Consider the $n \times n$ matrix

$$A_{ij} = \delta_{ij} + a_i b_j \,.$$

In this note, we shall provide three different proofs that

$$\det A = 1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}\,. \tag{1}$$

First proof

One definition of the determinant is

$$\det A = \epsilon_{i_1 i_2 i_3 \cdots i_n} A_{1i_1} A_{2i_2} A_{3i_3} \cdots A_{ni_n} \,,$$

where there is an implicit n-fold sum over i_1, i_2, $i_3, \ldots, i_n$. Thus,

$$\begin{aligned}\det A &= \epsilon_{i_1 i_2 i_3 \cdots i_n}(\delta_{1i_1} + a_1 b_{i_1})(\delta_{2i_2} + a_2 b_{i_2})(\delta_{1i_3} + a_3 b_{i_3}) \cdots (\delta_{ni_n} + a_n b_{i_n})\,. \\ &= \epsilon_{123\cdots n} + a_1 b_{i1}\epsilon_{i_1 23\cdots n} + a_2 b_{i_2}\epsilon_{1 i_2 3\cdots n} + a_3 b_{i_3}\epsilon_{12 i_3 \cdots n} + \ldots - a_n b_{i_n}\epsilon_{123\cdots i_n} + R\,, \end{aligned} \tag{2}$$

where R consists of a sum of terms that contain two or more factors of b. But $R = 0$, since for any two indices i_j and i_k ($1 \leq j < k \leq n$),

$$\epsilon_{i_1 i_2 i_3 \cdots i_n} b_{i_j} b_{i_k} = 0\,,$$

due to the fact that $b_{i_j} b_{i_k}$ is symmetric under the interchange of i_j and i_k, whereas $\epsilon_{i_1 i_2 i_3 \cdots i_n}$ is antisymmetric under the interchange of any pair of indices. Moreover, $\epsilon_{123\cdots n} = 1$. Hence, eq. (2) yields

$$\det A = 1 + a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots + a_n b_n = 1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}\,,$$

which confirms eq. (1).

Second proof

We shall compute the determinant by finding all of the eigenvalues and eigenvectors of A. One eigenvector is immediately apparent, namely $(a_1\,, a_2\,, a_3\,, \ldots\,, a_n)$. This can be easily verified since

$$A_{ij} a_j = (\delta_{ij} + a_i b_j) a_j = a_i(1 + a_j b_j) = (1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}) a_i\,.$$

The corresponding eigenvalue is $1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}$.

Remarkably, it is a simple matter to identify the remaining $n-1$ eigenvectors. A convenient choice is: $(b_2\,, -b_1\,, 0\,, \ldots\,, 0)$, $(b_3\,, 0\,, -b_1\,, \ldots\,, 0)\,, \ldots\,, (b_n\,, 0\,, 0\,, \ldots\,, -b_1)$. It is easy to check that the corresponding eigenvalues are degenerate and equal to 1. For example,

$$\begin{pmatrix} 1 + a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & 1 + a_2 b_2 & \cdots & a_2 b_n \\ \ddots & \ddots & \vdots & \ddots \\ a_n b_1 & a_n b_2 & \cdots & 1 + a_n b_n \end{pmatrix} \begin{pmatrix} b_2 \\ -b_1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} b_2 \\ -b_1 \\ \vdots \\ 0 \end{pmatrix}\,.$$

First, we assume that $\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}} \neq 0$. Then, the eigenvalue $1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}$ is not degenerate with the $(n-1)$-fold degenerate eigenvalue 1. Since the determinant of A is equal to the product of its eigenvalues, it immediately follows that $\det A = 1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}$, in agreement with eq. (1).

Second, suppose that $\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}} = 0$. Then $(a_1\,,\, a_2\,,\, a_3\,,\, \ldots\,,\, a_n)$ and $(b_1\,,\, b_2\,, b_3\,,\, \ldots\,,\, b_n)$ are orthogonal. But, note that $(b_1\,,\, b_2\,, b_3\,,\, \ldots\,,\, b_n)$ is also orthogonal to the $n-1$ vectors, $(b_2\,,\, -b_1\,,\, 0\,, \ldots\,,\, 0)$, $(b_3\,,\, 0\,,\, -b_1\,, \ldots\,,\, 0)$, …, $(b_n\,,\, 0\,,\, 0\,,\, \ldots\,,\, -b_1)$. Since there can be at most n mutually orthogonal vectors, we conclude that $(a_1\,,\, a_2\,,\, a_3\,,\, \ldots\,,\, a_n)$ must be a linear combination of $(b_2\,,\, -b_1\,,\, 0\,, \ldots\,,\, 0)$, $(b_3\,,\, 0\,,\, -b_1\,, \ldots\,,\, 0)\,, \ldots,\, (b_n\,,\, 0\,,\, 0\,,\, \ldots\,,\, -b_1)$. This means that if $\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}} = 0$, then there are only $n-1$ linearly independent eigenvectors with eigenvalue equal to 1. Nevertheless, the eigenvalue 1 in this case is n-fold degenerate. To prove this assertion, we evaluate the trace of A,

$$\mathrm{Tr}\, A = \delta_{ij} A_{ij} = \delta_{ij}(\delta_{ij} + a_i b_j) = n + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}\,.$$

If $\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}} = 0$, then $\mathrm{Tr}\, A = n$. Since the trace of A is equal to the sum of its eigenvalues, it then follows that the nth eigenvalue is 1, in which case the eigenvalue 1 is n-fold degenerate. The corresponding determinant is then $\det A = 1$, which is in agreement with eq. (1) in the case of $\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}} = 0$. The proof of eq. (1) is now complete.

Third proof

The proof is based on the formula,

$$\det \begin{pmatrix} F & C \\ B & D \end{pmatrix} = \det D\, \det\,(F - CD^{-1}B) = \det F\, \det\,(D - BF^{-1}C)\,, \tag{3}$$

where F is an invertible $n \times n$ matrix, B is an $m \times n$ matrix, C is an $n \times m$ matrix and D is an invertible $m \times m$ matrix. To prove eq. (3) we employ the following matrix identities:

$$\begin{pmatrix} F & C \\ B & D \end{pmatrix} = \begin{pmatrix} I_n & C \\ 0 & D \end{pmatrix} \begin{pmatrix} F - CD^{-1}B & 0 \\ D^{-1}B & I_m \end{pmatrix} = \begin{pmatrix} F & 0 \\ B & I_m \end{pmatrix} \begin{pmatrix} I_n & F^{-1}C \\ 0 & D - BF^{-1}C \end{pmatrix}\,, \tag{4}$$

where I_p is the $p \times p$ identity matrix and $\det I_p = 1$. We now make use of the following result,

$$\det \begin{pmatrix} F & 0 \\ B & D \end{pmatrix} = \det \begin{pmatrix} F & C \\ 0 & D \end{pmatrix} = \det F\, \det D\,, \tag{5}$$

which is proved in Appendix A. Taking the determinants of the two identities given in eq. (4) (noting that the determinant of a product of matrices is equal to the product of the individual determinants) and employing eq. (5), we end up with eq. (3).

The special case of eq. (3) that we will exploit to obtain eq. (1) is,

$$\det \begin{pmatrix} I_n & -A^{\mathsf{T}} \\ B & I_m \end{pmatrix} = \det\,(I_n + A^{\mathsf{T}} B) = \det\,(I_m + BA^{\mathsf{T}})\,, \tag{6}$$

where A and B are $m \times n$ matrices and A^{T} is the transpose of A. In particular, if $m = 1$ then $A = (a_1\,,\, a_2\,,\, a_3\,,\, \ldots\,,\, a_n)$ and $B = (b_1\,,\, b_2\,,\, b_3\,,\, \ldots\,,\, b_n)$. In particular,

$$\det\,(I_n + A^{\mathsf{T}} B) = \det\,(I_m + BA^{\mathsf{T}})$$

is equivalent to

$$\det\,(\delta_{ij} + a_i b_j) = 1 + \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}\,,$$

since the determinant of the 1×1 matrix $I_m + BA^{\mathsf{T}}$ (where $m = 1$) is equal to its matrix element. Once again, eq. (1) is verified.

APPENDIX A: The determinant of a matrix in block form

Here, I shall provide a proof of the following identity

$$\det\begin{pmatrix} F & 0 \\ B & D \end{pmatrix} = \det\begin{pmatrix} F & C \\ 0 & D \end{pmatrix} = \det F\,\det D\,, \tag{A.1}$$

where F is an invertible $n \times n$ matrix, B is an $m \times n$ matrix, C is an $n \times m$ matrix and D is an invertible $m \times m$ matrix. First, we consider the case of $n = 1$,

$$\det\begin{pmatrix} f & 0 \\ B & D \end{pmatrix}\,, \tag{A.2}$$

where f is a number. We can evaluate the determinant by using the cofactor expansion along the first row.* One then immediately obtains

$$\det\begin{pmatrix} f & 0 \\ B & D \end{pmatrix} = f\,\det D\,, \tag{A.3}$$

which verifies eq. (A.1) in the case of $n = 1$. The case of $n = 2$ is almost as simple,

$$\det\begin{pmatrix} f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ & & & \\ & B & & D \end{pmatrix} = f_{11}f_{22}\det D - f_{12}f_{21}\det D = (f_{11}f_{22} - f_{12}f_{21})\det D = \det F\,\det D\,, \tag{A.4}$$

after applying the cofactor expansion along the first row and making use of eq. (A.3).

To generalize to arbitrary n, we shall use the principle of mathematical induction. Assuming that eq. (A.1) holds for n, we shall prove the result for $n + 1$. In the case of $n + 1$, the cofactor expansion along the first row yields an expression for the determinant that is a sum of $n + 1$ terms,

$$\det\begin{pmatrix} F & 0 \\ B & D \end{pmatrix} = \sum_{k=1}^{n+1}(-1)^{k+1}f_{1k}\,\det A_{1k}\,, \tag{A.5}$$

where A_{1k} is the $n\times n$ matrix obtained by deleting row 1 and column k of the matrix $\left(\begin{smallmatrix} F & 0 \\ B & D \end{smallmatrix}\right)$, and f_{1k} is the matrix element corresponding to the first row and the kth column of the matrix F. You can easily verify that in the case of $n = 1$, eq. (A.5) reduces to eq. (A.4).

Since we are assuming that eq. (A.1) holds for n, it follows that

$$\det A_{1k} = \det F_{1k}\,\det D\,, \tag{A.6}$$

where F_{1k} is the $n \times n$ matrix obtained by deleting row 1 and column k of the matrix F. Thus, eqs. (A.5) and (A.6) yield

$$\det\begin{pmatrix} F & 0 \\ B & D \end{pmatrix} = \left(\sum_{k=1}^{n+1}(-1)^{k+1}f_{1k}\,\det F_{1k}\right)\det D = \det F\,\det D\,, \tag{A.7}$$

where we have recognized the expression for $\det F$ by the cofactor expansion along the first row. Thus, the proof by induction is now complete and the first part of eq. (A.1) is established. To prove the second part of eq. (A.1), simply follow the same steps as above while employing the cofactor expansion along column 1 to evaluate the relevant determinants.

*See, e.g., Section 3.2.3 of Nathaniel Johnson, *Introduction to Linear and Matrix Algebra* (Springer Nature Switzerland, Cham, Switzerland, 2021).