Properties of the differential operator $\vec{\boldsymbol{L}}$

The differential operator $\vec{\boldsymbol{L}}$ is defined as

$$\vec{\boldsymbol{L}} \equiv -i\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}} = i\left(\hat{\boldsymbol{\theta}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\right)\,, \tag{1}$$

where we have written out the explicit form in spherical coordinates (where θ is the polar angle and ϕ is the azimuthal angle) with respect to the spherical basis. In particular, if $\vec{\boldsymbol{x}} = r\hat{\boldsymbol{n}}$, where $r = |\vec{\boldsymbol{x}}|$, then $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta$. In these notes, we will employ the more common notation where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$.

It is sometimes useful to convert between the Cartesian basis and the spherical basis. For example,

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi\,, \tag{2}$$

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi\,, \tag{3}$$

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\,. \tag{4}$$

Inverting these results yields

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta\,, \tag{5}$$

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}}\cos\theta\cos\phi + \hat{\boldsymbol{y}}\cos\theta\sin\phi - \hat{\boldsymbol{z}}\sin\theta\,, \tag{6}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi\,. \tag{7}$$

The differential operator $\vec{\boldsymbol{L}}$ plays a critical role in Debye's decomposition theorem.[1]

Theorem. *Let $\vec{\boldsymbol{F}}$ be a divergenceless vector field, $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{F}} = 0$ (also called a solenoidal field). Then, there exist scalar functions ψ and χ (called the Debye potentials) such that*

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{L}}\psi + (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{L}})\chi\,, \tag{8}$$

where the Debye potentials are unique up to an arbitrary radial function. That is, $\vec{\boldsymbol{F}}$ is unchanged under the transformations,

$$\psi(\vec{\boldsymbol{x}}) \to \psi(\vec{\boldsymbol{x}}) + f(r)\,, \qquad\qquad \chi(\vec{\boldsymbol{x}}) \to \chi(\vec{\boldsymbol{x}}) + g(r)\,, \tag{9}$$

for arbitrary radial functions $g(r)$ and $g(r)$.

If in addition, $\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})$ satisfies the Helmholtz equation, $(\vec{\boldsymbol{\nabla}}^{\,2} + k^2))\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = 0$, then one can adjust the radial functions $f(r)$ and $g(r)$ such that

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\psi(\vec{\boldsymbol{x}}) = (\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\chi(\vec{\boldsymbol{x}}) = 0\,. \tag{10}$$

[1] See, e.g., Dietman Petrascheck and Franz Schwabl, *Electrodynamics* (Springer Nature, Berlin, Germany, 2025) p. 151. A proof of the first part of this theorem is given in C. G. Gray and B. Nickel, *Debye potential representation of vector fields*, American Journal of Physics **46**, 735–736 (1978).

In these notes, I will collect many useful results and identities related to the differential operator $\vec{\boldsymbol{L}}$. First, we record the following representations:

$$\vec{\boldsymbol{L}} = i\left(\hat{\boldsymbol{\theta}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\right)\,, \tag{11}$$

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\phi}\,, \tag{12}$$

$$L_\pm \equiv L_x \pm iL_y = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)\,, \tag{13}$$

$$\vec{\boldsymbol{L}}^{\,2} = L_x^2 + L_y^2 + L_z^2 = -\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\,, \tag{14}$$

$$-i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{L}} = r\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\,, \tag{15}$$

$$-i\,\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}} = \frac{\hat{\boldsymbol{r}}}{r}\,\vec{\boldsymbol{L}}^{\,2} + \hat{\boldsymbol{\theta}}\left(\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial r\partial\theta}\right) + \frac{\hat{\boldsymbol{\phi}}}{\sin\theta}\left(\frac{1}{r}\,\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial r\partial\phi}\right)\,. \tag{16}$$

The operators $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{L}}^{\,2}$ are purely angular operators. Moreover, as a consequence of eqs. (11), (14) and (16), it follows that for any radial function $f(r)$,

$$\vec{\boldsymbol{L}}f(r) = \vec{\boldsymbol{L}}^{\,2}f(r) = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\,f(r) = 0\,. \tag{17}$$

Next, we record the following useful operator identities:[2]

$$\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{L}} = 0, \tag{18}$$

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{L}} = 0, \tag{19}$$

$$\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}) = 0, \tag{20}$$

$$\vec{\boldsymbol{L}}\,\vec{\boldsymbol{L}}^{\,2} = \vec{\boldsymbol{L}}^{\,2}\vec{\boldsymbol{L}}, \tag{21}$$

$$\vec{\boldsymbol{L}}\,\vec{\boldsymbol{\nabla}}^{\,2} = \vec{\boldsymbol{\nabla}}^{\,2}\vec{\boldsymbol{L}}, \tag{22}$$

$$\vec{\boldsymbol{\nabla}} = \frac{\vec{\boldsymbol{x}}}{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{L}}, \tag{23}$$

$$\vec{\boldsymbol{\nabla}}^{\,2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} - \frac{\vec{\boldsymbol{L}}^{\,2}}{r^2}, \tag{24}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}} = -i\vec{\boldsymbol{x}}\,\vec{\boldsymbol{\nabla}}^{\,2} + i\vec{\boldsymbol{\nabla}}\left(1 + r\frac{\partial}{\partial r}\right), \tag{25}$$

$$\vec{\boldsymbol{x}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}) = i\vec{\boldsymbol{L}}^{\,2}, \tag{26}$$

$$\vec{\boldsymbol{x}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}) = -\vec{\boldsymbol{L}}\left(1 + r\frac{\partial}{\partial r}\right), \tag{27}$$

$$\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}) = -\vec{\boldsymbol{\nabla}}^{\,2}\vec{\boldsymbol{L}}, \tag{28}$$

$$\vec{\boldsymbol{L}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}) = 0. \tag{29}$$

[2]See, e.g., Appendix F of C.G. Gray, American *Multipole expansions of electromagnetic fields using Debye potentials*, Journal of Physics **46**, 169–179 (1978).

In particular, the differential operators L_i and L_j do not commute. Instead, they satisfy the following commutation relations,

$$\big[L_i\,,\,L_j\big] \equiv L_iL_j - L_jL_i = i\epsilon_{ijk}L_k\,, \tag{30}$$

with an implicit sum over the repeated index k, where $L_1 \equiv L_x$, $L_2 \equiv L_y$, and $L_3 \equiv L_z$. Eq. (30) is equivalent to the equation $\epsilon_{ijk}L_iL_j = iL_k$, which can be rewritten in vector notation as

$$\vec{\boldsymbol{L}} \times \vec{\boldsymbol{L}} = i\vec{\boldsymbol{L}}\,. \tag{31}$$

Eqs. (18)–(31) should be understood as operator equations that act on a function $f(\vec{\boldsymbol{x}})$. The following additional identities are also noteworthy:

$$\vec{\boldsymbol{L}}\boldsymbol{\cdot}\vec{\boldsymbol{F}} = -i\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}}), \tag{32}$$

$$\vec{\boldsymbol{L}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}}) = i\left[\vec{\boldsymbol{\nabla}}^{\,2}\big(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}\big) + \left(2 + r\frac{\partial}{\partial r}\right)\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}\right]. \tag{33}$$

Finally, we note that the spherical harmonics, $Y_{\ell m}(\theta,\phi)$ are simultaneous eigenfunctions of $\vec{\boldsymbol{L}}^{\,2}$ and L_z,

$$\vec{\boldsymbol{L}}^{\,2}Y_{\ell m}(\theta,\phi) = \ell(\ell+1)Y_{\ell m}(\theta,\phi)\,, \tag{34}$$

$$L_zY_{\ell m}(\theta,\phi) = mY_{\ell m}(\theta,\phi)\,, \tag{35}$$

where $\ell = 0, 1, 2, 3, \ldots$, and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. In addition, the operators $L_\pm$ [cf. eq. (13), when acting on the spherical harmonics, yield

$$L_+Y_{\ell m}(\theta,\phi) = \sqrt{(\ell-m)(\ell+m+1)}\,Y_{\ell,m+1}(\theta,\phi)\,, \tag{36}$$

$$L_-Y_{\ell m}(\theta,\phi) = \sqrt{(\ell+m)(\ell-m+1)}\,Y_{\ell,m-1}(\theta,\phi)\,, \tag{37}$$

which is why the $L_\pm$ are called raising and lowering operators.

Returning to the Debye's decomposition theorem, we note that the solenoidal field $\vec{\boldsymbol{F}}$ is unchanged under the transformations specified by eq. (9) as a consequence of eq. (17). Moreover, with $\vec{\boldsymbol{F}}$ given by eq. (8), we may use eqs. (18), (26) and (29) to obtain:

$$\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = i\vec{\boldsymbol{L}}^{\,2}\chi(\vec{\boldsymbol{x}})\,, \tag{38}$$

$$\vec{\boldsymbol{L}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{L}}^{\,2}\psi(\vec{\boldsymbol{x}})\,, \tag{39}$$

Using eq. (32), it follows that eq. (39) is equivalent to

$$\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}}) = i\vec{\boldsymbol{L}}^{\,2}\psi(\vec{\boldsymbol{x}})\,. \tag{40}$$

Expanding $\psi(\vec{\boldsymbol{x}})$ and $\chi(\vec{\boldsymbol{x}})$ in spherical harmonics, one can then solve for the Debye potentials. Note that the expansion in spherical harmonics starts at $\ell = 1$, since the $\ell = 0$ term is constant and thus is annihilated by the operator $\vec{\boldsymbol{L}}^{\,2}$.

Now, suppose that the solenoidal field $\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})$ also satisfies the Helmholtz equation,

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = 0\,. \tag{41}$$

Then, it follows that

$$\vec{\boldsymbol{\nabla}}^2\big(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}\big) = \vec{\boldsymbol{x}}\boldsymbol{\cdot}\big(\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{F}}\big) + 2\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{F}} = -k^2\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{F}}\,, \tag{42}$$

after using $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{F}} = 0$ and making use of eq. (41). Hence, we obtain

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{F}} = 0\,. \tag{43}$$

It then follows from eqs. (22) and (38) that

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)\vec{\boldsymbol{L}}^2\chi(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{L}}^2(\vec{\boldsymbol{\nabla}}^2 + k^2)\chi(\vec{\boldsymbol{x}}) = 0\,. \tag{44}$$

In light of eq. (17), we can conclude that

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)\chi(\vec{\boldsymbol{x}}) = h(r)\,, \tag{45}$$

for some radial function $h(r)$. If we now transform $\chi(\vec{\boldsymbol{x}}) \to \chi'(\vec{\boldsymbol{x}}) = \chi(\vec{\boldsymbol{x}}) + g(r)$ as specified in eq. (9) and choose $g(r)$ such that

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)g(r) = -h(r)\,, \tag{46}$$

then it follows that

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)\chi'(\vec{\boldsymbol{x}}) = (\vec{\boldsymbol{\nabla}}^2 + k^2)\big[\chi(\vec{\boldsymbol{x}}) + g(r)\big] = h(r) - h(r) = 0\,. \tag{47}$$

In fact, we can always find a function $g(r)$ such that eq. (46) is satisfied. By employing the Green function of the inhomogeneous Helmholtz equation [cf. eqs. (6.35) and (6.36) of Jackson], it follows that[3]

$$g(r) = \frac{1}{4\pi}\int \frac{e^{ik|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\,h(r')d^3x'\,. \tag{48}$$

A computation similar to that of eq. (42) yields

$$\vec{\boldsymbol{\nabla}}^2\big[\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}})\big] = -k^2\big[\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}})\big]\,. \tag{49}$$

In light of eqs. (22) and (40), it follows that

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)\vec{\boldsymbol{L}}^2\psi(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{L}}^2(\vec{\boldsymbol{\nabla}}^2 + k^2)\psi(\vec{\boldsymbol{x}}) = 0\,. \tag{50}$$

Using eq. (17), we can conclude that $(\vec{\boldsymbol{\nabla}}^2 + k^2)\psi(\vec{\boldsymbol{x}})$ is a radial function. A similar argument to the one given below eq. (45) implies that we can use the freedom to transform $\psi(\vec{\boldsymbol{x}}) \to \psi'(\vec{\boldsymbol{x}}) = \psi(\vec{\boldsymbol{x}}) + f(r)$ as indicated in eq. (9) to yield

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)\psi'(\vec{\boldsymbol{x}}) = 0\,. \tag{51}$$

Thus the second part of the theorem indicated by eq. (10) is proven.

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[3]For further details on the Green function of the inhomogeneous Helmholtz equation, see the class handout entitled: *The radial Green function.* In particular, the Green function satisfies:

$$(\vec{\boldsymbol{\nabla}}^2 + k^2)\left[\frac{e^{ik|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}}{4\pi|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\right] = -\delta^3(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,').$$

Inserting eq. (48) on the right hand side of eq. (46) and using the above result, we see that eq. (46) is satisfied.