big]_{\rm ret} \neq \big[\vec{\boldsymbol{\nabla}}' f(x',t')\big]_{\rm ret} . \tag{10}$$

In particular, $\vec{\boldsymbol{\nabla}}'\big[f(x',t')\big]_{\rm ret}$ must be evaluated at fixed t, since after applying the "ret" instruction to the evaluation of $f(x',t')$, which sets $t' = t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|/c$, the resulting expression $\big[f(x',t')\big]_{\rm ret}$ is a function of $\vec{\boldsymbol{x}}'$ and t. In contrast, $\big[\vec{\boldsymbol{\nabla}}' f(x',t')\big]_{\rm ret}$ must be evaluated at fixed t' since the "ret" instruction is applied *after* computing $\vec{\boldsymbol{\nabla}}' f(x',t')$. To relate the two quantities in eq. (10), one must employ the chain rule of partial differentiation.

The specific chain rules that we will require are given below. Suppose that we have a function $w = f\,(\vec{\boldsymbol{x}}',t')$, where t' is a function of $\vec{\boldsymbol{x}}'$ and t.

$$\left(\frac{\partial w}{\partial x'_i}\right)_t = \left(\frac{\partial w}{\partial x'_i}\right)_{t'} + \left(\frac{\partial w}{\partial t'}\right)_{\vec{\boldsymbol{x}}'}\left(\frac{\partial t'}{\partial x'_i}\right)_t , \tag{11}$$

$$\left(\frac{\partial w}{\partial t}\right)_{\vec{\boldsymbol{x}}'} = \left(\frac{\partial w}{\partial t'}\right)_{\vec{\boldsymbol{x}}'}\left(\frac{\partial t'}{\partial t}\right)_{\vec{\boldsymbol{x}}'} , \tag{12}$$

where the variable that is being held fixed is indicated explicitly by the subscript indicated by each partial derivative.

In the case of $w = \rho$, we can employ eq. (11) to obtain

$$\left(\vec{\boldsymbol{\nabla}}'\rho\right)_t = \left(\vec{\boldsymbol{\nabla}}'\rho\right)_{t'} + \left(\frac{\partial \rho}{\partial t'}\right)_{\vec{\boldsymbol{x}}'}\left(\vec{\boldsymbol{\nabla}}' t'\right)_t \tag{13}$$

Since $t' = t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|/c$, one can rewrite eq. (13) as

$$\vec{\boldsymbol{\nabla}}'[\rho]_{\rm ret} = \left[\vec{\boldsymbol{\nabla}}'\rho\right]_{\rm ret} + \left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret}\vec{\boldsymbol{\nabla}}'\left(-\frac{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}{c}\right) . \tag{14}$$

It is convenient to define

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}' \quad \text{and} \quad R = |\vec{\boldsymbol{R}}| = |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'| . \tag{15}$$

Note that when acting on a function $f(R)$, one can employ the chain rule to obtain

$$\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{x}}'} f(R) = -\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{x}}} f(R) = -\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{R}}}\, f(R) = -\hat{\boldsymbol{R}}\,\frac{\partial f}{\partial R} , \tag{16}$$

where $\hat{\boldsymbol{R}} \equiv \vec{\boldsymbol{R}}/R$, and we have employed the notation $\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{x}}'} \equiv \vec{\boldsymbol{\nabla}}'$. Hence, it follows that

$$\vec{\boldsymbol{\nabla}}'\left(-\frac{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}{c}\right) = \frac{\hat{\boldsymbol{R}}}{c} . \tag{17}$$

Hence, eqs. (14) and (17) yield

$$\big[\vec{\boldsymbol{\nabla}}'\rho\big]_{\rm ret} = \vec{\boldsymbol{\nabla}}'\big[\rho\big]_{\rm ret} - \frac{\hat{\boldsymbol{R}}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret} . \tag{18}$$

By a similar computation involving $\vec{\boldsymbol{J}}$, we obtain

$$\left[\vec{\boldsymbol{\nabla}}' \times \vec{\boldsymbol{J}}\right]_{\rm ret} = \vec{\boldsymbol{\nabla}}' \times [\vec{\boldsymbol{J}}]_{\rm ret} + \frac{1}{c}\left[\frac{\partial \vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret} \times \hat{\boldsymbol{R}}\,, \tag{19}$$

after reversing the order of the two vectors in the last cross product (which flips the sign of the last term above).

3. The Jefimenko equations

Starting with eq. (8), we make use of eq. (18) to obtain:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = -\int \frac{d^3x'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\left\{\vec{\boldsymbol{\nabla}}'[\rho]_{\rm ret} + \frac{1}{c^2}\left\{\frac{\partial \vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret} - \frac{\hat{\boldsymbol{R}}}{c}\left[\frac{\partial \rho}{\partial t'}\right]_{\rm ret}\right\}. \tag{20}$$

In regard to the first term inside the braces in eq. (20), we can integrate by parts. The surface term at infinity can be discarded under the assumption that $\rho(\vec{\boldsymbol{x}}\,', t')$ and is localized (and thus the integral over the surface at infinity vanishes). Using eq. (16),

$$\vec{\boldsymbol{\nabla}}'\frac{1}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|} = -\hat{\boldsymbol{R}}\frac{\partial}{\partial R}\left(\frac{1}{R}\right) = \frac{\hat{\boldsymbol{R}}}{R^2} \tag{21}$$

After the integration by parts, we obtain

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \int d^3x'\left\{\frac{\hat{\boldsymbol{R}}}{R^2}\,[\rho(\vec{\boldsymbol{x}}\,',t')]_{\rm ret} + \frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial\rho(\vec{\boldsymbol{x}}\,',t')}{\partial t'}\right]_{\rm ret} - \frac{1}{c^2R}\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t')}{\partial t'}\right]_{\rm ret}\right\}. \tag{22}$$

Likewise, starting with eq. (9), we make use of eq. (19) to obtain:

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{c}\int \frac{d^3x'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\left\{\vec{\boldsymbol{\nabla}}' \times [\vec{\boldsymbol{J}}]_{\rm ret} + \frac{1}{c}\left[\frac{\partial \vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret} \times \hat{\boldsymbol{R}}\right\}. \tag{23}$$

In regard to the first term inside the braces in eq. (23), we can again integrate by parts. The surface term at infinity can once again be discarded under the assumption that $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')$ and is localized. After employing eq. (16), we end up with[1]

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{c}\int d^3x'\left\{\left[\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t')\right]_{\rm ret} \times \frac{\hat{\boldsymbol{R}}}{R^2} + \left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t')}{\partial t'}\right]_{\rm ret} \times \frac{\hat{\boldsymbol{R}}}{cR}\right\}. \tag{24}$$

Eqs. (22) and (24) are known as the Jefimenko equations.[2] Note that these two equations [which are given in SI units in eqs. (6.55) and (6.56) of Jackson] are the time-dependent generalizations of the Coulomb and Biot-Savart laws, respectively.

[1]Eq. (24), in SI units, was obtained (prior to Jefimenko) using a Fourier transform technique in eq. (14-34) of Wolfgang K.H. Panofsky and Melba Phillips, *Classical Electricity and Magnetism*, 2nd edition (Addison-Wesley Publishing Company, Inc., Reading, MA, USA, 1962).

[2]See eqs. (15-7.5) and (15-7.6) in Oleg D. Jefimenko, *Electricity and Magnetism*, 2nd edition (Electric Scienetific Company, Star City, WV, USA, 1989).

4. Rewriting the Jefimenko equation for $\vec{E}$ following Panofsky and Phillips

We shall now perform further manipulations on eq. (22) which is reproduced below for the convenience of the reader:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\int d^3x'\left\{\frac{\hat{\boldsymbol{R}}}{R^2}\left[\rho(\vec{\boldsymbol{x}}',t')\right]_{\rm ret}+\frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial\rho(\vec{\boldsymbol{x}}',t')}{\partial t'}\right]_{\rm ret}-\frac{1}{c^2R}\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t')}{\partial t'}\right]_{\rm ret}\right\}. \tag{25}$$

The analog of eq. (18) is

$$\left[\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}\right]_{\rm ret}=\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\left[\vec{\boldsymbol{J}}\right]_{\rm ret}-\frac{\hat{\boldsymbol{R}}}{c}\boldsymbol{\cdot}\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}. \tag{26}$$

Using the continuity equation,

$$\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t')+\frac{\partial\rho(\vec{\boldsymbol{x}}',t')}{\partial t'}=0\,, \tag{27}$$

it then follows that

$$\left[\frac{\partial\rho(\vec{\boldsymbol{x}}',t')}{\partial t'}\right]_{\rm ret}=-\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\left[\vec{\boldsymbol{J}}\right]_{\rm ret}+\frac{\hat{\boldsymbol{R}}}{c}\boldsymbol{\cdot}\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}. \tag{28}$$

Inserting eq. (28) back into the second term on the right-hand side of eq. (25) yields

$$\int d^3x'\,\frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial\rho(\vec{\boldsymbol{x}}',t')}{\partial t'}\right]_{\rm ret}=-\frac{1}{c}\int d^3x'\,\frac{1}{R}\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\left[\vec{\boldsymbol{J}}\right]_{\rm ret}\hat{\boldsymbol{R}}+\frac{1}{c^2}\int d^3x'\,\frac{1}{R}\left(\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\right)\hat{\boldsymbol{R}}. \tag{29}$$

Focusing on the ith component of the first term on the right hand side of eq. (29),

$$-\frac{1}{c}\int d^3x'\,\frac{R_i}{R^2}\,\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\left[\vec{\boldsymbol{J}}\right]_{\rm ret}=\frac{1}{c}\int d^3x'\,\left[\vec{\boldsymbol{J}}\right]_{\rm ret}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}'\left(\frac{R_i}{R^2}\right), \tag{30}$$

after an integration by parts (and discarding the surface term at infinity). Noting that

$$\partial'_j\left(\frac{R_i}{R^2}\right)=-\partial_j\left(\frac{R_i}{R^2}\right)=-\partial_{R_j}\left(\frac{R_i}{R^2}\right)=-\frac{\delta_{ij}}{R^2}-R_i\hat{R}_j\frac{\partial}{\partial R}\left(\frac{1}{R^2}\right)=\frac{2\hat{R}_i\hat{R}_j-\delta_{ij}}{R^2}, \tag{31}$$

after writing $\hat{R}_i\equiv R_i/R$, it follows that

$$\begin{aligned}-\frac{1}{c}\int d^3x'\,\frac{1}{R}\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\left[\vec{\boldsymbol{J}}\right]_{\rm ret}\hat{\boldsymbol{R}}&=\frac{1}{c}\int d^3x'\,\frac{2\left(\left[\vec{\boldsymbol{J}}\right]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\right)\hat{\boldsymbol{R}}-\left[\vec{\boldsymbol{J}}\right]_{\rm ret}}{R^2}\\&=\frac{1}{c}\int d^3x'\,\frac{\left(\left[\vec{\boldsymbol{J}}\right]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\right)\hat{\boldsymbol{R}}+\left(\left[\vec{\boldsymbol{J}}\right]_{\rm ret}\times\hat{\boldsymbol{R}}\right)\times\hat{\boldsymbol{R}}}{R^2},\end{aligned} \tag{32}$$

after making use of the well-known triple vector product identity.

Hence, eq. (29) yields

$$\int d^3x'\,\frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial\rho(\vec{\boldsymbol{x}}\,',t')}{\partial t'}\right]_{\rm ret}=\frac{1}{c}\int d^3x'\,\frac{\Big([\vec{\boldsymbol{J}}]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\Big)\hat{\boldsymbol{R}}+\Big([\vec{\boldsymbol{J}}]_{\rm ret}\times\hat{\boldsymbol{R}}\Big)\times\hat{\boldsymbol{R}}}{R^2}$$

$$+\frac{1}{c^2}\int d^3x'\,\frac{1}{R}\left(\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\right)\hat{\boldsymbol{R}}\,. \tag{33}$$

Inserting this result back into eq. (25), we obtain

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\int d^3x'\left\{\frac{\hat{\boldsymbol{R}}}{R^2}\,[\rho(\vec{\boldsymbol{x}}\,',t')]_{\rm ret}+\frac{\Big([\vec{\boldsymbol{J}}]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\Big)\hat{\boldsymbol{R}}+\Big([\vec{\boldsymbol{J}}]_{\rm ret}\times\hat{\boldsymbol{R}}\Big)\times\hat{\boldsymbol{R}}}{cR^2}\right.$$

$$\left.+\frac{1}{c^2R}\left(\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\right)\hat{\boldsymbol{R}}-\frac{1}{c^2R}\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t')}{\partial t'}\right]_{\rm ret}\right\}. \tag{34}$$

Finally, we note that the second line of eq. (34) can be rewritten as a triple cross product. We therefore arrive at our final result:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\int d^3x'\left\{\frac{\hat{\boldsymbol{R}}}{R^2}\,[\rho(\vec{\boldsymbol{x}}\,',t')]_{\rm ret}+\frac{\Big([\vec{\boldsymbol{J}}]_{\rm ret}\boldsymbol{\cdot}\hat{\boldsymbol{R}}\Big)\hat{\boldsymbol{R}}+\Big([\vec{\boldsymbol{J}}]_{\rm ret}\times\hat{\boldsymbol{R}}\Big)\times\hat{\boldsymbol{R}}}{cR^2}\right.$$

$$\left.+\frac{1}{c^2R}\left(\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}\times\hat{\boldsymbol{R}}\right)\times\hat{\boldsymbol{R}}\right\}. \tag{35}$$

Eq. (35) was first obtained by Panofsky and Philips using a Fourier transform technique.[3]

5. The radiation fields

Consider the far (radiation) zone, corresponding to $d,\,\lambda\ll r$, where d is the length scale over which the charge and current sources are nonzero, $\lambda\equiv 2\pi/k$ is the wavelength of the emitted radiation, and $r\equiv|\vec{\boldsymbol{x}}|$ is the distance from the origin (where the sources are located) to the distant observer. In this parameter regime, one can neglect terms in the expressions for the electromagnetic fields that approach zero faster than $\mathcal{O}(1/r)$ as $r\to\infty$. Thus, the radiation fields correspond to the terms in eqs. (24) and (35) that scale as $1/R$, whereas the terms that scale as $1/R^2$ can usually be discarded. Thus,

$$\vec{\boldsymbol{E}}_{\rm rad}(\vec{\boldsymbol{x}},t)=\frac{1}{c^2}\int\frac{d^3x'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\left(\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t')}{\partial t'}\right]_{\rm ret}\times\hat{\boldsymbol{R}}\right)\times\hat{\boldsymbol{R}} \tag{36}$$

$$\vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}},t)=\frac{1}{c^2}\int\frac{d^3x'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t')}{\partial t'}\right]_{\rm ret}\times\hat{\boldsymbol{R}}\,. \tag{37}$$

[3]Eq. (35) can be found (using SI units) in eq. (14-42) of Wolfgang K.H. Panofsky and Melba Phillips, *Classical Electricity and Magnetism*, 2nd edition (Addison-Wesley Publishing Company, Inc., Reading, MA, USA, 1962). See also: Kirk T. McDonald, *The Relation Between Expressions for Time-Dependent Electromagnetic Fields Given by Jefimenko and by Panofsky and Phillips*, Am. J. Phys. **65**, 1074–1076 (1997) and `https://kirkmcd.princeton.edu/examples/jefimenko.pdf`.

In the limit of large $|\vec{\boldsymbol{x}}|$, we can approximate

$$\hat{\boldsymbol{R}} = \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|} = \hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r}\right) , \tag{38}$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/|\vec{\boldsymbol{x}}|$. Hence,

$$\vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}}, t) = -\frac{1}{c^2 r}\,\hat{\boldsymbol{n}} \times \int d^3x' \left[\frac{\partial \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\right]_{\rm ret} + \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{39}$$

Note that the minus sign in eq. (39) arises after reversing the order of the two vectors in the cross product. Finally, we make use of eq. (12), which implies that

$$\left[\frac{\partial \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')}{\partial t'}\right]_{\rm ret} = \frac{\partial}{\partial t}\big[\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t')\big]_{\rm ret} , \tag{40}$$

since $\partial t'/\partial t = 1$. Moreover, we can expand the expression for t' in the limit of large $|\vec{\boldsymbol{x}}|$ as follows:

$$\begin{aligned} |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'| &= \big[r^2 + r'^2 - 2\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}}\,'\big]^{1/2} = r\left[1 + \frac{r'^2 - 2\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}}\,'}{r^2}\right]^{1/2} \\ &= r\left[1 - \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}}\,'}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] = r - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}}\,' + \mathcal{O}\left(\frac{1}{r}\right) , \end{aligned} \tag{41}$$

where $r' \equiv |\vec{\boldsymbol{x}}\,'|$. Hence,

$$t' = t - \frac{r}{c} + \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}}\,'}{c} + \mathcal{O}\left(\frac{1}{r}\right) . \tag{42}$$

It is convenient to introduce the notation

$$\vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}}, t) \equiv \int d^3x'\, \vec{\boldsymbol{J}}\Big(\vec{\boldsymbol{x}}\,', t - \frac{r}{c} + \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}}\,'}{c}\Big) . \tag{43}$$

The final forms for the radiation fields are then given by:

$$\vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}}, t) = -\frac{1}{c^2 r}\,\hat{\boldsymbol{n}} \times \frac{\partial \vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}}, t)}{\partial t} + \mathcal{O}\left(\frac{1}{r^2}\right) , \tag{44}$$

$$\vec{\boldsymbol{E}}_{\rm rad}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}}, t) \times \hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{45}$$

Eqs. (44) and (45) can be used to compute the total power radiated out to a distant observer:[4]

$$P = \oint \vec{\boldsymbol{S}} \cdot \hat{\boldsymbol{n}}\, da\,, \qquad \text{where } da = r^2 d\Omega, \tag{46}$$

and $\vec{\boldsymbol{S}}$ is the Poynting vector,

$$\vec{\boldsymbol{S}} = \frac{c}{4\pi}\vec{\boldsymbol{E}}_{\rm rad} \times \vec{\boldsymbol{B}}_{\rm rad} . \tag{47}$$

[4]Note that the contributions of terms of $\mathcal{O}(1/r^2)$ in the electromagnetic fields to the radiated power vanish as $r \to \infty$, thereby justifying discarding such terms in defining the radiation fields $\vec{\boldsymbol{E}}_{\rm rad}$ and $\vec{\boldsymbol{B}}_{\rm rad}$.

In eq. (47), $\vec{\boldsymbol{E}}_{\rm rad}$ and $\vec{\boldsymbol{B}}_{\rm rad}$ are real physical electromagnetic fields. In this case, the more useful observable is the time-averaged power, which is obtained by averaging over a cycle,

$$\langle P\rangle = \frac{1}{T}\int_0^T P(t)\,dt\,, \tag{48}$$

where the period $T = 2\pi/\omega$. Note that

$$\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \frac{c}{4\pi}\Big[\big(\vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}},t)\times\hat{\boldsymbol{n}}\big)\times\vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}},t)\Big]\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \frac{c}{4\pi}\big[|\vec{\boldsymbol{B}}_{\rm rad}|^2 - \big(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}_{\rm rad}\big)^2\big]\,. \tag{49}$$

In light of eq. (44), $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}_{\rm rad}=0$. Hence,

$$\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \frac{c}{4\pi}|\vec{\boldsymbol{B}}_{\rm rad}|^2\,. \tag{50}$$

Inserting the result for $\vec{\boldsymbol{B}}_{\rm rad}$ from eq. (44),

$$\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,r^2 = \frac{1}{4\pi c^3}\left|\hat{\boldsymbol{n}}\times\frac{\partial\vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}},t)}{\partial t}\right|^2\,. \tag{51}$$

We therefore end up with

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left\{\left(\frac{\partial\vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}},t)}{\partial t}\right)^2 - \left(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{\partial\vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}},t)}{dt}\right)^2\right\}\,. \tag{52}$$

For example, if the sources are harmonically varying, then we define complex charge and current densities,

$$\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}\,, \qquad\qquad \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})e^{-i\omega t}\,. \tag{53}$$

In this case, eq. (43) yields

$$\vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}},t) = e^{i(kr-\omega t)}\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')e^{-ik\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}'}\,, \tag{54}$$

where $k\equiv\omega/c$.

Moreover, the electric and magnetic fields are also harmonically varying complex fields,

$$\vec{\boldsymbol{E}}_{\rm rad}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{E}}_{\rm rad}(\vec{\boldsymbol{x}})e^{-i\omega t}\,, \qquad\qquad \vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{B}}_{\rm rad}(\vec{\boldsymbol{x}})e^{-i\omega t}\,. \tag{55}$$

The corresponding physical fields are ${\rm Re}\,\vec{\boldsymbol{E}}_{\rm rad}$ and ${\rm Re}\,\vec{\boldsymbol{B}}_{\rm rad}$, respectively. In this case, the power is computed by employing the complex Poynting vector

$$\vec{\boldsymbol{S}}(\vec{\boldsymbol{x}}) = \frac{c}{8\pi}\vec{\boldsymbol{E}}_{\rm rad}(\vec{\boldsymbol{x}})\times\vec{\boldsymbol{B}}^{\,*}_{\rm rad}(\vec{\boldsymbol{x}})\,. \tag{56}$$

One advantage of this procedure is that the total power radiated out to a distant observer,

$$\langle P\rangle = r^2\ {\rm Re}\oint\vec{\boldsymbol{S}}(\vec{\boldsymbol{x}})\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,d\Omega\,. \tag{57}$$

is automatically averaged over a cycle. Consequently, in the case of harmonically varying sources, eq. (52) is replaced by

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{8\pi c^3}\left\{\left|\frac{\partial\vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}},t)}{\partial t}\right|^2 - \left|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{\partial\vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}},t)}{dt}\right|^2\right\}\,, \tag{58}$$

where $|z|^2\equiv z^*z$ for any complex number z.

6. The $\mathcal{O}(1/r^2)$ behavior of $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ and $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$

The radiated angular momentum per unit time in gaussian units is given by

$$\vec{\boldsymbol{\tau}} = -\frac{r^3}{4\pi}\int\big[(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) + (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{B}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}})\big]\,d\Omega\,, \tag{59}$$

where $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ and $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$ are the real physical electromagnetic fields. In light of the expressions given in eqs. (44) and (45), it follows that $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}_{\rm rad} = \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}_{\rm rad} = 0$ at $\mathcal{O}(1/r)$. In contrast, $\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}_{\rm rad}$ and $\hat{\boldsymbol{n}}\times\vec{\boldsymbol{B}}_{\rm rad}$ are both nonzero at $\mathcal{O}(1/r)$. In order to obtain a finite nonzero angular momentum radiated to a far distant observer (in the limit of $r\to\infty$), we will need to determine the $\mathcal{O}(1/r^2)$ behavior of $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ and $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$. This is the one case where eqs. (44) and (45) are not sufficient to obtain a result that can be measured by a distant observer. In particular, we will need to also consider the $\mathcal{O}(1/r^2)$ terms that were discarded in defining the radiation fields exhibited in eqs. (36) and (37). The goal of this section is to provide the $\mathcal{O}(1/r^2)$ expressions for $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ and $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$.

First, by using eq. (24), it follows that

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{c^2}\int\frac{d^3x'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t')}{\partial t'}\right]_{\rm ret}\times\hat{\boldsymbol{R}} - \frac{1}{cr^2}\hat{\boldsymbol{n}}\times\int d^3x'\,\big[\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t')\big]_{\rm ret} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{60}$$

Taking the dot product of eq. (60) with $\hat{\boldsymbol{n}}$, we obtain

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{c^2}\,\hat{\boldsymbol{n}}\cdot\int\frac{d^3x'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t')}{\partial t'}\right]_{\rm ret}\times\hat{\boldsymbol{R}} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{61}$$

We now make use of eq. (41) to write

$$\begin{aligned}\hat{\boldsymbol{R}} = \frac{\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|} &= \frac{\vec{\boldsymbol{x}}}{r-\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'} - \frac{\vec{\boldsymbol{x}}'}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) = \hat{\boldsymbol{n}}\left(1+\frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}{r}\right) - \frac{\vec{\boldsymbol{x}}'}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\\ &= \hat{\boldsymbol{n}} + \frac{\hat{\boldsymbol{n}}\big(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'\big)-\vec{\boldsymbol{x}}'}{r} + \mathcal{O}\left(\frac{1}{r^2}\right).\end{aligned} \tag{62}$$

Hence, it follows that

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{r^2c^2}\,\hat{\boldsymbol{n}}\cdot\int d^3x'\,\vec{\boldsymbol{x}}'\times\left[\frac{\partial\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}',t')}{\partial t'}\right]_{\rm ret} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{63}$$

We can then write eq. (63) more explicitly as

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \frac{1}{r^2c^2}\frac{\partial}{\partial t}\,\hat{\boldsymbol{n}}\cdot\int d^3x'\,\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}\Big(\vec{\boldsymbol{x}}',t-\frac{r}{c}+\frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}{c}\Big) + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{64}$$

Next, we employ eq. (35) to obtain

$$\begin{aligned}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \frac{1}{cr^2}\int d^3x'\Big\{c\hat{\boldsymbol{n}}\,[\rho(\vec{\boldsymbol{x}}',t')]_{\rm ret} + \Big(\big[\vec{\boldsymbol{J}}\big]_{\rm ret}\cdot\hat{\boldsymbol{n}}\Big)\hat{\boldsymbol{n}} + \Big(\big[\vec{\boldsymbol{J}}\big]_{\rm ret}\times\hat{\boldsymbol{n}}\Big)\times\hat{\boldsymbol{n}}\Big\}\\ +\frac{1}{c^2}\int\frac{d^3x'}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}\left(\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}\times\hat{\boldsymbol{R}}\right)\times\hat{\boldsymbol{R}} + \mathcal{O}\left(\frac{1}{r^3}\right).\end{aligned} \tag{65}$$

After using the identity

$$\hat{\boldsymbol{n}}\cdot\left\{\left(\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}\times\hat{\boldsymbol{R}}\right)\times\hat{\boldsymbol{R}}\right\}=\left(\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{n}}\right)\hat{\boldsymbol{R}}\cdot\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}-\hat{\boldsymbol{n}}\cdot\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}, \tag{66}$$

and making use of eq. (62), we end up with

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\frac{1}{c^2r^2}\int d^3x'\left\{c^2\left[\rho(\vec{\boldsymbol{x}}',t')\right]_{\rm ret}+c\hat{\boldsymbol{n}}\cdot\left[\vec{\boldsymbol{J}}\right]_{\rm ret}+(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}')\hat{\boldsymbol{n}}\cdot\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}-\vec{\boldsymbol{x}}'\cdot\left[\frac{\partial\vec{\boldsymbol{J}}}{\partial t'}\right]_{\rm ret}\right\}+\mathcal{O}\left(\frac{1}{r^3}\right). \tag{67}$$

We can then write eq. (67) more explicitly as

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)&=\frac{1}{r^2}\int d^3x'\,\rho\Big(\vec{\boldsymbol{x}}',t-\frac{r}{c}+\frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}{c}\Big)+\frac{1}{cr^2}\int d^3x'\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}}\Big(\vec{\boldsymbol{x}}',t-\frac{r}{c}+\frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}{c}\Big)\\&+\frac{1}{c^2r^2}\frac{\partial}{\partial t}\int d^3x'\left\{(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}')\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}}\Big(\vec{\boldsymbol{x}}',t-\frac{r}{c}+\frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}{c}\Big)-\vec{\boldsymbol{x}}'\cdot\vec{\boldsymbol{J}}\Big(\vec{\boldsymbol{x}}',t-\frac{r}{c}+\frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}{c}\Big)\right\}+\mathcal{O}\left(\frac{1}{r^3}\right).\end{aligned} \tag{68}$$

In the case of harmonically varying sources, we have $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}$ and $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)=\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})e^{-i\omega t}$. In this case, eqs. (64) and (68) take the following forms:

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)=-\frac{ik}{cr^2}\,e^{i(kr-\omega t)}\,\hat{\boldsymbol{n}}\cdot\int d^3x'\,\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}+\mathcal{O}\left(\frac{1}{r^3}\right), \tag{69}$$

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=&-\frac{ik}{cr^2}\,e^{i(kr-\omega t)}\int d^3x'\left[(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}')\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')-\vec{\boldsymbol{x}}'\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\right]e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}\\&+\frac{1}{cr^2}\,e^{i(kr-\omega t)}\int d^3x'\left[c\rho(\vec{\boldsymbol{x}}')+\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\right]e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}+\mathcal{O}\left(\frac{1}{r^3}\right),\end{aligned} \tag{70}$$

after putting $\omega=ck$. One can obtain the radiated angular momentum per unit time using the complex vectors $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$ by modifying eq. (59) appropriately,[5]

$$\langle\vec{\boldsymbol{\tau}}\rangle=-\frac{r^3}{8\pi}\,\mathrm{Re}\int\left[(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*)(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}})+(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{B}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}^*)\right]d\Omega\,, \tag{71}$$

As in the case of eq. (57), the radiated angular momentum per unit time is automatically averaged over a cycle in eq. (71).

The multipole expansion corresponds to an expansion of the exponential $e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}$. At lowest order, if we set $e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}=1$ in eq. (69), it then follows that

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)=-\frac{2ik}{r^2}\,e^{i(kr-\omega t)}\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{m}}+\mathcal{O}\left(\frac{1}{r^3}\right), \tag{72}$$

where

$$\vec{\boldsymbol{m}}\equiv\frac{1}{2c}\int d^3x'\,\vec{\boldsymbol{x}}'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')+\mathcal{O}\left(\frac{1}{r^3}\right), \tag{73}$$

is the complex magnetic dipole moment vector in gaussian units [and $\vec{\boldsymbol{m}}(t)=\vec{\boldsymbol{m}}\,e^{-i\omega t}$].

[5]An alternative form for $\langle\vec{\boldsymbol{\tau}}\rangle$ given in eq. (100), which provides a motivation for the choice of the complex conjugated fields in eq. (71), is presented at the end of this section.

In evaluating the electric field, we first note the following identities:

$$J_i(\vec{\boldsymbol{x}}\,') = \partial'_k\big[x'_i J_k(\vec{\boldsymbol{x}}\,')\big] - x'_i\,\vec{\boldsymbol{\nabla}}\,'\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\,, \tag{74}$$

$$x'_i J_j(\vec{\boldsymbol{x}}\,') + x'_j J_i(\vec{\boldsymbol{x}}\,') = \partial'_k\big[x'_i x'_j J_k(\vec{\boldsymbol{x}}\,')\big] - x'_i x'_j\,\vec{\boldsymbol{\nabla}}\,'\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\,. \tag{75}$$

Next, we make use of the continuity equation [eq. (27)] for harmonically varying charge and current densities [eq. (53)] to obtain

$$\vec{\boldsymbol{\nabla}}\,'\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,') = i\omega\rho(\vec{\boldsymbol{x}}\,')\,. \tag{76}$$

It then follows that

$$\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,') = -ick\vec{\boldsymbol{p}}\,, \tag{77}$$

after putting $\omega = ck$ and dropping the surface term at infinity (which vanishes for localized currents), where

$$\vec{\boldsymbol{p}} \equiv \int d^3x'\,\vec{\boldsymbol{x}}\,'\,\rho(\vec{\boldsymbol{x}}\,')\,, \tag{78}$$

is the complex electric dipole moment vector [and $\vec{\boldsymbol{p}}(t) = \vec{\boldsymbol{p}}\,e^{-i\omega t}$]. Similarly,

$$\int d^3x'\,\big[x'_i J_j(\vec{\boldsymbol{x}}\,') + x'_j J_i(\vec{\boldsymbol{x}}\,')\big] = -ick\int d^3x'\,x'_i x'_j\,\rho(\vec{\boldsymbol{x}}\,')\,. \tag{79}$$

Hence, after setting $e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}\,'} = 1$ in eq. (70) it follows that

$$\begin{aligned} -ik\int d^3x'\,\Big[\big(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}\,'\big)\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,') - \vec{\boldsymbol{x}}\,'\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\Big] &= \tfrac{1}{2}ik(\delta_{ij} - \hat{n}_i\hat{n}_j)\int d^3x'\,\big[x'_i J_j(\vec{\boldsymbol{x}}\,') + x'_j J_i(\vec{\boldsymbol{x}}\,')\big] \\ &= \tfrac{1}{2}ck^2(\delta_{ij} - \hat{n}_i\hat{n}_j)\int d^3x'\,x'_i x'_j\,\rho(\vec{\boldsymbol{x}}\,')\,, \end{aligned} \tag{80}$$

with an implicit double sum over the repeated indices i and j, respectively.

To make further progress, we now demonstrate that for harmonically varying charge and current densities [eq. (53)], the conservation of charge implies that for $\omega \neq 0$,

$$\int d^3x'\,\rho(\vec{\boldsymbol{x}}\,') = 0\,. \tag{81}$$

The continuity equation [eq. (27)] implies that

$$0 = \int d^3x'\,\vec{\boldsymbol{\nabla}}\,'\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,', t) = -\frac{\partial}{\partial t}\int d^3x'\,\rho(\vec{\boldsymbol{x}}\,', t)\,, \tag{82}$$

since the surface term at infinity vanishes for localized currents. That is, electric charge is conserved. Furthermore, in light of eq. (76), it follows that

$$0 = \int d^3x'\,\vec{\boldsymbol{\nabla}}\,'\cdot\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,') = i\omega\int d^3x'\,\rho(\vec{\boldsymbol{x}}\,')\,. \tag{83}$$

Thus, if $\omega \neq 0$, we can conclude that

$$\int d^3x'\,\rho(\vec{\boldsymbol{x}}\,') = 0\,. \tag{84}$$

We can now examine the consequence of setting $e^{-ik\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'}=1$ in the second line of eq. (70). The term proportional to $\rho(\vec{\boldsymbol{x}}\,')$ vanishes as a result of eq. (84). Using eq. (77), it follows that

$$\int d^3x'\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')=-ick\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,. \tag{85}$$

However, we have not been totally consistent in the multipole expansion. In particular, there will be additional terms proportional to $\vec{\boldsymbol{p}}$ and $\int d^3x'\,x_i'x_j'\,\rho(\vec{\boldsymbol{x}}\,')$ arising from the second term of the expansion of $e^{-ik\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'}$ in the second line of eq. (70). Finally, there will be one additional contribution proportional to $\int d^3x'\,x_i'x_j'\,\rho(\vec{\boldsymbol{x}}\,')$ arising from the third term of the expansion of $e^{-ik\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'}$. These contributions are easily evaluated.

First, after using eq. (78),

$$\int d^3x'\rho(\vec{\boldsymbol{x}}\,')\bigl[-ik\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'-\tfrac{1}{2}k^2\bigl(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'\bigr)^2\bigr]=-ik\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-\tfrac{1}{2}k^2\hat{n}_i\hat{n}_j\int d^3x'\,x_i'x_j'\rho(\vec{\boldsymbol{x}}\,')\,. \tag{86}$$

Second, in light of eq. (79),

$$\begin{aligned}\int d^3x'\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\bigl[-ik\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'\bigr]&=-\tfrac{1}{2}ik\hat{n}_i\hat{n}_j\int d^3x'\,\bigl[x_i'J_j(\vec{\boldsymbol{x}}\,')+x_j'J_i(\vec{\boldsymbol{x}}\,')\bigr]\\&=-\tfrac{1}{2}ck^2\hat{n}_i\hat{n}_j\int d^3x'\,x_i'x_j'\,\rho(\vec{\boldsymbol{x}}\,')\,.\end{aligned} \tag{87}$$

Collecting all of the results obtained above,

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=-e^{i(kr-\omega t)}\left\{\frac{2ik}{r^2}\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+\frac{k^2}{2r^2}(3\hat{n}_i\hat{n}_j-\delta_{ij})\int d^3x'\,x_i'x_j'\,\rho(\vec{\boldsymbol{x}}\,')\right\}\,. \tag{88}$$

The last term above can be expressed in terms of the complex electric quadrupole moment tensor,

$$Q_{ij}=\int d^3x'\,\bigl[3x_i'x_j'-\delta_{ij}|\vec{\boldsymbol{x}}\,'|^2\bigr]\rho(\vec{\boldsymbol{x}}\,')\,, \tag{89}$$

and $Q_{ij}(t)=Q_{ij}\,e^{-i\omega t}$. Note that Q_{ij} is a traceless symmetric tensor, i.e., $Q_{ij}=Q_{ji}$ and $\sum_{i,j}\delta_{ij}Q_{ij}=0$. It is convenient to introduce a vector $\vec{\boldsymbol{Q}}$, whose components are given by

$$Q_i=\sum_j Q_{ij}\hat{n}_j\,. \tag{90}$$

Then, our final results for leading multipole contributions to $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}\,',t)$ and $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}\,',t)$ at $\mathcal{O}(1/r^2)$ are:

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)=-\,\frac{2ik}{r^2}\,e^{i(kr-\omega t)}\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{m}}+\mathcal{O}\left(\frac{1}{r^3}\right)\,, \tag{91}$$

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)=-\,\frac{2ik}{r^2}\,e^{i(kr-\omega t)}\Bigl[\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-\tfrac{1}{4}ik\,\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{Q}}\Bigr]+\mathcal{O}\left(\frac{1}{r^3}\right)\,. \tag{92}$$

The next set of multipoles in the multipole expansion would be the magnetic quadrupole and the electric octopole, which are expected to be of equal importance (unless one of them vanishes). These terms can be identified by including higher order terms (beyond those already treated above) in the expansion of the exponential $e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}\,'}$. The computation of the electric octopole contribution is extremely tedious, and will not be given here. However, the computation of the magnetic quadrupole contribution is straightforward, as it simply corresponds to the second term in the expansion of the exponential in eq. (69).

In particular,

$$\begin{aligned}
-ik\,\hat{\boldsymbol{n}}\cdot\int d^3x'\,\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}\,' &= -ik\,\hat{n}_i\hat{n}_j\int d^3x'\,\Big\{x'_i\big[\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big]_j \\
&= -\tfrac{1}{2}ik\,\hat{n}_i\hat{n}_j\int d^3x'\,\Big\{x'_i\big[\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big]_j + x'_j\big[\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big]_i\Big\} \\
&= -\tfrac{1}{2}ick\hat{n}_i\hat{n}_jM_{ij}\,, \qquad (93)
\end{aligned}$$

with an implicit double sum over the repeated indices i and j, respectively, where the complex magnetic quadrupole moment tensor is defined in gaussian units as

$$M_{ij} = \frac{1}{c}\int d^3x'\,\Big\{x'_i\big[\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big]_j + x'_j\big[\vec{\boldsymbol{x}}\,'\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\big]_i\Big\}\,. \qquad (94)$$

Note that M_{ij} is a traceless symmetric tensor, i.e., $M_{ij} = M_{ji}$ and $\sum_{i,j}\delta_{ij}M_{ij} = 0$. It is convenient to introduce a vector $\vec{\boldsymbol{M}}$, whose components are given by

$$M_i = \sum_j M_{ij}\hat{n}_j\,. \qquad (95)$$

Hence, the contributions to $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$ arising from the magnetic dipole and magnetic quadrupole is given by

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = -\frac{2ik}{r^2}\,e^{i(kr-\omega t)}\Big[\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{m}} - \tfrac{1}{4}ik\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{M}}\Big] + \mathcal{O}\left(\frac{1}{r^3}\right)\,. \qquad (96)$$

Comparing this result with that of eq. (92), we see that the structure of eqs. (92) and (96) are identical after swapping the corresponding dipole moment vectors and quadrupole moment tensors.

One remarkable feature of eqs. (69) and (70) is that the multipole expansion of $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ contains only electric multipoles, whereas the multipole expansion of $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$ contains only magnetic multipoles. This should be contrasted with the multipole expansions of $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ and $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)$, which receive contributions from both the electric and magnetic multipoles.

Remarks on the quadrupole tensor normalization convention

I have seen the magnetic quadrupole tensor defined by some authors with an overall factor of 1/2 as compared to eq. (94). If you adopt this convention, then you lose the symmetry between Q_{ij} and M_{ij}. Other authors choose to define both the electric quadrupole tensor [eq. (89)] and magnetic quadrupole tensor [eq. (94)] by multiplying the corresponding definitions by an overall factor of 1/3. Such a choice does not upset the symmetry between expressions containing Q_{ij} and M_{ij}.

An alternative form[6] for the radiated angular momentum per unit time $\vec{\boldsymbol{\tau}}$

In light of eq. (45),

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t)\times\hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{97}$$

Since $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \mathcal{O}(1/r^2)$, it follows that

$$\begin{aligned}\vec{\boldsymbol{E}}\times\hat{\boldsymbol{n}} &= \big(\vec{\boldsymbol{B}}\times\hat{\boldsymbol{n}}\big)\times\hat{\boldsymbol{n}} + \mathcal{O}\left(\frac{1}{r^2}\right) = -\vec{\boldsymbol{B}} + \big(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}\big)\vec{\boldsymbol{B}} + \mathcal{O}\left(\frac{1}{r^2}\right) \\ &= -\vec{\boldsymbol{B}} + \mathcal{O}\left(\frac{1}{r^2}\right). \end{aligned} \tag{98}$$

Inserting the results of eqs. (97) and (98) into the integrand of eq. (71), we obtain

$$\begin{aligned}\big(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}^*\big)(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) + (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{B}})(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}^*) &= \vec{\boldsymbol{B}}^*(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) - \vec{\boldsymbol{E}}\,(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}^*) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ &= -\hat{\boldsymbol{n}}\times\big(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}^*\big) + \mathcal{O}\left(\frac{1}{r^4}\right), \end{aligned} \tag{99}$$

after noting that $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \mathcal{O}(1/r^2)$ and $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \mathcal{O}(1/r^2)$.

Finally, after writing $\hat{\boldsymbol{n}} = \vec{\boldsymbol{x}}/r$ and inserting eq. (99) back into eq. (71), we end up with

$$\langle\vec{\boldsymbol{\tau}}\rangle = \frac{r^2}{8\pi}\,\mathrm{Re}\int \vec{\boldsymbol{x}}\times\big(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}^*\big)\,d\Omega + \mathcal{O}\left(\frac{1}{r}\right). \tag{100}$$

Eq. (100) provides an alternative expression for the rate of angular momentum transport and is equivalent to eq. (71) in the limit of $r\to\infty$.[7] Note that the location of the complex conjugated fields in eq. (71) was chosen in order that one would obtain eq. (100) with the $\vec{\boldsymbol{B}}$ field complex conjugated. In this way, the integrand of $\langle\vec{\boldsymbol{\tau}}\rangle$ is closely related to the complex Poynting vector, $\vec{\boldsymbol{S}} = c\,(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}^*)/(8\pi)$.

The infinitesimal area element is $da = r^2\,d\Omega$, so eq. (100) can be rewritten as

$$\frac{d\vec{\boldsymbol{\tau}}}{da} = \frac{1}{8\pi}\,\mathrm{Re}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}^*)\,. \tag{101}$$

Since $\vec{\boldsymbol{\tau}} = d\vec{\boldsymbol{L}}/dt$, we interpret $d\vec{\boldsymbol{\tau}}/da$ as the angular momentum flux that is transported from the sources out to the observer located a long distance away. Eq. (101) should be compared with the expression for the angular momentum of a distribution of electromagnetic fields in vacuum given in problem 7.27 of Jackson (after conversion to gaussian units),

$$\vec{\boldsymbol{L}} \equiv \int \vec{\boldsymbol{\mathscr{L}}}\,d^3x = \frac{1}{4\pi c}\int d^3x\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})\,, \tag{102}$$

where $\vec{\boldsymbol{\mathscr{L}}}$ is the angular momentum density of a distribution of electromagnetic fields

[6] This added note was inspired by the treatment in Emil Jan Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw Hill Inc., New York, 1981). In particular, see the discussion on p. 226, including the very enlightening footnote at the bottom of that page.

[7] In this limit, $\vec{\boldsymbol{\tau}}$ approaches a constant value which is equal to the rate of angular momentum transported to the surface at infinity by the radiation.

in vacuum. Eq. (102) is applicable to the real fields. The corresponding result for the time-averaged angular momentum density of a distribution of harmonically varying electromagnetic fields is obtained by replacing $\vec{\boldsymbol{B}}$ with $\vec{\boldsymbol{B}}^*$ and multiplying the prefactor by 1/2. That is, the time-averaged angular momentum density of a distribution of harmonically varying electromagnetic fields is given by

$$\vec{\boldsymbol{\mathscr{L}}} = \frac{1}{8\pi c}\,\mathrm{Re}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}^*)\,. \tag{103}$$

We conclude that

$$\frac{d\vec{\boldsymbol{\tau}}}{da} = \frac{1}{8\pi}\,\mathrm{Re}\,\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}^*) = c\,\vec{\boldsymbol{\mathscr{L}}} + \mathcal{O}\left(\frac{1}{r^3}\right)\,. \tag{104}$$

That is, the angular momentum flux in the radiation zone is equal to c times the angular momentum density, although this identification is correct only at the lowest nontrivial order in the inverse distance expansion.

7. Radiation from harmonically varying sources by way of the vector potential

The traditional method employed by most textbooks in treating radiation from harmonically varying charge and current densities is to solve first for the vector potential using eq. (2). The disadvantage of this method is that after computing $\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)$, one must then perform a second computation to obtain the electric and magnetic fields. By using the Jefimenko equations and the improvement made by Panofsky and Phillips, one can obtain the electric and magnetic fields directly without the need to perform the intermediate step of determining $\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t)$. For completeness, in this section I will rederive the results obtained by Jackson for the expressions for the electric and magnetic fields up to and including terms of $\mathcal{O}(1/r^2)$ in the approximation where multipoles beyond electric dipole, magnetic dipole, and electric quadrupole are neglected.[8]

Starting from eq. (2) and assuming that

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,',t') = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')e^{-i\omega t'} = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')e^{i(kR-\omega t)}\,, \tag{105}$$

it follows that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{A}}(\vec{\boldsymbol{x}})e^{-i\omega t}$, where

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \frac{1}{c}\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}\,')\frac{e^{ikR}}{R}\,. \tag{106}$$

In the limit of large $r \equiv |\vec{\boldsymbol{x}}|$,

$$R = |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'| = r - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}\,' + \frac{|\vec{\boldsymbol{x}}\,'|^2 - (\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}\,')^2}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{107}$$

[8] Jackson does not provide the electric quadrupole fields at $\mathcal{O}(1/r^2)$, so the results obtained below in this case are new.

It suffices to make the following approximations in evaluating eq. (106):

$$\frac{1}{R} = \frac{1}{r} + \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}}'}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{108}$$

$$e^{ikR} = e^{ikr}e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}\left\{1 + \frac{ik}{2r}\big|\hat{\boldsymbol{n}} \times \vec{\boldsymbol{x}}'\big|^2 + \mathcal{O}\left(\frac{1}{r^2}\right)\right\}. \tag{109}$$

The end result is

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \frac{e^{ikr}}{cr}\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}\left\{1 + \frac{1}{r}\Big[\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}' + \tfrac{1}{2}ik\big|\hat{\boldsymbol{n}} \times \vec{\boldsymbol{x}}'\big|^2\Big]\right\} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{110}$$

Performing the multipole expansion by Taylor expanding the exponential $e^{-ik\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'}$, and keeping only the electric and magnetic dipole and electric quadrupole, we can neglect all terms in the integrand that involve a quadratic (or a higher power) of $\vec{\boldsymbol{x}}'$. Hence, we can approximate the scalar potential as follows

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \frac{e^{ikr}}{cr}\int d^3x'\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')\left\{1 + \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}'\left(\frac{1}{r} - ik\right)\right\} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{111}$$

in agreement with eq. (9.30) of Jackson (after converting to gaussian units).

In light of the identity

$$x'_i J_j(\vec{\boldsymbol{x}}') - x'_j J_i(\vec{\boldsymbol{x}}') = \epsilon_{ijk}\big(\vec{\boldsymbol{x}}' \times \vec{\boldsymbol{J}}\big)_k, \tag{112}$$

one can make use of eqs. (74)–(76) and to obtain

$$J_i(\vec{\boldsymbol{x}}') = \partial'_k\big[x'_i J_k(\vec{\boldsymbol{x}}')\big] - ickx'_i\,\rho(\vec{\boldsymbol{x}}'), \tag{113}$$

$$x'_i J_j(\vec{\boldsymbol{x}}') = \tfrac{1}{2}\partial'_k\big[x'_i x'_j J_k(\vec{\boldsymbol{x}}')\big] - \tfrac{1}{2}ickx'_i x'_j\,\rho(\vec{\boldsymbol{x}}') + \tfrac{1}{2}\epsilon_{ijk}\big(\vec{\boldsymbol{x}}' \times \vec{\boldsymbol{J}}\big)_k, \tag{114}$$

after putting $\omega = ck$. Hence, it follows that

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \frac{e^{ikr}}{r}\left\{-ik\vec{\boldsymbol{p}} - \left(\frac{1}{r} - ik\right)\left[\vec{\boldsymbol{n}} \times \vec{\boldsymbol{m}} + \tfrac{1}{6}ik\vec{\boldsymbol{Q}} + \tfrac{1}{6}ik\hat{\boldsymbol{n}}\int d^3x'|\vec{\boldsymbol{x}}'|^2\rho(\vec{\boldsymbol{x}}')\right]\right\}, \tag{115}$$

where the components of $\vec{\boldsymbol{Q}}$ are $Q_{ij}\hat{n}_j$ [cf. eq. (90)], where there is an implicit sum over the repeated index j.

We can now evaluate $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$ with the help of various vector identities appearing on the inside of the front cover of Jackson,

$$\vec{\boldsymbol{\nabla}} \times \big[f(r)\vec{\boldsymbol{v}}\big] = \vec{\boldsymbol{\nabla}}f(r) \times \vec{\boldsymbol{v}} = \frac{\partial f}{\partial r}\,\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}, \tag{116}$$

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \times \big[f(r)\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}\big] &= \vec{\boldsymbol{\nabla}}f(r) \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) + f(r)\vec{\boldsymbol{\nabla}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) \\ &= \frac{\partial f}{\partial r}\,\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) + f(r)\big[(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{\nabla}})\hat{\boldsymbol{n}} - \vec{\boldsymbol{v}}(\vec{\boldsymbol{\nabla}}\cdot\hat{\boldsymbol{n}})\big] \\ &= \frac{\partial f}{\partial r}\,\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) - \frac{f(r)}{r}\big[\vec{\boldsymbol{v}} + \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{v}})\big].\end{aligned} \tag{117}$$

for any constant vector $\vec{\boldsymbol{v}}$ and radial function $f(r)$.

We need one further identity:

$$\vec{\nabla}\times\big[f(r)\vec{Q}\big]=\vec{\nabla}f(r)\times\vec{Q}+f(r)\vec{\nabla}\times Q=\frac{\partial f}{\partial r}\,\hat{n}\times\vec{Q}+f(r)\vec{\nabla}\times Q\,. \tag{118}$$

Next, we evaluate:

$$\begin{aligned}\big(\vec{\nabla}\times\vec{Q}\big)_i&=\epsilon_{ijk}\frac{\partial}{\partial x_j}Q_k=\epsilon_{ijk}\frac{\partial}{\partial x_j}Q_{k\ell}\hat{n}_\ell=\epsilon_{ijk}Q_{k\ell}\frac{\partial}{\partial x_j}\left(\frac{x_\ell}{r}\right)=\epsilon_{ijk}Q_{k\ell}\left(\frac{r^2\delta_{j\ell}-x_jx_\ell}{r^3}\right)\\&=\epsilon_{ijk}Q_{k\ell}\left(\frac{\delta_{j\ell}-\hat{n}_j\hat{n}_\ell}{r}\right)=-\frac{1}{r}\epsilon_{ijk}Q_k\hat{n}_j=\frac{1}{r}\big(\hat{n}\times\vec{Q}\big)_i\end{aligned} \tag{119}$$

Hence, it follows that

$$\vec{\nabla}\times\big[f(r)\vec{Q}\big]=\left(\frac{\partial f}{\partial r}+\frac{f(r)}{r}\right)\hat{n}\times\vec{Q}\,. \tag{120}$$

After making use of all the identities collected above,

$$\begin{aligned}\vec{B}(\vec{x})=&\frac{k^2e^{ikr}}{r}\big[\hat{n}\times\vec{p}-\hat{n}\times(\hat{n}\times\vec{m})-\tfrac{1}{6}ik\,\hat{n}\times\vec{Q}\big]\\&+\frac{ike^{ikr}}{r^2}\big[\hat{n}\times\vec{p}+\vec{m}-3\hat{n}\big(\hat{n}\cdot\vec{m}\big)-\tfrac{1}{6}ik\,\hat{n}\times\vec{Q}\big]+\mathcal{O}\left(\frac{1}{r^3}\right)\,.\end{aligned} \tag{121}$$

Note that

$$\hat{n}\cdot\vec{B}(\vec{x})=-\frac{2ike^{ikr}}{r^2}\,\hat{n}\cdot\vec{m}+\mathcal{O}\left(\frac{1}{r^3}\right)\,, \tag{122}$$

in agreement with eq. (91).

The electric field $\vec{E}(\vec{x},t)=\vec{E}(\vec{x})e^{-i\omega t}$ can be computed by using eq. (9.5) of Jackson, which in gaussian units is given by

$$\vec{E}(\vec{x})=\frac{i}{k}\vec{\nabla}\times\vec{B}(\vec{x})\,. \tag{123}$$

We will need to work out a few more vector identities. Using $\vec{\nabla}\times\hat{n}=0$, it follows that

$$\vec{\nabla}\times\big[\hat{n}(\hat{n}\cdot\vec{v})\big]=\vec{\nabla}(\hat{n}\cdot\vec{v})\times\hat{n}=0\,, \tag{124}$$

for any constant vector $\vec{v}$. Thus,

$$\vec{\nabla}\times\big[f(r)\,\hat{n}\times(\hat{n}\times\vec{v})\big]=-\vec{\nabla}\times\big[f(r)\vec{v})\big]=-\frac{\partial f}{\partial r}\,\hat{n}\times\vec{v}\,, \tag{125}$$

after expanding out the triple vector product and making use of eqs. (116) and (124). Finally, we need to evaluate one more vector identity:

$$\begin{aligned}\vec{\nabla}\times\big[f(r)\hat{n}\times\vec{Q}\big]&=\vec{\nabla}f(r)\times(\hat{n}\times\vec{Q})+f(r)\vec{\nabla}\times(\hat{n}\times\vec{Q})\\&=\frac{\partial f}{\partial r}\,\hat{n}\times(\hat{n}\times\vec{Q})+f(r)\Big[\hat{n}(\vec{\nabla}\cdot\vec{Q})-\vec{Q}(\vec{\nabla}\cdot\hat{n})+(\vec{Q}\cdot\vec{\nabla})\hat{n}-(\hat{n}\cdot\vec{\nabla})\vec{Q}\Big]\,.\end{aligned} \tag{126}$$

Eq. (126) can be further simplified by using the following results:

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{Q}} = Q_{ij}\frac{\partial}{\partial x_i}\left(\frac{x_j}{r}\right) = Q_{ij}\left(\frac{\delta_{ij}-\hat{n}_i\hat{n}_j}{r}\right) = -\frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}}}{r}\,, \tag{127}$$

$$\vec{\boldsymbol{Q}}(\vec{\boldsymbol{\nabla}}\cdot\hat{\boldsymbol{n}}) = \frac{2\vec{\boldsymbol{Q}}}{r}\,, \tag{128}$$

$$(\vec{\boldsymbol{Q}}\cdot\vec{\boldsymbol{\nabla}})\hat{\boldsymbol{n}} = Q_{ij}\left(n_j\frac{\partial}{\partial x_i}\right)\left(\frac{\vec{\boldsymbol{x}}}{r}\right) = \frac{\vec{\boldsymbol{Q}}-\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}})}{r}\,, \tag{129}$$

$$(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\nabla}})Q_j = \hat{n}_i\frac{\partial}{\partial x_i}\big(Q_{jk}\hat{n}_k\big) = \hat{n}_iQ_{jk}\left(\frac{\delta_{ik}-\hat{n}_i\hat{n}_k}{r}\right) = 0\,. \tag{130}$$

Hence, we end up with

$$\vec{\boldsymbol{\nabla}}\times\big[f(r)\hat{\boldsymbol{n}}\times\vec{\boldsymbol{Q}}\big] = \frac{\partial f}{\partial r}\,\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{Q}}) - \frac{f(r)}{r}\big[\vec{\boldsymbol{Q}}+2\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}})\big]\,. \tag{131}$$

The rest of the computation involves some straightforward but tedious algebra. Using all the relevant identities derived above and inserting eq. (121) into eq. (123), we end up with

$$\begin{aligned}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = &-\frac{k^2e^{ikr}}{r}\Big[\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{p}})+\hat{\boldsymbol{n}}\times\vec{\boldsymbol{m}}-\tfrac{1}{6}ik\,\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{Q}})\Big]\\ &+\frac{ike^{ikr}}{r^2}\Big\{\vec{\boldsymbol{p}}-3\hat{\boldsymbol{n}}\big(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}\big)-\hat{\boldsymbol{n}}\times\vec{\boldsymbol{m}}+\tfrac{1}{6}ik\,\big[4\hat{\boldsymbol{n}}\big(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}}\big)-\vec{\boldsymbol{Q}}\,\big]\Big\}+\mathcal{O}\left(\frac{1}{r^3}\right).\end{aligned} \tag{132}$$

Note that

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = -\frac{2ike^{ikr}}{r^2}\Big[\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{p}}-\tfrac{1}{4}ik\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{Q}}\,\Big]+\mathcal{O}\left(\frac{1}{r^3}\right)\,, \tag{133}$$

in agreement with eq. (92).