The power spectrum of Cherenkov radiation

Consider a charge e moving along a trajectory, $\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{r}}(t')$, where t' is the retarded time. We choose the origin of our coordinate system to be in the region of space near the trajectory of the charge. First, suppose that the charge is moving with velocity $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ and acceleration $\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{v}}/dt = c\,d\vec{\boldsymbol{\beta}}/dt$. The radiation emitted by the charge is detected by an observer located at the point $\vec{\boldsymbol{x}}$. We define the unit vector $\hat{\boldsymbol{n}}$ by

$$\hat{\boldsymbol{n}} \equiv \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t)}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t)|}\,, \tag{1}$$

which points along the direction from the charge to the observer. Let us define $R \equiv |\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t)|$ and $r \equiv |\vec{\boldsymbol{x}}|$. Assuming that the observation point is very far away from the region of space where the trajectory of the charge is located, then

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}} + \mathcal{O}\left(\frac{1}{r}\right)\,. \tag{2}$$

That is, to paraphrase Jackson [see the text on p. 675 below eq. (14.62)], the unit vector $\hat{\boldsymbol{n}}$ is constant in time to a very good approximation.

In class, we showed that the power spectrum for a radiating charge e in vacuum is given by (in gaussian units)

$$\frac{d^2 I}{d\omega d\Omega} = \lim_{r\to\infty} c r^2 |\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}})|^2\,, \tag{3}$$

where[1]

$$\begin{aligned}\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t') e^{i\omega t'}\, dt' \\ &= \frac{e}{2\pi c^2 r}\, e^{i\omega r/c} \int_{-\infty}^{\infty} dt'\, e^{i\omega[t' - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{r}}(t')/c]}\, \frac{\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \vec{\boldsymbol{\alpha}}]}{(1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\beta}})^2} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \end{aligned} \tag{4}$$

where $\vec{\boldsymbol{\alpha}} \equiv d\vec{\boldsymbol{\beta}}/dt'$. Noting that:

$$\frac{d}{dt'}\left(\frac{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})}{1 - \vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{n}}}\right) = \frac{\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \vec{\boldsymbol{\alpha}}]}{(1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\beta}})^2}\,,$$

we may integrate eq. (4) by parts. Assuming that the surface term can be dropped,[2]

[1] Note that because the electric field, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t')$ is real, it follows that $\vec{\boldsymbol{E}}_{-\omega}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{E}}^*_\omega(\vec{\boldsymbol{x}})$. Hence it suffices to consider only positive frequencies.

[2] It is not obvious that the surface term can be legally dropped. One argument in its favor is based on treating $\lim_{t'\to\infty} e^{i\omega t'} = 0$ in the "sense of distributions" (cf. eq. (100) of the class handout entitled *Generalized Functions for Physics*). This is consequence of the Riemann-Lebesgue Lemma. The hand-waving argument is that in the $t' \to \infty$ limit, the oscillations of $e^{i\omega t'}$ are so fast that they effective washout (on average) any function that they multiply.

we find:

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{-ie\omega}{2\pi c^2 r}\, e^{i\omega r/c} \int_{-\infty}^{\infty} dt'\, e^{i\omega[t' - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{r}}(t')/c]}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) + \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{5}$$

We now suppose that the charge is moving at constant velocity $\vec{\boldsymbol{v}}$, with no acceleration. In this case, the particle trajectory is given by $\vec{\boldsymbol{r}}(t') = \vec{\boldsymbol{v}}t'$. Using the fact that we can approximate $\hat{\boldsymbol{n}}$ as being time-independent [cf. eq. (2)],

$$\begin{aligned}
\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) &= \frac{-ie\omega}{2\pi c^2 r}\, e^{i\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) \int_{-\infty}^{\infty} dt'\, e^{i\omega t'[1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c]} + \mathcal{O}\left(\frac{1}{r^2}\right) \\
&= \frac{-ie}{c^2 r}\, e^{i\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) ,
\end{aligned} \tag{6}$$

where in the last step, we used $\delta\left(\omega[1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c]\right) = \omega^{-1}\delta\left(1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c\right)$, since by assumption, ω is non-negative. We may use eq. (6) to determine $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$:

$$\begin{aligned}
\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) &= \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) \\
&= -\frac{2\pi ie}{c^2 r}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) \delta\left(t - \frac{r}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) .
\end{aligned}$$

Of course, since $|\vec{\boldsymbol{v}}| < c$, it follows that $\delta\left(1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c\right) = 0$. Hence as expected, there is no radiation from a charge moving at constant velocity.

In an isotropic, homogeneous medium where $\epsilon \neq 1$, the above results apply if we make the following transformations:[3] $\vec{\boldsymbol{E}} \to n_r \vec{\boldsymbol{E}}$, $c \to c/n_r$ and $e \to e/n_r$, where the index of refraction is $n_r \equiv \sqrt{\epsilon}$. In this case,

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{-ie\omega}{2\pi c^2 r}\, e^{in_r\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) \int_{-\infty}^{\infty} dt'\, e^{i\omega t'[1 - n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c]} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{7}$$

Evaluating the above integral, we obtain:

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{-ie}{c^2 r}\, e^{in_r\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{8}$$

If $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}} = c/n_r$ (which is possible if the charged particle is moving faster than the speed of light in the medium, c/n_r), then $\delta(1 - n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c) \neq 0$ and the resulting electric field does exhibit an $\mathcal{O}(1/r)$ behavior at large r. Indeed, one can verify that

$$\begin{aligned}
\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) &= \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) \\
&= -\frac{ie}{c^2 r}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t - n_r r/c)} + \mathcal{O}\left(\frac{1}{r^2}\right) \\
&= -\frac{2\pi ie}{c^2 r}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(t - \frac{n_r r}{c}\right) \delta\left(1 - \frac{n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) .
\end{aligned} \tag{9}$$

[3]For simplicity, we assume that the magnetic permeability $\mu = 1$.

Thus, radiation can occur—this is Cherenkov radiation. Note that the delta functions in eq. (9) imply that the electric field is singular on the surface of the Mach cone where $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}} = c/n_r$ and $r = ct/n_r$. These singularities arise due to an idealization of the problem (e.g. the assumption of a point charge); in a more realistic setting these singularities are smoothed out.[4]

To compute the power spectrum, one may be tempted to insert eq. (8) into eq. (3). However, this results in a square of a delta-function which requires a careful interpretation. Here, we provide a calculational method that avoids the square of the delta-function. Insert eq. (7) into eq. (3) [after replacing $\vec{\boldsymbol{E}} \to n_r\vec{\boldsymbol{E}}$ and $c \to c/n_r$ in the latter], and note the identity $|\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{v}})|^2 = |\hat{\boldsymbol{n}}\times\vec{\boldsymbol{v}}|^2$. Then,

$$\frac{d^2I}{d\omega d\Omega} = n_r c r^2 |\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}})|^2 = \frac{n_r e^2\omega^2}{4\pi^2 c^3}|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{v}}|^2 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, e^{i\omega(t'-t'')(1-n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c)}\,.$$

It is convenient to introduce two new integration variables:

$$T \equiv \tfrac{1}{2}(t'+t'')\,, \qquad\qquad t \equiv t'-t''\,.$$

Note that the Jacobian of the transformation from (t', t'') to (t, T) is unity. This change of variables is inspired by the treatment of Cherenkov radiation given in *Classical Electrodynamics* by Julian Schwinger et al. I quote from this textbook:

> [The time] t is of order $1/\omega$, thus setting the time scale for the emission of radiation.[5] This microscopic time scale may be much smaller than macroscopic time intervals; for example, for visible light, $t \sim 10^{-15}$ sec. The time T is then interpreted as the average (macroscopic) time of emission...

We therefore define the *power* distribution of the radiation by:

$$\frac{d^2I}{d\omega d\Omega} = \int_{-\infty}^{\infty} dT\, \frac{dP}{d\omega d\Omega}\,.$$

It then immediately follows that:

$$\begin{aligned}\frac{dP}{d\omega d\Omega} &= \frac{n_r e^2\omega^2}{4\pi^2 c^3}\,|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{v}}|^2 \int_{-\infty}^{\infty} dt\, e^{i\omega t(1-n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c)} \\ &= \frac{n_r e^2\omega^2}{2\pi c^3}\,|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{v}}|^2 \delta\left(\omega\left(1-\frac{n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right)\right)\,.\end{aligned}$$

Note that $dP/d\omega d\Omega$ is independent of T; that is, the rate of energy emission is constant in (macroscopic) time.

If we introduce the wave number vector in the medium, $\vec{\boldsymbol{k}} \equiv (n_r\omega/c)\hat{\boldsymbol{n}}$, then the above result can be rewritten as:

$$\frac{dP}{d\omega d\Omega} = \frac{n_r e^2\omega^2}{2\pi c^3}\,|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{v}}|^2\delta(\omega - \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{v}})\,.$$

[4]For example, see Glenn S. Smith, *Cherenkov radiation from a charge of finite size or a bunch of charges*, American Journal of Physics **61**, 147–155 (1993).

[5]In Fourier integrals of the form $\int_{-\infty}^{\infty} e^{i\omega t}F(t)dt$, where $F(t)$ is a well behaved function, the important range of t that contributes to the integral is of order $1/\omega$.

Finally, if we integrate over $d\Omega = 2\pi\, d\cos\psi$, where $\cos\psi \equiv \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{v}}$ and $v \equiv |\vec{\boldsymbol{v}}|$, we arrive at the Tamm-Frank formula:

$$\boxed{\frac{dP}{d\omega} = \frac{e^2\omega\, v}{c^2}\left(1 - \frac{c^2}{n_r^2 v^2}\right)\Theta(n_r v - c)} \tag{10}$$

Note that I have employed the step function,

$$\Theta(x) \equiv \begin{cases} 1\,, & \text{for } x > 0\,, \\ 0\,, & \text{for } x < 0\,, \end{cases}$$

in eq. (10) to emphasize that radiation only occurs if $v > c/n_r$. One should in mind that $n_r = n_r(\omega)$ depends on the frequency. In general, $n_r(\omega) \to 1$ as $\omega \to \infty$. Thus, Cherenkov radiation operates only over a narrow (finite) band of ω in which $n_r(\omega)v > c$.

Finally, one may wonder why Cherenkov radiation is possible given that the numerator of the integrand in eq. (4) vanishes when $\vec{\boldsymbol{\alpha}} \equiv d\vec{\boldsymbol{\beta}}/dt' = 0$. However, after making the replacement $\vec{\boldsymbol{E}} \to n_r\vec{\boldsymbol{E}}$, $c \to c/n_r$ and $e \to e/n_r$, we note that the denominator factor in eq. (4) is modified to $(1 - n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c)^2$. Thus, when $c/n_r < v < c$, the denominator can be zero, and one can no longer argue that setting $\vec{\boldsymbol{\alpha}} = 0$ will yield a vanishing electric field. The indeterminate 0/0 was resolved via the integration by parts used in deriving eq. (5), although this step depended on the validity of dropping the "surface term" in obtaining eq. (5) from eq. (4).

Perhaps a more satisfying derivation of Cherenkov radiation is based on the observation that a point charge moving with constant velocity in a material medium generates time-dependent elemental dipoles. If the velocity of the charge is larger than c/n_r (i.e. the speed of light in the medium), then there is a coherent superposition of the radiation emitted by each elemental dipole. The total coherent radiation can be identified as the Cherenkov radiation. This analysis provides for a very physical picture for the origin of Cherenkov radiation. For further details, see the article entitled *Induced time-dependent polarization and the Cherenkov effect* by J.A.E. Roa-Neri, J.L. Jimenez and M. Villavicencio, European Journal of Physics, **16**, 191–194 (1995).

I will end this short note with one final observation. In the vacuum, consider an observer at time t. The observer measures the radiation of a charge that was emitted at the retarded time $t_{\rm ret}$. Let us define the vector $\vec{\boldsymbol{R}}$ that points from the observer to the charge at the retarded time and the vector $\vec{\boldsymbol{x}}$ that points from the observer to the location of the charge at time t. If the charge is moving at constant velocity $\vec{\boldsymbol{v}}$, then $\vec{\boldsymbol{x}} = \vec{\boldsymbol{R}} + \vec{\boldsymbol{v}}(t - t_{\rm ret})$. Noting that $|\vec{\boldsymbol{R}}| = c(t - t_{\rm ret})$, it follows that

$$c(t - t_{\rm ret}) = |\vec{\boldsymbol{x}} + \vec{\boldsymbol{v}}(t - t_{\rm ret})|\,. \tag{11}$$

Eq. (11) yields a quadratic equation for $t - t_{\rm ret}$:

$$c^2(t - t_{\rm ret})^2 = r^2 + v^2(t - t_{\rm ret})^2 + 2\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}}(t - t_{\rm ret})\,, \tag{12}$$

where $r \equiv |\vec{\boldsymbol{x}}|$ and $v \equiv |\vec{\boldsymbol{v}}|$. It follows that

$$t - t_{\rm ret} = \frac{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}} \pm \sqrt{(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}})^2 + (c^2 - v^2)r^2}}{c^2 - v^2} \,. \tag{13}$$

By definition, $t_{\rm ret} < t$. Thus eq. (13) yields a unique positive real solution for $t - t_{\rm ret}$, since $v < c$ for any particle of nonzero mass.

In the case of a charged particle moving at constant velocity in a medium, we must replace $c \to c/n_r$. If $v < c/n_r$, then the modified version of eq. (13) with $c \to c/n_r$ still yields a unique positive real solution for $t - t_{\rm ret}$. But if $c/n_r < v < c$, then the modified version of eq. (13) with $c \to c/n_r$ has no positive real solutions if $\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}} > 0$. On the other hand, if $\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}} < 0$, then real roots exist if

$$(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}})^2 > r^2\left(v^2 - \frac{c^2}{n_r^2}\right) \,. \tag{14}$$

If we denote the angle between $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{v}}$ by θ, then $\cos\theta$ must lie in the range,

$$-1 \leq \cos\theta < -\sqrt{1 - \frac{c^2}{n_r^2 v^2}} \,. \tag{15}$$

Moreover, there are *two* positive roots if $\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}} < 0$ (and $v > c/n_r$), since

$$|\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}}| > \sqrt{(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{v}})^2 - (v^2 - c^2/n_r^2)r^2}.$$

That is, there are two retarded times!

To be more precise, we should designate the two solutions by $t_{\rm ret}$ and $t_{\rm adv}$, corresponding to the usual retarded time and the "advanced time," where $t_{\rm ret} < t_{\rm adv} < t$. One should then carefully reconsider some of the derivations presented for radiation problems in vacuum, when applied to a moving charge in a medium, in light of this observation.